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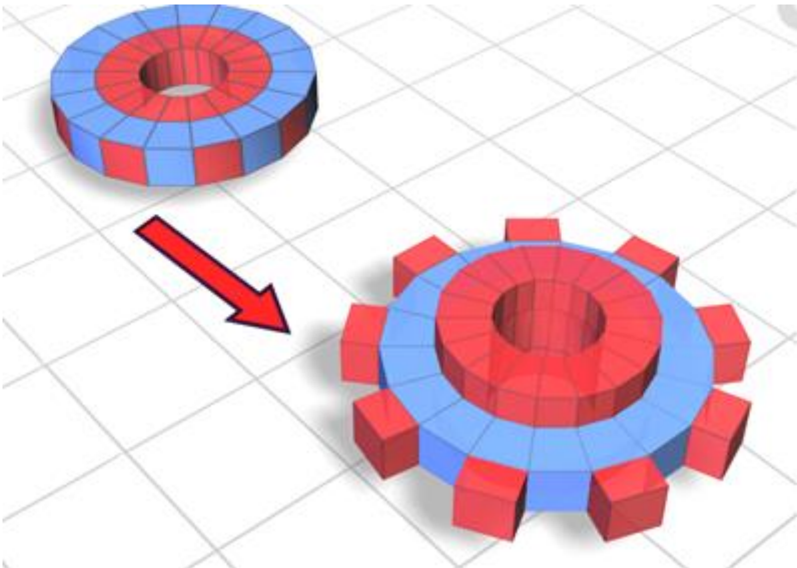

Solve for integers:

$$y^3 = x^2 + 2x$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Bedri Jajrizi-Mitrovica-Kosovo

Let: $x + 1 = k \Rightarrow y^3 = (k-1)(k+1);\ (k-1, k+1) \in \{1, 2\}$

Case i) $(k-1, k+1) = 1 \Rightarrow k$ *–even.*

$(k-1, k+1) = cube \Rightarrow \exists a, b \in \mathbb{Z}$ *such that* $k+1 = a^3, k-1 = b^3 \Rightarrow a^3 - b^3 = 2.$

It's clear that: $a^3 = 1, b^3 = -1 \Rightarrow a = 1, b = -1$

So: $k + 1 = 1 \Rightarrow k = 0, y = -1$

Solutions is: $x = 1, y = -1$

Case ii) $(k-1, k+1) = 2 \Rightarrow k$ *is odd,* y *is even.*

It's clear that $k \equiv 1, 3 (mod 4).$

Also (x, y) *is solution*$\Leftrightarrow$ $(-k, y)$ *is solution.*

So assume that $k \equiv 1 (mod 4) \Rightarrow k + 1 \equiv 2 (mod 4), k - 1 \equiv 0 (mod 4)$

We can write the equation in the form:

$$\left(\frac{y}{2}\right)^3=\frac{k+1}{2}\cdot\frac{k-1}{4} \textit{ where } \left(\frac{k+1}{2},\frac{k-1}{4}\right)=1 \textit{ because } (k+1,k-1)=1$$

$$\textit{Let: } \frac{k+1}{2}=a^3,\frac{k-1}{2}=b^3\Rightarrow 2a^3-1=k=4b^3+1\Rightarrow a^3-2b^3=1$$

$$(a,b)\in\{(0,1),(-1,1)\}\Rightarrow k\in\{-1,-3\}\Rightarrow x\in\{-2,-4\}$$

$$\textit{Solutions are: } (x,y)\in\{(-2,0),(-4,2),(0,0),(2,2)\}$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.