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In ΔABC the following relationship holds:

$$\left(\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)\left(\sqrt[3]{\frac{h_a}{r_a}}+\sqrt[3]{\frac{h_b}{r_b}}+\sqrt[3]{\frac{h_c}{r_c}}\right)\ge 9$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Tran Hong-Dong Thap-Vietnam

$$h_a=\frac{2S}{a};\ r_a=\frac{S}{s-a};\ h_b=\frac{2S}{b};\ r_b=\frac{S}{s-b};\ h_c=\frac{2S}{c};\ r_c=\frac{S}{s-c}$$

Let: $x=\frac{h_a}{r_a}=\frac{2(s-a)}{a}>0;\ y=\frac{h_b}{r_b}=\frac{2(s-b)}{b}>0;\ z=\frac{h_c}{r_c}=\frac{2(s-c)}{c}>0$

$$\Rightarrow \frac{1}{x+2}+\frac{1}{y+2}+\frac{1}{z+2}=\frac{1}{2\cdot\frac{s}{a}-2+2}+\frac{1}{2\cdot\frac{s}{b}-2+2}+\frac{1}{2\cdot\frac{s}{c}-2+2}$$

$$=\frac{a+b+c}{2s}=1\Leftrightarrow\frac{1}{x+2}+\frac{1}{y+2}+\frac{1}{z+2}=1;\ \ (*)$$

Now,

$$LHS=(x+y+z)(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z})$$

$$=\left(\sqrt{x^2}+\sqrt{y^2}+\sqrt{z^2}\right)\left(\sqrt[6]{x^2}+\sqrt[6]{y^2}+\sqrt[6]{z^2}\right)\overset{CBS}{\geq}\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{z^2}\right)^2$$

$$=\left(\sqrt[3]{x\cdot 1\cdot 1}+\sqrt[3]{y\cdot 1\cdot 1}+\sqrt[3]{z\cdot 1\cdot 1}\right)^2$$

$$\overset{Am-Hm}{\geq}\left(\frac{3}{\frac{1}{x}+\frac{1}{x}+1}+\frac{3}{\frac{1}{y}+\frac{1}{y}+1}+\frac{3}{\frac{1}{z}+\frac{1}{z}+1}\right)^2=\left(\frac{3x}{x+2}+\frac{3y}{y+2}+\frac{3z}{z+2}\right)^2$$

$$=3^2\left(\frac{x}{x+2}+\frac{y}{y+2}+\frac{z}{z+2}\right)^2\overset{(*)}{=}9$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.