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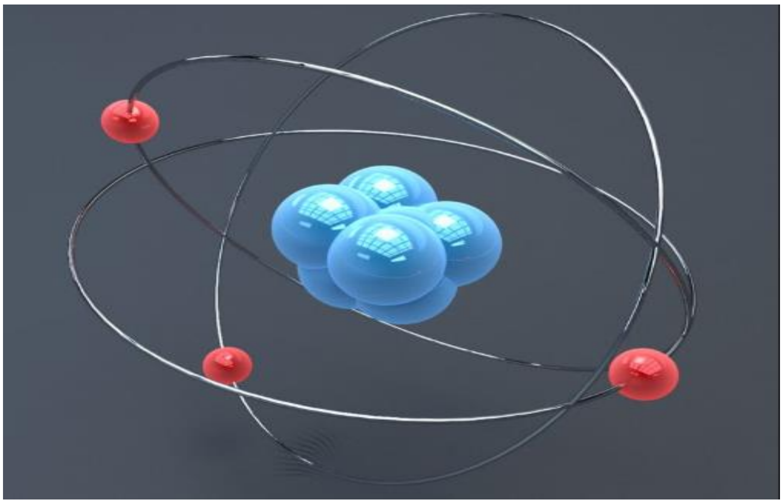

UP.297. Let $(a_n)_{n\geq 1}$ be a sequence real numbers such that $a_n \in \mathbb{R}_+^* = (0,\infty), \forall n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*. \text{ Find:}$$

$$\lim_{n\to\infty} \sqrt[n^2]{a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \cdot \sqrt[n]{a_n}}$$

Proposed By D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Solution 1,2 by proposers

Solution 1 by proposers:

Let be $b_n = a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \cdot \sqrt[n]{a_n}, \forall n \geq 1$.

From Cauchy-D'Alembert Criterion (C-D'A), we have:

$$\lim_{n\to\infty} \sqrt[n^2]{b_n} = \lim_{n\to\infty} \sqrt[n]{\sqrt[n]{b_n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}; (1)$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{\sqrt[n+1]{a_{n+1}}}{n+1} = \frac{1}{e} \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n}$$

$$=\frac{1}{e}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\stackrel{C-D'A}{=}\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}=\frac{1}{e}\lim_{n\to\infty}\frac{a_{n+1}}{na_n}\left(\frac{n}{n+1}\right)^n=\frac{a}{e^2}\stackrel{(1)}{\Rightarrow}$$

$$\lim_{n\to\infty}\sqrt[n^2]{b_n}=\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{n+1}{n}\right)=\frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1=1.$$

Solution 2 by proposers

$$\lim_{n\to\infty}\frac{1}{n^2}\log b_{n+1}=\lim_{n\to\infty}\frac{\log b_{n+1}-\log b_n}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{n}{2n+1}\cdot\frac{\log b_{n+1}-\log b_n}{n}\stackrel{C-S}{=}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{\log b_{n+2}-2\log b_{n+1}+\log b_n}{(n+1)-n}=\frac{1}{2}\log\left(\lim_{n\to\infty}\frac{b_{n+2}\cdot b_n}{b_{n+1}^2}\right)=$$

$$=\frac{1}{2}\log\left(\lim_{n\to\infty}\frac{\sqrt[n+2]{a_{n+2}}\cdot\sqrt[n+1]{a_{n+1}}}{\sqrt[n+1]{a_{n+1}^2}}\right)=\frac{1}{2}\log\left(\lim_{n\to\infty}\frac{\sqrt[n+2]{a_{n+2}}}{n+2}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n+2}{n+1}\right)=$$

$$=\frac{1}{2}\log\left(\frac{a}{e^2}\cdot\frac{e^2}{a}\cdot 1\right)=\frac{1}{2}\log 1=0\Rightarrow\lim_{n\to\infty}\sqrt[n^2]{b_n}=e^{\lim_{n\to\infty}\frac{1}{n^2}\log b_{n+1}}=e^0=1.$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.