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In ΔABC the following relationship holds:

$$\frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}}-\sqrt{\frac{R}{r}-2}\right)\le\sum\sqrt{\frac{b+c-a}{a}}\le\frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{R}{r}-2}\right)$$

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$$\sqrt{\frac{b+c-a}{a}}\le\frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{R}{r}-2}\right)\Leftrightarrow\frac{b+c-a}{a}\le\frac{1}{2}\left(\frac{R}{r}+\frac{R}{r}-2+\frac{2\sqrt{R(R-2r)}}{r}\right)$$

$$=\frac{R}{r}-1+\frac{\sqrt{R(R-2r)}}{r}$$

$$\Leftrightarrow\frac{b+c-a}{a}+1\le\frac{AO+IO}{r}\Leftrightarrow\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}\le\frac{AO+IO}{r}$$

$$\Leftrightarrow AI.\cos\frac{B-C}{2}\overset{(i)}{\le}AO+IO$$

Now, $\because -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2} \therefore 0 < \cos\frac{B-C}{2} \leq 1 \Rightarrow AI.\cos\frac{B-C}{2}$

$$\leq AI \overset{\text{triangle inequality}}{\leq} AO + IO \Rightarrow \text{(i) is true}$$

$$\Rightarrow \sqrt{\frac{b+c-a}{a}} \leq \frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r}-2}\right) \text{ and analogs}$$

$$\Rightarrow \sum\sqrt{\frac{b+c-a}{a}} \overset{(1)}{\leq} \frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r}-2}\right)$$

Again, $\sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2}\right) \Leftrightarrow \frac{b+c-a}{a}$

$$\geq \frac{1}{2}\left(\frac{R}{r}+\frac{R}{r}-2-\frac{2\sqrt{R(R-2r)}}{r}\right) = \frac{R}{r}-1-\frac{\sqrt{R^2-2Rr}}{r}$$

$$\Leftrightarrow \frac{b+c}{a} \geq \frac{R-\sqrt{R^2-2Rr}}{r} = \frac{\left(R+\sqrt{R^2-2Rr}\right)\left(R-\sqrt{R^2-2Rr}\right)}{r\left(R+\sqrt{R^2-2Rr}\right)} = \frac{2Rr}{r\left(R+\sqrt{R^2-2Rr}\right)}$$

$$\Leftrightarrow \frac{2a}{b+c} - 1 \leq \frac{R+\sqrt{R^2-2Rr}}{R} - 1$$

$$\Leftrightarrow \frac{2a-b-c}{b+c} \overset{(ii)}{\leq} \frac{\sqrt{R^2-2Rr}}{R}$$

If $2a-b-c \leq 0$, then, LHS of (ii) $\leq 0 \overset{\text{Euler}}{\leq} \frac{\sqrt{R^2-2Rr}}{R} \Rightarrow$ (ii) is true $\Rightarrow \sqrt{\frac{b+c-a}{a}}$

$$\geq \frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2}\right)$$

Let us now consider the case when $2a-b-c > 0 \therefore$ (ii) $\Leftrightarrow \left(\frac{2a-b-c}{b+c}\right)^2$

$$\leq \frac{R^2-2Rr}{R^2} \Leftrightarrow \frac{2r}{R} \leq 1 - \left(\frac{2a-b-c}{b+c}\right)^2$$

$$\Leftrightarrow \frac{2r}{R} \leq \left(1-\frac{2a-b-c}{b+c}\right)\left(1+\frac{2a-b-c}{b+c}\right) = \frac{4a(b+c-a)}{(b+c)^2} = \frac{8a(s-a)}{(b+c)^2}$$

$$\Leftrightarrow \left(\frac{r}{R}\right)(b+c)^2 \leq 4a(s-a)$$

$$\Leftrightarrow 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(4R\cos\frac{A}{2}\cos\frac{B-C}{2}\right)^2 \leq 4.4R\sin\frac{A}{2}\cos\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow \cos^2\frac{B-C}{2} \leq 1 \to \text{true} \because 0 < \cos\frac{B-C}{2} \leq 1$$

$$\Rightarrow \text{(ii) is true} \Rightarrow \sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r} - 2}\right)$$

$$\therefore \text{ combining both cases, } \sqrt{\frac{b+c-a}{a}}$$

$$\geq \frac{\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r} - 2}\right) \text{ and analogs}$$

$$\Rightarrow \sum \sqrt{\frac{b+c-a}{a}} \overset{(2)}{\geq} \frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r} - 2}\right) \therefore (1),(2) \Rightarrow \frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r} - 2}\right)$$

$$\leq \sum \sqrt{\frac{b+c-a}{a}} \leq \frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r} - 2}\right) \text{ (Proved)}$$