R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

In ΔABC the following relationship holds:

$$\frac{4R+r}{6}\sqrt{\frac{4R+r}{r}} \geq \frac{h_a^2}{w_a+m_a}+\frac{h_b^2}{w_b+m_b}+\frac{h_c^2}{w_c+m_c} \geq \frac{r^2}{R}$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Bogdan Fuştei-Romania; Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Bogdan Fuştei-Romania

$$\sum_{cyc}\frac{h_a^2}{w_a+m_a} \geq \sum_{cyc}\frac{h_a^2}{2m_a}=\frac{1}{2}\cdot\sum_{cyc}\frac{h_a^2}{m_a};(1)$$

From $\frac{R}{2r} \geq \frac{m_a}{h_a}$ (Panaitopol inequality) we have: $\frac{h_a}{2m_a} \geq \frac{r}{R} \Leftrightarrow \frac{h_a^2}{2m_a} \geq \frac{r}{R}\cdot h_a$ and analogs.

$$\frac{1}{2}\cdot\sum_{cyc}\frac{h_a^2}{m_a} \geq \frac{r}{R}(h_a+h_b+h_c) \geq \frac{r}{R}\cdot 9r=\frac{9r^2}{R} > \frac{r^2}{R};\quad (2)$$

$h_a^2 \le s(s-a)$ and analogs (because: $h_a \le w_a \le \sqrt{s(s-a)}$)

$m_a \ge \frac{b+c}{2} \cdot \cos\frac{A}{2}$ and analogs, then we have:

$m_a \cdot w_a \ge s(s-a) \Rightarrow \ 2m_a \cdot w_a \ge 2s(s-a)$ and with

$m_a^2 + w_a^2 \ge 2m_a w_a) \Rightarrow (m_a + w_a)^2 = m_a^2 + w_a^2 + 2m_a w_a \ge 4s(s-a)$

$m_a + w_a \ge 2\sqrt{s(s-a)}$ and analogs.

$\frac{s(s-a)}{2\sqrt{s(s-a)}} \ge \frac{h_a^2}{m_a+w_a} \Rightarrow \frac{1}{2}\sqrt{s(s-a)} \ge \frac{h_a^2}{m_a+w_a}$ and analogs.

So,we have: $\frac{1}{2}\sum_{cyc}\sqrt{s(s-a)} \ge \sum_{cyc}\frac{h_a^2}{m_a+w_a}$

Applying C-B-S and we have: $s\sqrt{3} \ge \sum_{cyc}\sqrt{s(s-a)}$

$$\frac{s\sqrt{3}}{2} \ge \frac{1}{2}\sum_{cyc}\sqrt{s(s-a)}$$

But: $4R + r \ge s\sqrt{3} \Rightarrow \frac{4R+r}{2} \ge \sum_{cyc}\frac{h_a^2}{m_a+w_a}$

We must show that: $\frac{4R+r}{6}\sqrt{\frac{4R+r}{r}} \ge \frac{4R+r}{2} \Leftrightarrow$

$\sqrt{\frac{4R+r}{r}} \ge 3 \Leftrightarrow \frac{4R+r}{r} \ge 9 \Leftrightarrow R \ge 2r$ (Euler).Proved.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\Omega = \frac{h_a^2}{w_a + m_a} + \frac{h_b^2}{w_b + m_b} + \frac{h_c^2}{w_c + m_c} \overset{C-B-S}{\ge}$$

$$\frac{(h_a + h_b + h_c)^2}{(m_a + m_b + m_c) + (w_a + w_b + w_c)} \overset{\sum w_a \le \sum m_a \le \frac{9R}{2}}{\ge}$$

$$\geq \frac{(h_a+h_b+h_c)^2}{2\cdot\frac{9R}{2}} = \frac{\left(\frac{s^2+r^2+4Rr}{2R}\right)^2}{9R} = \frac{(s^2+r^2+4Rr)^2}{36R^2} \overset{(*)}{\geq} \frac{r^2}{R}$$

$$(*) \Leftrightarrow (s^2+r^2+4Rr)^2 \geq 36r^2R^2 \Leftrightarrow s^2+r^2+4Rr \geq 6Rr$$

$$\Leftrightarrow s^2 \geq 2Rr - r^2$$

But: $s^2 \geq 16Rr - 5r^2$ **(Mitrinovic inequality), then**

$$16Rr - 5r^2 \overset{(**)}{\geq} 2Rr - r^2$$

$(**) \Leftrightarrow 14Rr \geq 4r^2 \Leftrightarrow R \geq \frac{2}{7}r$ **(true because:** $R \geq 2r \geq \frac{2}{7}r$ **Euler)**
$\Leftrightarrow (**)$ **−is true**$\Leftrightarrow (*)$ **−is true.**

$w_a, m_a \geq h_a \Rightarrow w_a + m_a \geq 2h_a$ **and analogs.**

$$\Omega = \sum_{cyc}\frac{h_a^2}{w_a+m_a} \leq \sum_{cyc}\frac{h_a^2}{2h_a} = \frac{1}{2}(h_a+h_b+h_c) = \frac{s^2+r^2+4Rr}{4R} \leq$$

$$\leq \frac{4R^2+4Rr+3r^2+4Rr+r^2}{4R} = \frac{R^2+2Rr+r^2}{R} = \frac{(R+r)^2}{R}$$

We need to prove:

$$\frac{(R+r)^2}{R} \overset{(1)}{\leq} \frac{4R+r}{6}\sqrt{\frac{4R+r}{r}} \overset{t=\frac{R}{r}\geq 2}{\Longrightarrow} \frac{(t+1)^2}{t} \leq \frac{4t+1}{6}\sqrt{4t+1} \Leftrightarrow$$

$$36(t+1)^4 \leq t^2(4t+1)^3 \Leftrightarrow$$

$$64t^5 + 12t^4 - 132t^3 - 215t^2 - 144t - 36 \geq 0 \Leftrightarrow$$

$(t-2)(4t^2+4t+3)(16t^2+19t+6) \geq 0$ **true from** $t \geq 2 \Rightarrow (1)$ **is true.Proved.**

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.