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In ΔABC the following relationship holds:

$$\frac{m_a w_a + m_b w_b + m_c w_c}{\sqrt{m_a r_a} + \sqrt{m_b r_b} + \sqrt{m_c r_c}} \le \frac{(4R+r)^2}{9r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Tran Hong-Dong Thap-Vietnam

In any ΔABC:

$$m_a^2 + m_b^2 + m_c^2 + w_a^2 + w_b^2 + w_c^2 \le \frac{3}{4}(a^2+b^2+c^2) + [s(s-a)+s(s-b)+s(s-c)]$$

$$= \frac{3}{2}(s^2 - 4Rr - r^2) + s^2 = \frac{5s^2 - 12Rr - 3r^2}{2} \overset{(1)}{\le} \frac{2(4R+r)^2}{3}$$

$$(1) \Leftrightarrow 3(5s^2 - 12Rr - 3r^2) \le 4(4R+r)^2$$

But:

$$s^2 \le 4R^2 + 4Rr + 3r^2\ (Gerretsen) \Rightarrow 5s^2 \le 20R^2 + 20Rr + 15r^2 \Rightarrow$$

$$3(5s^2 - 12Rr - 3r^2) \le 60R^2 + 60Rr + 45r^2 - 36Rr - 9r^2 \overset{(2)}{\le} 4(16R^2 + 8Rr + r^2)$$

$(2) \Leftrightarrow 4R^2 + 8Rr - 32r^2 \geq 0 \Leftrightarrow R^2 + 2Rr - 8r^2 \geq 0$ true because:

$R \geq 2r\ (Euler) \Rightarrow R^2 + 2Rr - 8r^2 \geq 4r^2 + 4r^2 - 8r^2 = 0 \Rightarrow$(2) is true $\Rightarrow$(1) is true

Now,

$$m_a w_a + m_b w_b + m_c w_c \overset{AM-GM}{\leq} \frac{\sum_{cyc}(m_a^2 + w_a^2)}{2} \leq \frac{(4R + r)^2}{3}$$

$$\sqrt{m_a r_a} + \sqrt{m_b r_b} + \sqrt{m_c r_c} \overset{AM-GM}{\geq} 3\sqrt[3]{\sqrt{m_a m_b m_c r_a r_b r_c}} \geq 3\sqrt[3]{s^2 r} \overset{Mitrinovic}{\geq}$$

$$\geq 3\sqrt[3]{27r^3} = 9r;$$

$$\Rightarrow LHS = \frac{m_a w_a + m_b w_b + m_c w_c}{\sqrt{m_a r_a} + \sqrt{m_b r_b} + \sqrt{m_c r_c}} \leq \frac{(4R + r)^2}{3 \cdot 9r} < \frac{(4R + r)^2}{9r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a w_a \overset{A-G}{\leq} \frac{m_a^2 + w_a^2}{2} \leq \frac{m_a^2 + s(s-a)}{2} \text{ and analogs} \overset{\text{summing up}}{\Rightarrow} \sum m_a w_a \leq \frac{\frac{3}{4}\sum a^2 + s^2}{2}$$

$$= \frac{\frac{3(s^2 - 4Rr - r^2)}{2} + s^2}{2}$$

$$\Rightarrow \sum m_a w_a \overset{(1)}{\leq} \frac{5s^2 - 12Rr - 3r^2}{4} \text{ and } \sum \sqrt{m_a r_a} \geq \sum \sqrt{h_a r_a} = \sum \sqrt{\frac{2rs.\, s\sin\frac{A}{2}}{4R\sin\frac{A}{2}\cos^2\frac{A}{2}}}$$

$$= s\sqrt{\frac{r}{2R}} \sum \sec\frac{A}{2} \overset{\text{Jensen}}{\geq} 3s\sqrt{\frac{r}{2R}} \sec\frac{\pi}{6}$$

$$= 2\sqrt{3}s\sqrt{\frac{r}{2R}} \left(\because f(x) = \sec\frac{x}{2}\ \forall x \in (0, \pi) \text{ is convex as } f''(x) > 0\right)$$

$$\Rightarrow \sum \sqrt{m_a r_a} \overset{(2)}{\geq} 2\sqrt{3}s\sqrt{\frac{r}{2R}} \therefore (1), (2) \Rightarrow \frac{m_a w_a + m_b w_b + m_c w_c}{\sqrt{m_a r_a} + \sqrt{m_b r_b} + \sqrt{m_c r_c}}$$

$$\leq \frac{\frac{5s^2 - 12Rr - 3r^2}{4}}{2\sqrt{3}s\sqrt{\frac{r}{2R}}} \overset{?}{\leq} \frac{(4R + r)^2}{18r} \Leftrightarrow 4.81r^2\left(\frac{5s^2 - 12Rr - 3r^2}{4}\right)^2 \overset{?}{\leq} \frac{6rs^2}{R}(4R + r)^4$$

$$\Leftrightarrow 8s^2(4R + r)^4 \underset{(i)}{\overset{?}{\geq}} 27Rr(5s^2 - 12Rr - 3r^2)^2$$

$$\text{Now, } 2s^2 \overset{\text{Gerretsen}}{\geq} 27Rr + 5r(R - 2r) \overset{\text{Euler}}{\geq} 27Rr \text{ and}$$

$\therefore$ in order to prove (i), it suffices to prove

$$: 4(4R + r)^4 \overset{?}{\geq} (5s^2 - 12Rr - 3r^2)^2$$

$$\Leftrightarrow 2(4R+r)^2 \overset{?}{\gtreqless} 5s^2 - 12Rr - 3r^2 \Leftrightarrow 2(4R+r)^2 - 5s^2 + 12Rr + 3r^2 \overset{?}{\gtreqless} 0 \to \text{true}$$

$$\because 2(4R+r)^2 - 5s^2 + 12Rr + 3r^2$$

$$\overset{\text{Trucht}}{\gtreqless} s^2 + 12Rr + 3r^2 > 0 \therefore (i)\ \textit{is true} \Rightarrow \frac{m_a w_a + m_b w_b + m_c w_c}{\sqrt{m_a r_a} + \sqrt{m_b r_b} + \sqrt{m_c r_c}} < \frac{(4R+r)^2}{18r}$$

$$< \frac{(4R+r)^2}{9r}\ (\text{Proved})$$

Note by editor:

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