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JP.299 If $a, b, c > 0; abc = 4$ then:

$$\frac{2(a^5+b^5)+c^5}{(a+b)^2}+\frac{2(b^5+c^5)+a^5}{(b+c)^2}+\frac{2(c^5+a^5)+b^5}{(c+a)^2}\geq 15$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$3a^5+b^5+c^5 \overset{AM-GM}{\geq} 5\cdot\sqrt[5]{(a^5)^3b^5c^5}=5a^3bc \quad (1)$$

$$a^5+3b^5+c^5 \overset{AM-GM}{\geq} 5\cdot\sqrt[5]{a^5(b^5)^3c^5}=5ab^3c \quad (2)$$

$$2a^5+2b^5+c^5 \overset{AM-GM}{\geq} 5\cdot\sqrt[5]{(a^5)^2\cdot(b^5)^2\cdot c^5}=5a^2b^2c \quad (3)$$

$$2a^5+2b^5+c^5 \overset{AM-GM}{\geq} 5\cdot\sqrt[5]{(a^5)^2\cdot(b^5)\cdot c^5}=5a^2b^2c \quad (4)$$

By adding (1); (2); (3); (4):

$$8a^5+8b^5+4c^5\geq 5abc(a^2+2ab+b^2)$$

$$2a^5+2b^5+c^5\geq\frac{5}{4}abc(a+b)^2, \qquad \frac{2(a^5+b^5)+c^5}{(a+b)^2}\geq\frac{5}{4}abc$$

$$\sum_{cyc}\frac{2(a^5+b^5)+c^5}{(a+b)^2}\geq\frac{5}{4}\sum_{cyc}abc=\frac{5}{4}\cdot 3abc=\frac{15}{4}\cdot 4=15$$

Equality holds for $a=b=c=\sqrt[3]{4}$