A Simple Approach To Finding A Relationship Between Series Of Binomial Central Coefficients And The Fractional Order Derivative 1/2

Lucas Paes Barreto
lucaspaes222@hotmail.com.br
Pernambuco, Brazil

April 3, 2020

Let's start with the followings definitions

$$\therefore\ D_*^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 < \alpha < n,\ n \in \mathbb{N}\ \wedge\ t > a \in \mathbb{R}. \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

$$\therefore\ D_*^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \qquad (*)$$

where D_*^α is the Caputo Fractional Differential Operator (F.D.O.) of the function x^n and $\Gamma(n+1)$ is the Gamam Function. But here we will give the fractional sense for the order of this differential operator, i.e. $\alpha \in \mathbb{Q}$. Now, let's take $\alpha = \frac{1}{2}$ then from $(*)$ we have

$$D^{\frac{1}{2}} x^n = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} x^{n-\frac{1}{2}} = \frac{(4x)^n}{\binom{2n}{n}\sqrt{\pi x}} \qquad (1)$$

where the following properties were used: $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ and $\frac{(2n)!}{(n!)^2} = \binom{2n}{n}$, being $\binom{2n}{n}$ the Central Coefficient Binomial. It is valid to let the reader know that the notations are equivalent $D^{\frac{1}{2}} \equiv \sqrt{D}$. Now we go to the objective of this work taking the sum on both sides of (1):

$$\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{\boldsymbol{D}} x^n = \frac{1}{\sqrt{\pi x}} \sum_{n=1}^{\infty} \frac{(4x)^n}{n\binom{2n}{n}}$$

From Calculus, we know that D^α has linear operations then

$$\Rightarrow \sqrt{\boldsymbol{D}}\sum_{n=1}^{\infty}\frac{x^n}{n}=\frac{1}{\sqrt{\pi x}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} \quad\Longleftrightarrow\quad \sqrt{\boldsymbol{D}}\left(-\ln(1-x)\right)=\frac{1}{\sqrt{\pi x}}\sum_{n=1}^{\infty}\frac{(4x)^n}{\binom{2n}{n}}$$

$$\therefore\ -\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)=\frac{1}{\sqrt{\pi x}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} \tag{2}$$

In this excerpt from the work, we will stick to solving the RHS (Right Hand Side) series. So from the Beta Function we have the following property:

$$\therefore\ B(n,m)=\int_0^1 t^{n-1}(1-x)^{m-1}\mathrm{d}t=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

making the change $n\to n+1$ and $m\to n$ we get

$$\Longrightarrow\ B(n{+}1,n)=\int_0^1 t^n(1{-}t)^{n-1}\mathrm{d}t=\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}=\frac{\Gamma(n+1)n\Gamma(n)}{n\Gamma(2n+1)}=\frac{\Gamma^2(n+1)}{n\Gamma(2n+1)}$$

given the definition $\Gamma(n+1)=n!$ we get

$$\therefore\ B(n+1,n)=\frac{(n!)^2}{n(2n)!}=\frac{1}{n\binom{2n}{n}}=\int_0^1 t^n(1-t)^{n-1}\mathrm{d}t.$$

Therefore (2) assume the form

$$-\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)=\sum_{n=1}^{\infty}\frac{(4x)^n}{\sqrt{\pi x}}\int_0^1 t^n(1-t)^{n-1}\mathrm{d}t$$

As the sum varies in n we have from the geometric series that

$$-\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)=\frac{1}{\sqrt{\pi x}}\int_0^1\frac{1}{1-t}\sum_{n=1}^{\infty}(4xt(1-t))^n\mathrm{d}t=\frac{1}{\sqrt{\pi x}}\int_0^1\frac{1}{1-t}\cdot\frac{4xt(1-t)}{1-4xt(1-t)}\mathrm{d}t$$

$$\therefore\ -\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)=\frac{1}{\sqrt{\pi x}}\int_0^1\frac{4xt}{1-4xt+4xt^2}\mathrm{d}t \tag{3}$$

The last integral can be solved as follows

$$\Longrightarrow\ \int_0^1\frac{4xt}{1-4xt+4xt^2}\mathrm{d}t=\frac{1}{2}\int_0^1\frac{8xt+4x-4x}{1-4xt+4xt^2}\mathrm{d}t=$$

$$=\frac{1}{2}\int_0^1\frac{8xt-4x}{1-4xt+4xt^2}\mathrm{d}t+2x\int_0^1\frac{\mathrm{d}t}{1-4xt+4xt^2}=\underbrace{\frac{1}{2}\ln(1-4xt+4xt^2)\Big|_0^1}_{=0}+2x\mathcal{I}$$

where $\mathcal{I} = \displaystyle\int_0^1 \frac{\mathrm{d}t}{1-4xt+4xt^2}$, solving $\mathcal{I}$:

$$\implies \mathcal{I} = \frac{1}{x}\int_0^1 \frac{\mathrm{d}t}{(2t-1)^2+\frac{1}{x}-1} = \frac{1}{1-x}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{2t-1}{\sqrt{1-x}}\sqrt{x}\right)^2+1} =$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}\arctan\left(\frac{2t-1}{\sqrt{1-x}}\sqrt{x}\right)\Big|_0^1 = \frac{1}{\sqrt{x}\sqrt{1-x}}\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right).$$

then

$$\implies \int_0^1 \frac{4xt}{1-4xt+4xt^2}\mathrm{d}t = 2x\cdot\frac{1}{\sqrt{x}\sqrt{1-x}}\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) = \frac{2\sqrt{x}}{\sqrt{1-x}}\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right).$$

as soon

$$\implies -\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right) = \frac{1}{\sqrt{\pi x}}\cdot\frac{2\sqrt{x}}{\sqrt{1-x}}\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) = \frac{1}{\sqrt{\pi}}\cdot\frac{2}{\sqrt{1-x}}\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right)$$

note that $\arctan\left(\frac{\sqrt{x}}{\sqrt{1-x}}\right) = \arcsin(\sqrt{x})$, and (3) takes the following form

$$\therefore\ -\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right) = \frac{1}{\sqrt{\pi}}\frac{2\arcsin(\sqrt{x})}{\sqrt{1-x}}$$

Then we finally have two strong equalities

$$\therefore \boxed{\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} = -\sqrt{\pi x}\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right) \quad\text{and}\quad \sqrt{\boldsymbol{D}}\left(\ln(1-x)\right) = -\frac{2}{\sqrt{\pi}}\frac{\arcsin(\sqrt{x})}{\sqrt{1-x}},\ x\neq 1}\ (**)$$

Some numerical results

In this excerpt we will take the points $x_0 = \frac{3}{4}$, $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{4}$ as an example starting from $(**)$. then follows that

$$\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)_{x=x_0} = -\frac{4\sqrt{\pi}}{3} \implies \sum_{n=1}^{\infty}\frac{3^n}{n\binom{2n}{n}} = \frac{2\pi}{\sqrt{3}}$$

$$\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)_{x=x_1} = -\sqrt{\frac{\pi}{2}} \implies \sum_{n=1}^{\infty}\frac{2^n}{n\binom{2n}{n}} = \frac{\pi}{2}$$

$$\sqrt{\boldsymbol{D}}\left(\ln(1-x)\right)_{x=x_2} = -\frac{2}{3}\sqrt{\frac{\pi}{3}} \implies \sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}} = \frac{\pi}{3\sqrt{3}}$$

Main result from work

Let's do the following, let

$$\sqrt{\boldsymbol{D}}\,(\ln(1-x)) := g(x)$$
$$\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} := S^1(x)$$

Thus

$$\therefore\ g(x) = -\frac{1}{\sqrt{\pi x}}S^1(x) \tag{4}$$

Now we need to introduce the definition of Fractional Integral in order to obtain the result of the integration of the function $g(x)$ and thereby obtain a closed form for the integral of the series $S^1(x)$. The following definition is known as Riemann – Liouville fractional integral:

$$\therefore\ J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau, \quad \alpha>0,\ t>a,\ \alpha,t,a\in\mathbb{R}.$$

Multiplying by $\frac{1}{\sqrt{x}}$ and taking the operator J^{α} in both sides from (4)

$$\Longrightarrow\ J^{\alpha}\left(\frac{1}{\sqrt{x}}g(x)\right) = -\frac{1}{\sqrt{\pi}}J^{\alpha}\left(\frac{1}{x}S^1(x)\right)$$

However, putting $\alpha=1$ and $a=0$ at the point $x_0=\frac{1}{4}$

$$\Longrightarrow\ J^{1}\left(\frac{1}{\sqrt{x}}g(x)\right) = -\frac{1}{\sqrt{\pi}}J^{1}\left(\frac{1}{x}S^1(x)\right) = -\frac{1}{\sqrt{\pi}}\underbrace{\frac{1}{\Gamma(1)}}_{=1}\int_0^x (x-\tau)^0\frac{1}{\tau}S^1(\tau)\mathrm{d}\tau =$$

$$= -\frac{1}{\sqrt{\pi}}\int_0^x\frac{1}{\tau}S^1(\tau)\mathrm{d}\tau \ \Longrightarrow\ -\frac{1}{\sqrt{\pi}}\int_0^x\frac{1}{\tau}S^1(\tau)\mathrm{d}\tau = -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{4^n}{n\binom{2n}{n}}\int_0^x\tau^{n-1}\mathrm{d}\tau =$$
$$= -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{4^n}{n\binom{2n}{n}}\cdot\frac{x^n}{n} = -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}} \ \Longrightarrow\ \therefore\ J^1\left(\frac{1}{\sqrt{x}}g(x)\right) = -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}}. \tag{5}$$

By other side, through the result from (∗∗)

$$-\frac{1}{\sqrt{\pi x}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} = -\frac{2}{\sqrt{\pi}}\frac{\arcsin(\sqrt{x})}{\sqrt{1-x}} \ \Longrightarrow \therefore\sum_{n=1}^{\infty}\frac{(4x)^n}{n\binom{2n}{n}} = \frac{2\sqrt{x}\arcsin(\sqrt{x})}{\sqrt{1-x}}$$

This means

$$\implies -\frac{1}{\sqrt{\pi}}J^1\left(\frac{1}{x}S^1(x)\right) = -\frac{1}{\sqrt{\pi}}J^1\left(\frac{1}{x}\cdot\frac{2\sqrt{x}\arcsin(\sqrt{x})}{\sqrt{1-x}}\right)$$

$$\implies -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}} = -\frac{2}{\sqrt{\pi}}J^1\left(\frac{\arcsin(\sqrt{x})}{\sqrt{x}\sqrt{1-x}}\right)$$

$$\implies \sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}} = 2J^1\left(\frac{\arcsin(\sqrt{x})}{\sqrt{x}\sqrt{1-x}}\right)$$

$$\implies \sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}} = 2\int_0^x\frac{\arcsin(\sqrt{\tau})}{\sqrt{\tau}\sqrt{1-\tau}}\mathrm{d}\tau \overbrace{=}^{\arcsin(\sqrt{\tau})\to y} 4\int_0^{\arcsin(\sqrt{x})} y\mathrm{d}y = 2\arcsin^2(\sqrt{x})$$

$$\therefore \sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}} = 2\arcsin^2(\sqrt{x}) \tag{6}$$

Thus we conclude from (5) and (6) that

$$\therefore J^1\left(\frac{1}{\sqrt{x}}g(x)\right) = -\frac{2\arcsin^2(\sqrt{x})}{\sqrt{\pi}}. \tag{7}$$

Taking the point $x_0 = \frac{1}{4}$ we get

$$\implies J^1\left(\frac{1}{\sqrt{x}}g(x)\right)_{x_0=\frac{1}{4}} = -\frac{\sqrt{\pi^3}}{36}$$

As $g(x) = \sqrt{\boldsymbol{D}}\,(\ln(1-x))$ and coming back to the notation for J^α we finally have two other strong results

$$\therefore \boxed{\int_0^x\frac{\sqrt{\boldsymbol{D}}\,(\ln(1-\tau))}{\sqrt{\tau}}\mathrm{d}\tau = -\frac{2\arcsin^2(\sqrt{x})}{\sqrt{\pi}} \quad \text{and} \quad -\sqrt{\pi}\int_0^x\frac{\sqrt{\boldsymbol{D}}\,(\ln(1-\tau))}{\sqrt{\tau}}\mathrm{d}\tau = \sum_{n=1}^{\infty}\frac{(4x)^n}{n^2\binom{2n}{n}}}$$

Therefore, taking the results above at point $x_0 = \frac{1}{4}$ we can extract the following final results:

$$\therefore \boxed{\int_0^{1/4}\frac{\sqrt{\boldsymbol{D}}\,(\ln(1-\tau))}{\sqrt{\tau}}\mathrm{d}\tau = -\frac{\zeta(2)}{6\sqrt{\pi}} \implies \therefore \sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = \frac{\zeta(2)}{3}}$$

where $\zeta(2) = \frac{\pi^2}{6}$ being $\zeta(s)$ is the Riemann Zeta Function.

Discussions

I consider these small steps relevant to understand an intrinsic relationship between the series of inverse binomial central coefficients and the fractional calculation provided by Leibniz. The Caputo Fractional Differential Operator of order $\frac{1}{2}$ allows us to relate an important class of infinite series, which are the Central Binomial Coefficients, with $\zeta(2)$ with the help of the function $f(t) = \ln(1-t)$ which receives the action of the operator D_*^{α}, which brings us pleasant results. Therefore, what excited me to the point of elaborating the present work was the desire to show that the calculations above demonstrate that when solving a series of such a nature we may actually be calculating integrals or fractional derivatives and, if that is the case, we will be finding beautiful relationships between Fractional Calculus and Riemann's Zeta Function. Thus, the execution of the calculations above is subject to several generalizations, including taking other functions since with the support that the expansion in Taylor Series allows us almost always in comfortable manipulations with the Gamma and Beta Functions enabling them.

Constructive criticism and praise will always be welcome. Grateful for the attention!