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NESBITT'S TYPE INEQUALITIES WITH CONSTRAINT

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Let $a,b,c=[0,\infty)$ such that at most one is zero. Prove that:

1. If $8(a^2+b^2+c^2)=11(ab+bc+ca)$ then:

$$\frac{69}{40}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{51}{28}$$

2. If $a^2+b^2+c^2=2(ab+bc+ca)$ then:

$$2\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{12}{5}$$ **(CRUX 46-3 Lorian Saceanu 4524)**

3. If $5(a^2+b^2+c^2)=17(ab+bc+ca)$ then:

$$\frac{17}{5}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{15}{4}$$

4. If $a^2+b^2+c^2=4(ab+bc+ca)$ then:

$$4\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3(3\sqrt{2}+16)}{4}$$

We will prove that:

Lemma 1:

If $a,b,c\in[0,\infty)$ such that at most one is zero then denoting $a+b+c=p$, $ab+bc+ca=q$ and $abc=r$, we have:

(1) $p^2\ge 3q$

(2) $p^2\in[3q,4q]\Rightarrow r\in\left[\frac{9pq-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27};\frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

(3) $p\in[4q,\infty)\Rightarrow r\in\left[0,\frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

Proof.

The equation that has a,b,c as roots is

$x^3-px^2+qx-r=0$. We consider the function $f:\mathbb{R}\to\mathbb{R}$

$f(x)=x^2-px^2+qx-r$. In order that $f(x)=0$ to have three real positive roots it must that:

$f'(x)=3x^2-2px+q=0$ to have real positive roots.

Then $\Delta=4p^2-12q\geq 0\Rightarrow p^2\geq 3q$ (1)

Let $x_{1,2}=\frac{2p\pm 2\sqrt{p^2-3q}}{6}=\frac{p\pm\sqrt{p^2-3q}}{3}$ the root of $f'(x)=0$

Then $f\left(\frac{p-\sqrt{p^2-3q}}{3}\right)=\left(\frac{p-\sqrt{p^2-3q}}{3}\right)^3-p\left(\frac{p-\sqrt{p^2-3q}}{3}\right)^2+q\left(\frac{p-\sqrt{p^2-3q}}{3}\right)-r=$

$=\frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}-r$ and

$$f\left(\frac{p+\sqrt{p^2-3q}}{3}\right)=\left(\frac{p+\sqrt{p^2-3q}}{3}\right)^3-p\left(\frac{p+\sqrt{p^2-3q}}{3}\right)^2+q\left(\frac{p+\sqrt{p^2-3q}}{3}\right)+r$$

$$=\frac{9pq-3p^3-2(p^2-3q)^{\frac{3}{2}}}{27}-r$$

Obviously $f(0)=-r$ and $\lim_{x\to\infty}f(x)=\infty$

Then, according to Rolle's sequence and taking into account that:

$\frac{p-\sqrt{p^2-3q}}{3}\leq\frac{p+\sqrt{p^2-3q}}{3}$ we obtain:

$$f\left(\frac{p+\sqrt{p^2-3q}}{3}\right)\leq 0\Rightarrow r\geq\frac{9p^2-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27}$$

$$f\left(\frac{p-\sqrt{p^2-3q}}{3}\right)\geq 0\Rightarrow r\leq\frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}$$

$$f(0)\leq 0\Rightarrow -r\leq 0\Rightarrow r\geq 0$$

In conclusion $r\in\left[\frac{9pq-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27};\frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]\cap[0,\infty)$

Taking into account that $p^2\geq 3q$ we prove that $9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}\geq 0$, namely

$2(p^2-3q)^{\frac{3}{2}}\geq 2p^3-9pq$

We prove that $2(p^2-3q)^{\frac{3}{2}}\geq p(2p^2-9q)$

If $2p^2 - 9q \leq 0$ namely $p^2 \leq \frac{9}{2}q$ the inequality is obvious. If $p^2 \geq \frac{9}{2}q$ the inequality is equivalent with $4(p^2-3q)^3 \geq p^2(2p^2-9q)^2 \Leftrightarrow q^2(p^2-4q) \geq 0$ true.

We study the expression's sign $9pq - 2p^3 - 2(p^2-3q)^{\frac{3}{2}}$

We establish the relationship between p^2 and q in order that the expression to be positive.

$$9pq - 2p^3 - 2(p^2-3q)^{\frac{3}{2}} \geq 0 \Leftrightarrow 9pq - 2p^3 \geq 2(p^2-3q)^{\frac{3}{2}} \Leftrightarrow$$

$$\Leftrightarrow p(9q-2p^2) \geq 2(p^2-3q)^{\frac{3}{2}}$$

Obviously for $p^2 > \frac{9}{2}q$ the expression is negative.

For $p^2 \leq \frac{9}{2}q$ (and $p^2 \geq 3q$) the inequality is equivalent with $p^2(9q-2p^2)^2 \geq 4(p^2-3q)^3$

$$\Leftrightarrow q^2(4q-p^2) \geq 0 \Leftrightarrow p^2 \leq 4q$$

Hence for $p^2 \in [3q, 4q]$ the expression we study is positive and for $p^2 \leq [4q, \infty)$ the expression is negative.

We obtain:

For $p^2 \in [3q, 4q] \Rightarrow r \in \left[\frac{9pq-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27}; \frac{9pq-3p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

For $p^2 \in [4q, \infty) \Rightarrow r \in \left[0, \frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

Lemma 2.

If $a, b, c \in [0, \infty)$ such that at most one is zero, then denoting $a+b+c=p$, $ab+bc+ca=q$ and $abc=r$ we have:

(1) It exist $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$

(2) If $t \in \left[0, \frac{1}{2}\right] \Rightarrow \frac{r}{p^3} \in \left[\frac{(t+1)^2(1-2t)}{27}; \frac{(t-1)^2(2t+1)}{27}\right]$

(3) If $t \in \left[\frac{1}{2}; 1\right) \Rightarrow \frac{r}{s^3} \in \left[0, \frac{(t-1)^2(2t+1)}{27}\right]$

Proof.

According to Lemma 1 (1) $\frac{p^2}{q} \geq 3$. We prove that it exists $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$

$$\Leftrightarrow 1-t^2=\frac{3q}{p^2}\Leftrightarrow t^2=\frac{p^2-3q}{p^2}\Rightarrow t=\pm\frac{\sqrt{p^2-3q}}{p}$$

We choose $t\in\frac{\sqrt{p^2-3q}}{p}$ and we prove $t\in[0,1)$

Obviously $t\geq 0$. We prove $t<1\Leftrightarrow\frac{\sqrt{p^2-3q}}{p}<1\Leftrightarrow p^2-3q<p^2\Leftrightarrow -3q<0\Leftrightarrow q>0$

true $(a,b,c\in[0,\infty)$ at most one is zero)

So, it exists $t\in[0,1)$ such that $\frac{p^2}{q}=\frac{3}{1-t^2}$ (1), $t\in\frac{\sqrt{p^2-3q}}{p}$

According to Lemma 1

(2) $\frac{p^2}{q}\in[3,4]\Rightarrow\frac{r}{p^3}\in\left[\frac{9\frac{q}{p^2}-2-2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27};\frac{9\frac{q}{p^2}-2+2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27}\right]$

(3) $\frac{p^2}{q}\in[4,\infty)\Rightarrow\frac{r}{p^3}\in\left[0,\frac{9\frac{q}{p^2}-2+2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27}\right]$

We replace in these relationships $\frac{p^2}{q}=\frac{3}{1-t^2}$ and we obtain:

(2) $\frac{3}{1-t^2}\in[3,4]\Rightarrow\frac{3}{p^3}\in\left[\frac{9\frac{1-t^2}{3}-2-2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27};\frac{9\frac{1-t^2}{3}-2+2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27}\right]$

(3) $\frac{3}{1-t^2}\in[4,\infty)\Rightarrow\frac{r}{p^3}\in\left[0,\frac{9\frac{1-t^2}{3}-2+2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27}\right]$

After we make the calculus the relationship becomes:

(2) $t\in\left[0,\frac{1}{2}\right]\Rightarrow\frac{r}{p^3}\in\left[\frac{-2t^3-3t^2+1}{27};\frac{2t^3-3t^2+1}{27}\right]$

(3) $t\in\left[\frac{1}{2},1\right)\Rightarrow\frac{r}{s^3}\in\left[0,\frac{2t^3-3t^2+1}{27}\right]$

or

(2) $t\in\left[0,\frac{1}{2}\right]\Rightarrow\frac{r}{p^3}\in\left[\frac{(t+1)^2(1-2t)}{27};\frac{(t-1)^2(2t+1)}{27}\right]=\mathrm{D}_1$

(3) $t\in\left[\frac{1}{2},1\right)\Rightarrow\frac{r}{p^3}\in\left[0,\frac{(t-1)^2(2t+1)}{27}\right]=\mathrm{D}_2$

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Theorem.

Let $a,b,c\in[0,\infty)$ such that at most one is zero. Let $a+b+c=p$,

$ab+bc+ca=q, abc=r$ and $t\in[0,1)$ such that $\frac{p^2}{q}=\frac{3}{1-t^2}$

(1) If $t\in\left[0,\frac{1}{2}\right]$ then:

$$\frac{3}{2}\frac{2t^2+t+2}{(t+1)(2-t)}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(t+2)}$$

(2) If $t\in\left[\frac{1}{2},1\right)$ then:

$$\frac{1+2t^2}{1-t^2}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(t+2)}$$

Proof.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{p-a}+\frac{b}{p-b}+\frac{c}{p-c}=$$

$$=\frac{a(p-b)(p-c)+b(p-a)(p-c)+c(p-a)(p-b)}{(p-a)(p-b)(p-c)}=$$

$$=\frac{p^3-2pq+3r}{pq-r}=\frac{1-2\frac{q}{p^2}+3\frac{r}{p^3}}{\frac{q}{p^2}-\frac{r}{p^3}}=$$

$$=\frac{1-2\frac{1-t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}=\frac{\frac{3-2+2t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}=\frac{\frac{1+t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}$$

We denote $\frac{r}{p^3}=x$ and we consider the function:

$f(x)=\frac{\frac{1+2t^2}{3}+3x}{\frac{1-t^2}{3}-x}; f:D_1\to\mathbb{R}$ if $t\in\left[0,\frac{1}{2}\right]$ or $f:D_2\to$ if $t\in\left[\frac{1}{2},1\right)$

$$f'(x)=\frac{3\left(\frac{1-t^2}{3}-x\right)+\left(\frac{1+2t^2}{3}+3x\right)}{\left(\frac{1-t^2}{3}-x\right)^2}=\frac{1-t^2-3x+\frac{1+2t^2}{3}+3x}{\left(\frac{1-t^2}{3}-x\right)^2}$$

$$=\frac{3-3t^2+1+2t^2}{3\left(\frac{1-t^2}{3}-x\right)^2}=\frac{4-t^2}{3\left(\frac{1-t^2}{3}-x\right)^2}>0$$

So, f strictly increasing on D_1 and D_2

Then

(1) for $t\in\left[0,\frac{1}{2}\right]$

$$f\left(\frac{(t+1)^2(1-2t)}{27}\right)\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le f\left(\frac{(t-1)^2(2t+1)}{27}\right)$$

(2) for $t\in\left[\frac{1}{2},1\right)$

$$f(0)\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le f\left(\frac{(t-1)^2(2t+1)}{27}\right)$$

namely, after calculus

(1) For $t\in\left[0,\frac{1}{2}\right]$

$$\frac{3}{2}\frac{2t^2+t+2}{(t+1)(2-t)}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(2+t)}$$

(2) For $t\in\left[\frac{1}{2},1\right)$

$$\frac{1+2t^2}{1-t^2}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(2+t)}$$

Observation. For $t=\frac{1}{3}$ we obtain ex 1), $t=\frac{1}{2}$ we obtain ex 2), $t=\frac{2}{3}$ we obtain ex 3) and for $t=\frac{\sqrt{2}}{2}$ we obtain ex 4).

Refference:

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