RMM - Abstract Algebra Marathon 1 - 100

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1. Let A be a ring with identity. For each $a \in A$ we define

$$E_a \coloneqq \{x \in A : xa = 1\}$$

Show that if $c \in E_a$ and $|E_a| \geq 2$ then the function $\varphi_a : E_a \to E_a$ defined by $\varphi_a(x) = ax + c - 1$ is injective but not surjective.

Proposed by Oleg Țurcan-Lisbon-Portugal

Solution by Kelvin Hong-Rawang-Malaysia

To show injective, suppose $\varphi_a(x) = \varphi_a(y)$, then:

$$ax + c - 1 = ay + c - 1 \Rightarrow ax = ay \Rightarrow cax = cay \Rightarrow x = y.$$

To show it is not surjective, we argue by contradiction. So, we suppose it is surjective, there is $x \in E_a$ such that: $ax + c - 1 = c \in E_a$ therefore $ax = 1$ since $|E_a| \geq 2$, we can choose $y \in E_a, y \neq x$ such that: $ya = 1$ multiplying x on the right on both sides we have: $yax = x \Rightarrow y = x$ which is a contradiction. Hence the mapping is injective but not surjective.

2. $x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}, x \circ y = xy - 5x - 5y + 30, G = (5, \infty)$

Prove that $(\mathbb{R}, *) \cong (G, \circ)$ as abelian groups.

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We first show that $(\mathbb{R}, *)$, where $x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}$ is an abelian group. Clearly, $x * y \in \mathbb{R}, \forall x, y \in \mathbb{R}$

$*$ is associative suppose $x, y, z \in \mathbb{R}$. Let $x = \tan\alpha, y = \tan\beta, z = \tan\gamma$

$$-\frac{\pi}{2} < \alpha, \beta, \gamma < \frac{\pi}{2}$$

$$x * y = (\tan\alpha)\sqrt{1 + \tan^2\beta} + \tan\beta\sqrt{1 + \tan^2\alpha} = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}$$

$$(x * y) * z = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\sqrt{1 + \tan^2\gamma} + \tan\alpha\sqrt{1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2}$$

But $1+\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha\cos\beta}\right)^2=\frac{(1-\sin^2\alpha)(1-\sin^2\beta)+\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta}=\frac{(1+\sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}$

Thus, $(x*y)*z=\frac{\overset{\oplus}{\sin\alpha\sin\beta}}{\cos\alpha\cos\beta\cos\gamma}+\frac{\sin\gamma(1+\sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma}=$

$$=\frac{\sin\alpha+\sin\beta+\sin\gamma+\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$$

Similarly, $x*(y*z)=\frac{\sin\alpha+\sin\beta+\sin\gamma+\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

Thus, $(x*y)*z=x*(y*z);\forall x,y,z\in\mathbb{R}$

$*$ *is commutative is obvious. Identity Element* $=0$

$$x*0=x\sqrt{1+0^2}+0\sqrt{1+x^2}=x;\forall x\in\mathbb{R}$$

Inverse Element. For each $x\in\mathbb{R},-x\in\mathbb{R}$ *is inverse of* x*.*

Indeed $x*(-x)=0$

$\therefore(\mathbb{R},*)$ *is an abelian group. Next, we show that if* $G=(5,\infty)$*, and*

$a\circ b=ab-5a-5b+30;\forall a,b\in\mathbb{G}$*, then* $(\mathbb{G},\circ)$ *is an abelian group.*

Note $a\circ b=(a-5)(b-5)+5$

$\circ$ *is commutative and its identity element is* 6*.*

$\circ$ *is associative*

Let $a,b,c\in\mathbb{G}$*,*

$$(a\circ b)\circ c=\big((a-5)(b-5)+5\big)\circ c=\big((a-5)(b-5)+5-5\big)(c-5)+5$$

$$=(a-5)(b-5)(c-5)+5$$

Similarly, $a\circ(b\circ c)=(a-5)(b-5)(c-5)+5$

$\therefore(a\circ b)\circ c=a\circ(b\circ c);\forall a,b,c\in\mathbb{G}$

Finally, if $a\in 5$*, then* $a>5$*, and* $b=5+\frac{1}{a-5}$ *is inverse of* a*. Indeed,*

$a\circ b=(a-5)(b-5)+5=(a-5)\left(\frac{1}{a-5}\right)+5=1+5=6=$ *identity element.*

We now show that $\Phi:\mathbb{R}\to\mathbb{G}$ *defined by* $\Phi(x)=5+5^{\sinh^{-1}x}$

is the required isomorphism of $\mathbb{R}$ *onto* $\mathbb{G}$

As $5^{\sinh^{-1}x}>0,\forall x\in\mathbb{R},\Phi(x)\in\mathbb{G};\forall x\in\mathbb{R}$

For $x,y\in\mathbb{R}$

$$\Phi(x * y) = 5^{\sinh^{-1}\left(x\sqrt{1+y^2}+y\sqrt{1+x^2}\right)} + 5 \quad (1)$$

and $\Phi(x) \circ \Phi(y) = 5^{\sinh^{-1} x} \cdot 5^{\sinh^{-1} y} + 5 \quad (2)$

$$= 5^{\sinh^{-1} x + \sinh^{-1} y} + 5$$

But $\sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right) \quad (3)$

$\therefore$ *from (1), (2), (3):* $\Phi(x * y) = \Phi(x) \circ \Phi(y)$

Thus, Φ *is a homomorphism from* $(\mathbb{R}, *)$ *to* $(\mathbb{G}, a)$

Φ *is one – to – one*

Let $x, y \in \mathbb{R}$ *and* $\Phi(x) = \Phi(y)$

$$\Rightarrow 5^{\sinh^{-1} x} + 5 = 5^{\sinh^{-1} y} + 5 \Rightarrow \sinh^{-1} x = \sinh^{-1} y \Rightarrow x = y$$

$\therefore$ Φ *is one – to – one*

Φ *is onto*

Let $y \in \mathbb{G} \Rightarrow y > 5 \Rightarrow y - 5 > 0$

Let $t = \log_5(y - 5) \Rightarrow 5^t = y - 5$

As $t \in \mathbb{R}, \exists x \in \mathbb{R}$ *such that* $\sinh^{-1} x = t$ *or take* $x = \sinh t$.

Then $\Phi(x) = 5^{\sinh^{-1} x} + 5 = 5^t + 5 = y - 5 + 5 = y$

$\therefore$ Φ *is onto.*

Hence, $(\mathbb{R}, *) \cong (\mathbb{G}, \circ)$ *as abelian groups.*

3. Find $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $t \in \mathbb{R}$, **such that:**

$$A^4 - 4A^3 + 6A^2 - 4A + I_2 = O_2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\det(A) = \cos^2 t + \sin^2 t = 1 \Rightarrow A^{-1}$ *exists and* $A^{-1} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

Let $B = A + A^{-1} = \begin{pmatrix} 2\cos t & 0 \\ 0 & 2\cos t \end{pmatrix}$

Given equation is $A^4-4A^3+6A^2-4A+I_2=O_2$

$$\Rightarrow A^2-4A+6I_2-4A^{-1}+A^{-2}=O_2\Rightarrow A^2+A^{-2}-4(A+A^{-1})+6I_2=O_2$$

$$\Rightarrow (A+A^{-1})^2-4(A+A^{-1})+4I_2=O_2\Rightarrow (A+A^{-1}-2I_2)^2=0\Rightarrow (B-2I_2)^2=O_2$$

$$\therefore \begin{pmatrix}2\cos t-2 & 0\\ 0 & 2\cos t-2\end{pmatrix}^2=O_2\Rightarrow 4\begin{pmatrix}(\cot t-1)^2 & 0\\ 0 & (\cos t-1)^2\end{pmatrix}=0$$

$$\Rightarrow (\cot t-1)^2=0\Rightarrow \cot t=1,\sin t=0\therefore A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I_2$$

Solution 2 by Adrian Popa-Romania

$$A=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix};A^4-4A^3+6A^2-4A+I_2=O_2;\ A=?$$

$$A^n=\begin{pmatrix}\cos nt & -\sin nt\\ \sin nt & \cos nt\end{pmatrix}\Rightarrow$$

$$\Rightarrow\begin{pmatrix}\cos 4t & \sin 4t\\ \sin 4t & \cos 4t\end{pmatrix}-4\begin{pmatrix}\cos 3t & -\sin 3t\\ \sin 3t & \cos 3t\end{pmatrix}+6\begin{pmatrix}\cos 2t & -\sin 2t\\ \sin 2t & \cos 2t\end{pmatrix}-$$

$$-4\begin{pmatrix}\cos t & -\sin t\\ \sin 5 & \cos t\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\Rightarrow$$

$$\Rightarrow \left.\begin{matrix}I\ \ \cos 4t-4\cos 3t+6\cos 2t-4\cos t+1=0\\ II\ \ \sin 4t-4\sin 3t+6\sin 2t-4\sin t+0=0\end{matrix}\right\}$$

$$I+iII:\cos 4t+i\sin 4t-4(\cos 3t+i\sin 3t)+6(\cos 2t+i\sin 2t)-$$

$$-4(\cos t+i\sin t)+1=0\Rightarrow$$

$$\Rightarrow (\cos t+i\sin t)^4-4(\cos t+i\sin t)^3+6(\cos t+i\sin t)^2-4(\cos t+i\sin t)+1=0$$

$$\cos t+i\sin t=z\Rightarrow z^4-4z^3+6z^2-4z+1=0;\ (z-1)^4=0\Rightarrow z=1$$

$$\cos t+i\sin t=1\Rightarrow \left.\begin{matrix}\cos t=1\\ \sin t=0\end{matrix}\right\}\Rightarrow t=2k\pi;k\in\mathbb{Z}$$

$$A=\begin{pmatrix}\cos 2k\pi & -\sin 2k\pi\\ \sin 2k\pi & \cos 2k\pi\end{pmatrix}$$

4. Find $A\in M_2((0,\infty)), A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ such that:

$$a^2+d^2=4,\frac{1}{b}+\frac{1}{c}=2,(\det A)^3-6(\det A)^2+11\det A=0$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

$$(\det A)^3 - 6(\det A)^2 + 11\det A = 0 \Rightarrow (\det A)[(\det A - 3)^2 + 2] = 0$$

As, $\det(A)$ *is real, we get* $\det(A) = 0 \Rightarrow ad - bc = 0$ *or* $ad = bc$

Now, $2ad \le a^2 + d^2 = 4 \Rightarrow ad \le 2$ ***(1)***

Also, $\sqrt{bc} \ge \frac{1}{\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)} = \frac{1}{\frac{1}{2}} = 2 \Rightarrow bc \ge 4 \Rightarrow ad = bc \ge 4$ ***(2)***

As (1) and (2) contradict each other, no such matrix exists.

5. Find $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}, a^2 + b^2 \neq 0$ **such that:**

$$A^4 - 3A^3 + 4A^2 - 3A + I_2 = O_2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

If we use $z = a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$ *as field isomorphism of* $\mathbb{C}$ *onto*

$M = \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R}\right\}$ *then* $z \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *will make the equation as:*

$$z^4 - 3z^3 + 4z^2 - 3z + 1 = 0 \Rightarrow z^2 + \frac{1}{z^2} - 3\left(z + \frac{1}{z}\right) + 4 = 0$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^2 - 3\left(z + \frac{1}{z}\right) + 2 = 0 \Rightarrow z + \frac{1}{z} = 1 \text{ or } z + \frac{1}{z} = 2$$

$$z = -\omega, -\omega^2 \text{ or } z = 1$$

In this case $|z| = 1$*, and corresponding matrices became:*

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ as identified earlier.}$$

6. If $A \in M_4(\mathbb{R}), \det A \neq 0, (Tr\ A)^2 = 3Tr(A^2)$ **then:**

$$Tr(A^3) = 3 \cdot \det A \cdot Tr(A^{-1})$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Andrew Okukura-Romania

We will note the eigenvalues of $A: \lambda_1, \lambda_2, \lambda_3, \lambda_4$*. We have:*

$$\begin{matrix} Tr\ A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ Tr\ (A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \end{matrix} \Rightarrow (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2 = 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \Rightarrow$$

$$\Rightarrow 2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)=2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)$$

$$\Rightarrow \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2 \,|\cdot(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$

$$\Rightarrow (\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)=\sum\lambda^2\cdot\sum\lambda$$

$$\Rightarrow \lambda_1\lambda_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_1\lambda_3(\lambda_1+\lambda_3)+\lambda_1\lambda_3(\lambda_2+\lambda_4)+\lambda_1\lambda_4(\lambda_1+\lambda_4)+$$
$$+\lambda_1\lambda_4(\lambda_2+\lambda_3)+\lambda_2\lambda_3(\lambda_2+\lambda_3)+\lambda_2\lambda_3(\lambda_1+\lambda_4)+\lambda_2\lambda_4(\lambda_2+\lambda_4)+\lambda_2\lambda_4(\lambda_1+\lambda_3)+$$
$$+\lambda_3\lambda_4(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2)=\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3+\sum_{1\le i<j\le 4}\lambda_i\lambda_j(\lambda_i+\lambda_j)$$

$$\Rightarrow \lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3=3\lambda_1\lambda_2\lambda_3+3\lambda_1\lambda_2\lambda_4+3\lambda_1\lambda_3\lambda_4+3\lambda_2\lambda_3\lambda_4$$

$$Tr\,A^3=\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3$$

$$\Rightarrow \begin{matrix}\det A=\lambda_1\lambda_2\lambda_3\lambda_4\\ Tr\,A^{-1}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\end{matrix}\Rightarrow (\det A)(Tr\,A^{-1})=\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4$$

$$\Rightarrow Tr\,A^3=3(\det A)(Tr\,A^{-1})$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the eigenvalues of A:

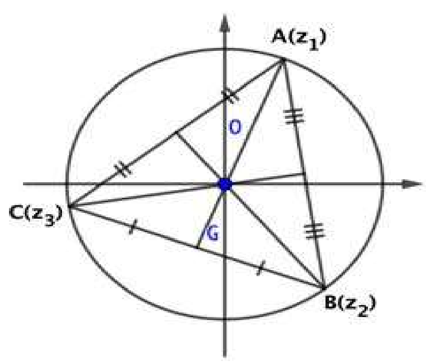
$$\therefore 3Tr\,(A)\,Tr\,(A^2)-(Tr\,(A))^3+6(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)=$$
$$=2(\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3)$$

$$\Rightarrow \lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3=3\lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right)=3\det(A)\,Tr\,(A^{-1})=Tr\,(A^3)$$

7. $z_1,z_2,z_3\in\mathbb{C}-\{0\}$, **different in pairs,** $A(z_1),B(z_2),C(z_3)$,

$|z_1|=|z_2|=|z_3|=1$. **If** $\frac{z_1}{z_2+z_3-z_1}+\frac{z_2}{z_3+z_1-z_2}+\frac{z_3}{z_1+z_2-z_3}+\frac{3}{2}=0$

then: $AB=BC=CA$



Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } |z_1| = |z_2| = |z_3| = 1$$

$$\frac{z_1}{z_2+z_3-z_1} + \frac{z_2}{z_3+z_1-z_2} + \frac{z_3}{z_1+z_2-z_3} + \frac{3}{2} = 0 \quad (1)$$

Let $z = z_1 + z_2 + z_3$, (1) can be written as

$$\left(\frac{z_1}{z-2z_1} + \frac{1}{2}\right) + \left(\frac{z_2}{z-2z_2} + \frac{1}{2}\right) + \left(\frac{z_3}{z-2z_3} + \frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{z}{z-2z_1} + \frac{z}{z-2z_2} + \frac{z}{z-2z_3} = 0 \Rightarrow zw = 0$$

where $\omega = \frac{1}{z-2z_1} + \frac{1}{z-2z_2} + \frac{1}{z-2z_3}$ we assert $z = 0$. Assume $z \neq 0$; then $\omega = 0$

$$\Rightarrow z^2 - 2(z_2 + z_3)z + 4z_2z_3 + z^2 - 2(z_3 + z_1)z + 4z_3z_1 +$$

$$+z^2 - 2(z_1 - z_2)z + 4z_1z_2 = 0$$

$$\Rightarrow 3z^2 - 4(z_1 + z_2 + z_3)z + 4z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$

$$\Rightarrow z^2 = 4z_1z_2z_3(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 4z_1z_2z_3\bar{z} \Rightarrow |z|^2 = 4|z_1||z_2||z_3||\bar{z}|$$

$$\Rightarrow |z| = 4 \quad \left[\because |z_1| = |z_2| = |z_3| \overset{(1)}{\Rightarrow} \text{and } |\bar{z}| = |z| \neq 0\right]$$

But $|z| = |z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3| = 3 \Rightarrow |z| \leq 3 \quad (2)$

But (1) and (2) contradict each other. Thus, our assumption $z \neq 0$, is incorrect.

$$\therefore z = 0$$

We have: $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2$

$$\Rightarrow |z_2 - z_3|^2 + |z_1|^2 = 2|z_2|^2 + 2|z_3|^2 = 4 \Rightarrow |z_2 - z_3|^2 = 3$$

$$AB^2 = 3$$

Similarly, $BC^2 = 3$, $CA^2 = 3$. Thus, $AB^2 = BC^2 = CA^2 \Rightarrow AB = BC = CA$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\left(\frac{z_1}{z_2+z_3-z_1} + \frac{1}{2}\right) + \left(\frac{z_2}{z_3+z_1-z_2} + \frac{1}{2}\right) + \left(\frac{z_3}{z_1+z_2-z_3} + \frac{1}{2}\right) = 0$$

$$\left(\frac{2z_1 + z_2 + z_3 - z_1}{2(z_2 + z_3 - z_1)}\right) + \left(\frac{2z_2 + z_3 + z_1 - z_2}{2(z_3 + z_1 - z_2)}\right) + \left(\frac{2z_3 + z_1 + z_2 - z_3}{2(z_1 + z_2 - z_3)}\right) = 0$$

$$\frac{z_1 + z_2 + z_3}{2}\left(\frac{1}{z_2 + z_3 - z_1} + \frac{1}{z_3 + z_1 - z_2} + \frac{1}{z_1 + z_2 - z_3}\right) = 0$$

$$\frac{1}{2}z_G\left(\frac{1}{z_1+z_2+z_3-2z_1}+\frac{1}{z_1+z_2+z_3-2z_2}+\frac{1}{z_1+z_2+z_3-2z_3}\right)=0$$

$$\frac{1}{2}z_G\left(\frac{1}{3z_G-2z_1}+\frac{1}{3z_G-2z_2}+\frac{1}{3z_G-2z_3}\right)=0$$

$$\frac{1}{2}z_G\left[\frac{9z_G^2-6z_3z_G-6z_2z_G+4z_2z_3+9z_G^2-6z_Gz_3-6z_1z_G+4z_1z_3+9z_G^2-6z_Gz_2-6z_1z_G+4z_1z_2}{(3z_G-2z_1)(3z_G-2z_2)(3z_G-3z_3)}\right]=0$$

$$\frac{1}{2}z_G\left[\frac{27z_G^2-12z_3z_6-12z_2z_G-12z_1z_G+4z_1z_3+4z_1z_2+4z_2z_3}{(3z_G-2z_1)(3z_G-2z_2)(3z_G-2z_3)}\right]=0$$

There are two cases: $z_G=0$

or: $27z_G^2-12(z_3+z_2+z_1)\cdot z_G+4z_1z_3+4z_1z_2+4z_2z_3=0$

$$27z_G^2-36z_G^2+4(z_1z_2+z_2z_3+z_1z_3)=0$$

$$9z_G^2=4(z_1z_2+z_2z_3+z_1z_3)\Rightarrow z_G^2=\frac{4}{9}(z_1z_2+z_2z_3+z_1z_3)$$

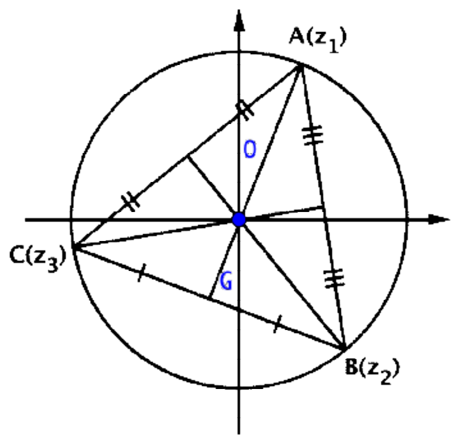
$$z_G^2=\frac{4}{9}(z_1z_2z_3)\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\Rightarrow z_G^2=\frac{4}{9}(z_1z_2z_3)(\overline{z_1}+\overline{z_2}+\overline{z_3})$$

$$|z_G|^2=\frac{4}{9}|\overline{z_1}+\overline{z_2}+\overline{z_3}|\Rightarrow|z_G|^2=\frac{4}{9}\cdot3|\overline{z_G}|$$

$$|z_G|^2=\frac{4}{3}|z_G|\Rightarrow|z_G|=\frac{4}{3}\Rightarrow\frac{4}{3}=\frac{|z_1+z_2+z_3|}{3}\le|z_1|+|z_2|+|z_3|$$

$4\le3$***. This is impossible, so:*** $|z_G|=0\Rightarrow z_G=0\Rightarrow z_1+z_2+z_3=0.z_G=0$***, so:*** $G\equiv0$

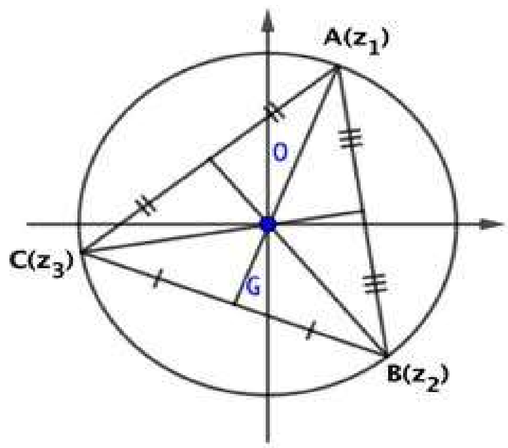
Note: center of circle and centroid has the same point so the triangle ABC ***is equilateral triangle:***$AB=BC=CA$



8. $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, **different in pairs,** $A(z_1), B(z_2), C(z_3)$,

$$|z_1| = |z_2| = |z_3| = 1. \text{ If } \frac{z_1}{z_2+z_3-z_1} + \frac{z_2}{z_3+z_1-z_2} + \frac{z_3}{z_1+z_2-z_3} + \frac{3}{2} = 0$$

then: $AB = BC = CA$



Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $|z_1| = |z_2| = |z_3| = 1$

$$\frac{z_1}{z_2+z_3-z_1} + \frac{z_2}{z_3+z_1-z_2} + \frac{z_3}{z_1+z_2-z_3} + \frac{3}{2} = 0 \quad (1)$$

Let $z = z_1 + z_2 + z_3$, *(1) can be written as*

$$\left(\frac{z_1}{z-2z_1} + \frac{1}{2}\right) + \left(\frac{z_2}{z-2z_2} + \frac{1}{2}\right) + \left(\frac{z_3}{z-2z_3} + \frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{z}{z-2z_1} + \frac{z}{z-2z_2} + \frac{z}{z-2z_3} = 0 \Rightarrow zw = 0$$

where $\omega = \frac{1}{z-2z_1} + \frac{1}{z-2z_2} + \frac{1}{z-2z_3}$ *we assert* $z = 0$. *Assume* $z \neq 0$; *then* $\omega = 0$

$$\Rightarrow z^2 - 2(z_2 + z_3)z + 4z_2z_3 + z^2 - 2(z_3 + z_1)z + 4z_3z_1 +$$

$$+z^2 - 2(z_1 - z_2)z + 4z_1z_2 = 0$$

$$\Rightarrow 3z^2 - 4(z_1 + z_2 + z_3)z + 4z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$

$$\Rightarrow z^2 = 4z_1z_2z_3(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 4z_1z_2z_3\bar{z} \Rightarrow |z|^2 = 4|z_1||z_2||z_3||\bar{z}|$$

$$\Rightarrow |z| = 4 \left[\because |z_1| = |z_2| = |z_3| \overset{(1)}{\Rightarrow} and\ |\bar{z}| = |z| \neq 0\right]$$

But** $|z| = |z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3| = 3 \Rightarrow |z| \le 3$ **(2)

But (1) and (2) contradict each other. Thus, our assumption** $z \neq 0$**, is incorrect.

$\therefore z = 0$***. We have:*** $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2$

$$\Rightarrow |z_2 - z_3|^2 + |z_1|^2 = 2|z_2|^2 + 2|z_3|^2 = 4 \Rightarrow |z_2 - z_3|^2 = 3$$

$$AB^2 = 3$$

Similarly, $BC^2 = 3, CA^2 = 3$***. Thus,*** $AB^2 = BC^2 = CA^2 \Rightarrow AB = BC = CA$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\left(\frac{z_1}{z_2 + z_3 - z_1} + \frac{1}{2}\right) + \left(\frac{z_2}{z_3 + z_1 - z_2} + \frac{1}{2}\right) + \left(\frac{z_3}{z_1 + z_2 - z_3} + \frac{1}{2}\right) = 0$$

$$\left(\frac{2z_1 + z_2 + z_3 - z_1}{2(z_2 + z_3 - z_1)}\right) + \left(\frac{2z_2 + z_3 + z_1 - z_2}{2(z_3 + z_1 - z_2)}\right) + \left(\frac{2z_3 + z_1 + z_2 - z_3}{2(z_1 + z_2 - z_3)}\right) = 0$$

$$\frac{z_1 + z_2 + z_3}{2}\left(\frac{1}{z_2 + z_3 - z_1} + \frac{1}{z_3 + z_1 - z_2} + \frac{1}{z_1 + z_2 - z_3}\right) = 0$$

$$\frac{1}{2}z_G\left(\frac{1}{z_1 + z_2 + z_3 - 2z_1} + \frac{1}{z_1 + z_2 + z_3 - 2z_2} + \frac{1}{z_1 + z_2 + z_3 - 2z_3}\right) = 0$$

$$\frac{1}{2}z_G\left(\frac{1}{3z_G - 2z_1} + \frac{1}{3z_G - 2z_2} + \frac{1}{3z_G - 2z_3}\right) = 0$$

$$\frac{1}{2}z_G\left[\frac{9z_G^2 - 6z_3z_G - 6z_2z_G + 4z_2z_3 + 9z_G^2 - 6z_Gz_3 - 6z_1z_G + 4z_1z_3 + 9z_G^2 - 6z_Gz_2 - 6z_1z_G + 4z_1z_2}{(3z_G - 2z_1)(3z_G - 2z_2)(3z_G - 3z_3)}\right] = 0$$

$$\frac{1}{2}z_G\left[\frac{27z_G^2 - 12z_3z_6 - 12z_2z_G - 12z_1z_G + 4z_1z_3 + 4z_1z_2 + 4z_2z_3}{(3z_G - 2z_1)(3z_G - 2z_2)(3z_G - 2z_3)}\right] = 0$$

There are two cases: $z_G = 0$

or: $27z_G^2 - 12(z_3 + z_2 + z_1) \cdot z_G + 4z_1z_3 + 4z_1z_2 + 4z_2z_3 = 0$

$$27z_G^2 - 36z_G^2 + 4(z_1z_2 + z_2z_3 + z_1z_3) = 0$$

$$9z_G^2 = 4(z_1z_2 + z_2z_3 + z_1z_3) \Rightarrow z_G^2 = \frac{4}{9}(z_1z_2 + z_2z_3 + z_1z_3)$$

$$z_G^2 = \frac{4}{9}(z_1z_2z_3)\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) \Rightarrow z_G^2 = \frac{4}{9}(z_1z_2z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3})$$

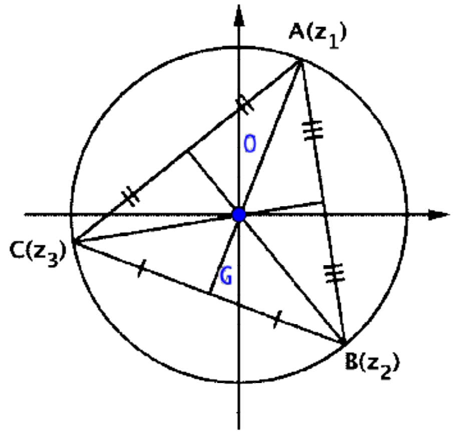
$$|z_G|^2 = \frac{4}{9}|\overline{z_1} + \overline{z_2} + \overline{z_3}| \Rightarrow |z_G|^2 = \frac{4}{9} \cdot 3|\overline{z_G}|$$

$$|z_G|^2 = \frac{4}{3}|z_G| \Rightarrow |z_G| = \frac{4}{3} \Rightarrow \frac{4}{3} = \frac{|z_1 + z_2 + z_3|}{3} \le |z_1| + |z_2| + |z_3|$$

$4 \le 3$. ***This is impossible, so:*** $|z_G| = 0 \Rightarrow z_G = 0 \Rightarrow z_1 + z_2 + z_3 = 0$

$z_G = 0$, ***so:*** $G \equiv 0$

Note: center of circle and centroid has the same point so the triangle ABC ***is equilateral triangle.*** $AB = BC = CA$



9. Find $x, y, z, w \in \mathbb{R}$ **such that:**

$$\begin{pmatrix} \sin x & \cos y \\ \tan z & \cot w \end{pmatrix}^n = \begin{pmatrix} \sin^n x & \cos^n y \\ \tan^n z & \cot^n w \end{pmatrix}, \forall n \in \mathbb{N} - \{0\}$$

Proposed by Daniel Sitaru – Romania

Solution by Andrew Okukura-Romania

For simplicity, we will note in $x = a, \cos y = b, \tan z = c, \cot w = d$. ***Thus, the condition can be written as:*** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}, \forall n \in \mathbb{N} \setminus \{0\}$

For $n = 2$ ***we have:*** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$

$\Rightarrow bc = 0 \Rightarrow b = 0$ ***or*** $c = 0$.

I. $b = 0$. ***That means the only equality left is*** $(ca + d) = c^2$

If $c = 0$ ***then the matrix*** $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ ***satisfies the identity in the hypothesis. (for any diagonal matrix*** $\begin{pmatrix} 0^n & 0 \\ 0 & d^n \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^n$***)***

If $c \neq 0 \Rightarrow c = a + d$. For $n = 3$ we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & b^3 \\ c^3 & d^3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & 0 \\ a+d & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a^3 & 0 \\ & d^3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a^3 & 0 \\ a(a+d)^2 + d^2(a+d) & d^3 \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix}$$

Thus, we have:

$$a(a+d)^2 + d^2(a+d) = (a+d)^3 | : (a+d) \Rightarrow a^2 + ad + d^2 = a^2 + 2ad + d^2$$

$$\Rightarrow ad = 0 \Rightarrow a = 0 \text{ or } d = 0$$

If $a = b = 0$ or $d = b = 0$ then the matrices $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ satisfy the identity in the hypothesis.

Thus, in this case the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ satisfy the identity.

By applying the same algorithm we obtain the solutions:

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *(duplicate)*, $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$

Thus, the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ are the only ones which satisfy the identity above, $a, b, c, d \in \mathbb{R}$. Thus, the solutions for x, y, z, w are:

I. $x, w \in \mathbb{R}, y = k\pi + \frac{\pi}{2} \wedge z = t\pi, k, z \in \mathbb{Z}, w \neq q\pi, \forall q \in \mathbb{Z}$

II. $z, w \in \mathbb{R}, x = k\pi \wedge y = t\pi + \frac{\pi}{2}, k, t \in \mathbb{Z}, w \neq q\pi$ and $z \neq p\pi + \frac{\pi}{2}, \forall q, p \in \mathbb{Z}$

III. $x, z \in \mathbb{R}, y = t\pi + \frac{\pi}{2}, w = k\pi + \frac{\pi}{2}, k, t \in \mathbb{Z}, z \neq p\pi + \frac{\pi}{2}, \forall p \in \mathbb{Z}$

IV. $x, y \in \mathbb{R}, z = t\pi, w = k\pi + \frac{\pi}{2}, t, k \in \mathbb{Z}$

V. $y, w \in \mathbb{R}, x = k\pi, z = t\pi, t, k \in \mathbb{Z}, w \neq q\pi, \forall q \in \mathbb{Z}$

10. If $A \in M_n(\mathbb{R}), n \geq 2, A^3 + A^2 + 7A = 9I_n$ then find:

$$\Omega = \det(A + 2I_n)$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vișescu-Romania

$$A^3 + A^2 + 7A = 9I_n \Rightarrow A^3 + A^2 + 7A - 9I_n = 0$$

$$\Rightarrow \underbrace{\text{the minimal polynom of } A}_{m_A} | x^3 + x^2 + 7x - 9$$

$$m_A | (x-1)\left(\underbrace{x^2 + 2x + 9}_{\Delta<0}\right)$$

$$x^3 + x^2 + 7x - 9 \,|\, \underline{x-1}$$

$$-x^3 + x^2 \qquad\qquad x^2 + 2x + 9$$

— — — — —

$$/ \quad 2x^2 + 7x - 9$$

$$-2x^2 + 2x$$

— — — — — — —

$$/ \; 9x - 9$$

$$-9x - 9$$

— — — — —

= =

Then (Frobenius)

$P_A(x) = \det(xI_n - A)$ ***and*** m_A ***have the same ireductible divisors.***

So, $P_A(x) = (x-1)^p(x^2 + 2x + 9)^q = (-1)^n \det(A - xI_n)$

with $p + 2q = n$

$$P(-2) = (-3)^p(4 - 4 + 9)^q = (-1)^n \det(A + 2I_n)$$

$$(-1)^n \det(A + 2I_n) = (-3)^p \cdot 3^{2q};\ \det(A + 2I_n) = (-1)^n \cdot (-3)^p \cdot (-3)^{2q}$$

$$\det(A + 2I_n) = (-1)^n(-3)^{2q+p}; \det(A + 2I_n) = (-1)^n(-3)^n$$

$$\det(A + 2I_n) = 3^n$$

11. $X, Y \in M_2(\mathbb{R}), X^{19} + X^{17} = Y^{21} + Y^{19} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Marian Ursărescu – Romania

$$X \in M_2(\mathbb{R}); x^{19} + x^{17} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = A$$

$$X^{19} + X^{17} = A \Rightarrow \left.\begin{matrix} X^{20} + X^{18} = AX \\ X^{20} + X^{18} = XA \end{matrix}\right\} \Rightarrow AX = XA$$

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \left.\begin{matrix} AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & -a+b \\ c-d & -c+d \end{pmatrix} \\ XA = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ -a+c & -b+d \end{pmatrix} \end{matrix}\right\} \Rightarrow$$

$$b = c; a = d \Rightarrow X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Rightarrow X^n = \begin{pmatrix} \frac{(a+b)^n + (a-b)^n}{2} & \frac{(a+b)^n - (a-b)^n}{2} \\ \frac{(a+b)^n - (a-b)^n}{2} & \frac{(a+b)^n + (a-b)^n}{2} \end{pmatrix}$$

$$\Rightarrow X^{19} + X^{17} = A \Rightarrow \begin{cases} \frac{(a+b)^{19} + (a-b)^{19}}{2} + \frac{(a+b)^{17} + (a-b)^{17}}{2} = 1 \\ \frac{(a+b)^{19} - (a-b)^{19}}{2} + \frac{(a+b)^{17} - (a-b)^{17}}{2} = -1 \end{cases}$$

$$\Rightarrow (a+b)^{19} + (a+b)^{17} = 0 \Rightarrow a + b = 0 \text{ unique solution } b = -a$$

$$\Rightarrow (a-b)^{19} + (a-b)^{17} = 2 \Rightarrow a - b = 1 \Rightarrow a = \frac{1}{2}, b = \frac{1}{2}$$

$$X = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \text{ The same for } Y = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}} = \sqrt[n]{\frac{1}{1}} = 1 \text{ constant sequence.}$$

$$\Omega = 1$$

Solution 2 by Florentin Vișescu – Romania

$$Z = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} XZ = XZ \\ YZ = ZY \end{matrix}$$

$$\left.\begin{matrix} XZ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & -a+b \\ c-d & -c+d \end{pmatrix} \\ ZX = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ -a+c & -b+d \end{pmatrix} \end{matrix}\right| \Rightarrow \begin{matrix} a-b = +a-c \Rightarrow b = c \\ -a+b = b-d \Rightarrow a = d \\ c-d = -a+c \Rightarrow a = d \\ -c+d = -b+d \Rightarrow c = b \end{matrix}$$

$$X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}; Y = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$$

We know that: $X^n = \begin{pmatrix} \frac{(a+b)^n+(a-b)^n}{2} & \frac{(a+b)^n-(a-b)^n}{2} \\ \frac{(a+b)^n-(a-b)^n}{2} & \frac{(a+b)^n+(a-b)^n}{2} \end{pmatrix}$

$$X^{19} + X^{17} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Rightarrow \frac{(a+b)^{19} + (a-b)^{19}}{2} + \frac{(a+b)^{17} + (a-b)^{17}}{2} = 1$$

$$\frac{(a+b)^{19} - (a-b)^{19}}{2} + \frac{(a+b)^{17} - (a-b)^{17}}{2} = -1$$

$$\begin{cases} (a+b)^{19} + (a+b)^{17} = 0 \\ (a-b)^{19} + (a-b)^{17} = 2 \end{cases} \begin{cases} (a+b)^{17}((a+b)^2 + 1) = 0 \\ (a-b)^{17}((a-b)^2 + 1) = 2 \end{cases} \Rightarrow b = -a$$

$$(2a)^{17}((2a)^2 + 1)2 \quad t^{19} + t^{17} - 2 = 0$$

$$t^{19} - 1 + t^{17} - 1 = 0$$

$$(t-1)(t^{18} + t^{17} + \cdots + t + 1) + (t-1)\left(t^{16} + t^{15} + \cdots + t + 1\right) = 0$$

$$t - 1 = 0 \quad t = 1 \Rightarrow 2a = 1\,;\, a = \frac{1}{2}; b = -\frac{1}{2}$$

$$\underbrace{t^{18} + t^{17} + \cdots + t + 1}_{>0} + \underbrace{t^{16} + t^{15} + \cdots + t + 1}_{>0 \ im\mathbb{R}} = 0$$

So, $x = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ ***Analog*** $Y = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow$

$$\Rightarrow X^{n+1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad Tr\, X^{n+1} = 1$$

$Y^{n+2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $Tr\, Y^{n+2} = 1 \Rightarrow \sqrt[n^2]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}} = \sqrt[n^2]{\frac{1}{1}} = 1$ ***constant sequence***

$$\Omega = 1$$

12. If $A \in M_n(\mathbb{Q})$, $\det A \neq 0, A^2 + (A^{-1})^2 = I_n$ then:

$$n \,\vdots\, 4$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

$$A \in M_n(\mathbb{Q}), \det A \neq 0$$

$$A^2 + (A^{-1})^2 = I_n \Rightarrow n \vdots 4$$

$$A^2 + (A^{-1})^2 = I_n | \cdot A^2 \Rightarrow A^4 + I_n = A^2$$

$$A^4 - A^2 + I_n = 0_n$$

$\Rightarrow m_a | x^4 - x^2 + 1$. *But* $x^4 - x^2 + 1$ *etc*

Irreducible in $\mathbb{Q}[x]$

$$(x^4 - x^2 + 1 = (x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1) =$$

$$= \left(x - \frac{-\sqrt{3}+i}{2}\right)\left(x - \frac{-\sqrt{3}-i}{2}\right)\left(x - \frac{\sqrt{3}+i}{2}\right)\left(x - \frac{\sqrt{3}-i}{2}\right)) \Rightarrow m_a = x^4 - x^2 + 1$$

Then, according to Frobenius $\left.\begin{matrix} P_A = (x^4 - x^2 + 1)^k \\ But\ grade\ P_A = n \end{matrix}\right| \Rightarrow 4k = n \Rightarrow n \vdots 4$

13. $A, B \in M_n(\mathbb{R}), \left(4 + \sqrt{10 + 2\sqrt{5}}\right)(A^2 + B^2) = (\sqrt{5} - 1)(AB - BA)$

If $\det(AB - BA) \neq 0$ then n is divisible with 20.

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

We know that $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$

$$\sin^2\frac{\pi}{10} + \cos^2\frac{\pi}{10} = 1 \Rightarrow \left(\frac{\sqrt{5}-1}{4}\right)^2 + \cos^2\frac{\pi}{10} = 1 \Rightarrow \cos^2\frac{\pi}{10} = \frac{16 - 6 + 2\sqrt{5}}{16}$$

$$\cos^2\frac{\pi}{10} = \frac{10 + 2\sqrt{5}}{16} \Rightarrow \cos\frac{\pi}{10} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Then $\left(1 + \cos\frac{\pi}{10}\right)(A^2 + B^2) = \sin\frac{\pi}{10}(AB - BA)$

$$2\cos^2\frac{\pi}{20}(A^2 + B^2) = 2\sin\frac{\pi}{20}\cos\frac{\pi}{20}(AB - BA)$$

$$\cos\frac{\pi}{20}(A^2 + B^2) = \sin\frac{\pi}{20}(AB - BA)$$

$$A^2 + B^2 = \tan\frac{\pi}{20}(AB - BA)$$

$$A^2 + B^2 = (A + iB)(A - iB) + iAB - iBA = (A + iB)(A - iB) + i(AB - BA)$$

So, $(A+iB)(A-iB) = A^2 + B^2 + i(BA-AB)$

$$= \tan\frac{\pi}{20}(AB-BA) - i(AB-BA) = \left(\tan\frac{\pi}{20} - i\right)(AB-BA)$$

$$\det[(A+iB)(A-iB)] = \left(\tan\frac{\pi}{20} - 1\right)^n \underbrace{\det(AB-BA)}_{\substack{\not\parallel \\ 0}}$$

$$\det(A+iB)\det(A-iB) = \left(\frac{\sin\frac{\pi}{20} - i\cos\frac{\pi}{20}}{\cos\frac{\pi}{20}}\right)^n \det(AB-BA)$$

$$\det(A+iB)\overline{\det(A+iB)} = \left(\frac{-i^2\sin\frac{\pi}{20} - i\cos\frac{\pi}{20}}{\cos\frac{\pi}{20}}\right)^n \det(AB-BA)$$

$$\Rightarrow \det(AB-BA)\cdot\frac{(-i)^n\cdot\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)^n}{\left(\cos\frac{\pi}{20}\right)^n} \in \mathbb{R}$$

$$\Rightarrow \sin\frac{n\pi}{20} = 0 \Rightarrow \frac{n\pi}{20} = n\pi \Rightarrow n = 20k$$

14. $A, B \in M_3(\mathbb{R}), \det(I_3 + (AB-BA)^2) = 0$. **Find:**

$$\Omega_1 = \det(AB-BA),\ \Omega_2 = Tr\left((AB-BA)^2\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

Let be $A = (a_{ij})_{\substack{i=\overline{1,3} \\ j=1,3}};\ B = (b_{ij})_{\substack{i=\overline{1,3} \\ j=1,3}}$

Let be $C = AB \Rightarrow c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj} \Rightarrow c_{ii} = \sum_{k=1}^{3} a_{ik}\, b_{kj}$

Let be $D = BA \Rightarrow d_{ij} = \sum_{k=1}^{3} b_{ik}\, a_{kj} \Rightarrow d_{ii} = \sum_{k=1}^{3} b_{ik}a_{ki}$

Let be $E = C - D(= AB - BC) \Rightarrow e_{ii} = \sum_{k=1}^{3}(a_{ik}b_{ki} - b_{ik}a_{ki})$

$$\Rightarrow Tr(AB-BA) = Tr\ E = \sum_{i=1}^{3}\sum_{k=1}^{3}(a_{ik}b_{ki} - b_{ik}a_{ki}) =$$

$$= \sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}b_{ki} - 3\sum_{i=1}^{3}\sum_{k=1}^{3} b_{ik}a_{ki} = 0$$

So, $Tr\ E = 0$

$$\det(I_3 + E^2) = \det(E^2 + I_3) = \det(E^2 - (iI_3)^2) = \det(E - iI_3)\det(E + iI_3) = 0 \Rightarrow$$

$$\Rightarrow P_E(x) = (x^2 + 1)(x - r) \text{ but}$$

$$P_E(x) = x^3 - TrEx^2 + \frac{(Tr\ E)^2 - Tr\ E^2}{2}x - \det E$$

$$P_E(x) = x^3 - rx^2 + x - r \Rightarrow Tr\ E = r = 0$$

$$1 = \frac{(Tr\ E)^2 - Tr\ E^2}{2} \Rightarrow 2 = -Tr\ E^2 \Rightarrow Tr\ E^2 = -2$$

$$\det E = r = 0$$

$$\Omega_1 = 0 \text{ and } \Omega_2 = -2$$

15. $A \in M_4(\mathbb{R}), \det A = -1, \det(A^2 + I_4) = 0$. **Find:**

$$\Omega = Tr\ (A^*)$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vișescu-Romania

$$\det(A^2 + I_4) = \det(A - iI_4)\det(A + iI_4) = 0$$

$\Rightarrow i$ or $-i$ root of $P_a(x)$, the characteristic polynomial of A. But $P_a(x) \in R[x]$

Then, i and $-i$ roots of $P_a(x)$

Let be λ_1 and λ_2 the other roots of $P_A(x)$

We have $P_A(x) = x^4 + ax^3 + bx^2 + cx + d$

$$-a = i - i + \lambda_1 + \lambda_2 = \lambda_1 + \lambda_2$$

$$b = i\lambda_1 + i\lambda_2 - i^2 - i\lambda_1 - i\lambda_2 + \lambda_1\lambda_2 = 1 + \lambda_1\lambda_2$$

$$-c = -i\lambda_1\lambda_2 + i\lambda_i\lambda_2 - i^2\lambda_1 - i^2\lambda_2 = \lambda_1 + \lambda_2$$

$$d = -i^2\lambda_1\lambda_2 = \lambda_1\lambda_2$$

$$P_A(x) = x^4 - (\lambda_1 + \lambda_2)x^3 + (1 + \lambda_1\lambda_2)x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$$

$$\left.\begin{matrix} P_A(0) = \lambda_1\lambda_2 \\ P_A(0) = \det A \end{matrix}\right\} \Rightarrow \lambda_1\lambda_2 = -1 \Rightarrow$$

$$P_A(x) = x^4 - (\lambda_1 + \lambda_2)x^3 - (\lambda_1 + \lambda_2)x - 1$$

$$P_A(x) = x^4 - Tr\ (A)x^3 + Tr(A^x)x^2 + cx + \det A$$

So, $Tr\ A^* = 0 = \Omega$

16. Find $X \in M_3(\mathbb{R})$ such that:

$$X^{2019} + X = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vișescu-Romania

$$X^{2020} + X^2 = X\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}X$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\Rightarrow \begin{cases} a = e = i \\ d = h = g = 0 \\ b = f \end{cases} \Rightarrow X = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}. \text{ So,}$$

$$X = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = aI_3 + B, \text{where } B = \begin{pmatrix} a & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, B^2 = \begin{pmatrix} 0 & 0 & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =, B^3 = O_3$$

$$\Rightarrow X^n = (aI_3 + B)^n = \sum_{k=0}^{n} \binom{n}{k} (aI_3)^{n-k} B^k$$

$$= \binom{n}{0} a^n I_3^n + \binom{n}{1} a^{n-1} I_3^{n-1} B + \binom{n}{2} a^{n-2} I_3^{n-2} B^2$$

$$= a^{n-2}\begin{pmatrix} a^2 & nab & nac + \frac{n(n-1)}{2}ab^2 \\ 0 & a^2 & nab \\ 0 & 0 & a^2 \end{pmatrix}$$

$$X^{2019} + X = a^{2017}\begin{pmatrix} a^2 & 2019ab & 2019ac + \frac{2019 \cdot 2018)}{2}ab^2 \\ 0 & a^2 & 2019ab \\ 0 & 0 & a^2 \end{pmatrix} + \begin{pmatrix} a^2 & 2ab & 2ac + ab^2 \\ 0 & a^2 & 2ab \\ 0 & 0 & a^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a^{2019} + a^2 = 2 \\ 2019a^{2018}b + 2ab = 2 \\ 2019a^{2018}c + 2019 \cdot 1009a^{2018}b^2 + 2ac + ab^2 = 0 \end{cases}$$

We have: $a^{2019} + a^2 - 2 = 0$

Let $f(a) = a^{2019} + a^2 - 2, f'(x) = 2019a^{2018} + 2a$

$$a_1 = 0, a_2 = -\sqrt[2017]{\frac{2}{2019}}$$

a	$-\infty$ $\quad a_2 \quad$ 0 $\quad \infty$
f(a)	$-\infty$ $- \ - - - - - - - - - (-2) \ +$ $\quad +\infty$

$$a = 1 \Rightarrow 2019b + 2b = 2 \Rightarrow b = \frac{2}{2021}$$

$$2019c + 2019 \cdot 1009 \cdot \left(\frac{2}{2021}\right)^2 + 2c + \left(\frac{2}{2021}\right)^2 = 0$$

$$c = -\frac{4}{2021^3}(1 + 1009 \cdot 2019)$$

So,

$$X = \begin{pmatrix} 1 & \frac{2}{2021} & -\frac{4}{2021^3}(1 + 1009 \cdot 2019) \\ 0 & 1 & \frac{2}{2021} \\ 0 & 0 & 1 \end{pmatrix}$$

17. $A \in M_4(\mathbb{R}), A \cdot A^T = I_4, Tr\ A = 0$**. Find:**

$$\Omega = Tr(A^{2019})$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

$$A \in M_4(\mathbb{R}); A \cdot {}^TA = I_4, Tr\ A = 0;\ \Omega = Tr\ (A^{2019})$$

$$\det(A^T A) = 1 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1 \Rightarrow \exists A^{-1} = {}^TA$$

$$Tr\ {}^TA = Tr\ A = 0 \Rightarrow Tr\ A^{-1} = 0$$

$$A^{-1} = \frac{1}{\det A} \cdot A^* \Rightarrow A^* = \det A \cdot A^{-1}$$

$$Tr\ A^* = \det A\, Tr\ A^{-1} = 0$$

$$Tr\ A^* = 0$$

1) $\det A = 1$

$$A^4 - OA^3 + OA^2 - XA + I_4 = O_4$$

$$A^4 - XA + I_4 = O_4$$

2) *If* $\det A = -1$

$$A^4 - XA - I_4 = 0_4$$

Let be the spectrum $A = \{A_1, A_2, A_3, A_4\} \not\ni 0$

Then the spectrum $A^{-1} = \left\{\frac{1}{\lambda_1}; \frac{1}{\lambda_2}; \frac{1}{\lambda_3}; \frac{1}{\lambda_4}\right\}$

and as ${}^TA = A^{-1} \Rightarrow$ *the spectrum of* ${}^TA = \left\{\frac{1}{\lambda_1}; \frac{1}{\lambda_2}; \frac{1}{\lambda_3}; \frac{1}{\lambda_4}\right\}$

But $Tr\ {}^TA = 0 \Rightarrow \frac{1}{\lambda_1}; \frac{1}{\lambda_2}; \frac{1}{\lambda_3}; \frac{1}{\lambda_4} = 0$

$$\Rightarrow \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_3 = 0 \Rightarrow x = 0$$

So, if **1)** $\det A = 1 \quad A^4 = -I_4$

if **2)** $\det A = -1 \quad A^4 = I_4$

1) $A^{2019} = (A^4)^{504} \cdot A^3 = (-I_4)^{504} \cdot A^3 = A^3$

2) $A^{2019} = (A^4)^{504} A^3 = I_4^{504} \cdot A^3 = A^3$

So, $Tr\ A^{2019} = Tr\ A^3$

1) $A^3 = -A^{-1} \Rightarrow A^3 = -{}^TA \Rightarrow Tr\ A^3 = -Tr\ {}^TA = 0$

2) $A^3 = A^{-1} \Rightarrow A^3 = {}^TA \Rightarrow Tr\ A^3 = Tr\ {}^TA = 0$

18. Let be $A \in M_3(\mathbb{R})$.Prove that:

$$det(A^2 + I_3) = 0 \Leftrightarrow detA = TrA \text{ and } TrA^* = 1, A^* - \text{adjoint of } A$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Florentin Vişescu-Romania

$$A \in M_3(\mathbb{R})$$

$$det(A^2 + I_3) = 0 \Leftrightarrow det(A^2 - i^2 I_3) = 0 \Leftrightarrow$$

$$det(A - iI_3)(A + iI_3) = 0 \Leftrightarrow det(A - iI_3) = 0 \text{ or } det(A + iI_3) = 0$$

$\Leftrightarrow i$ *or* $-i$ *are roots for* P_A

$$\Leftrightarrow P_A(X) = X^3 - TrA \cdot X^2 + TrA^* - detA \cdot I_3$$

$$P_A(X) = (X - r)(X + i)(X - i), r \in \mathbb{R} \Leftrightarrow$$

$$P_A(X) = (X - r)(X^2 + 1) \Leftrightarrow P_A(X) = X^3 - rX^2 + X - r$$

$$\Leftrightarrow TrA = detA \text{ and } TrA^* = 1$$

Solution 2 by Julio Cesar Monhsam-Pelotas-Brazil

$$(\Rightarrow)\ \ det(A^2+I_3)=0 \Leftrightarrow det(A^2-i^2I_3)=0 \Leftrightarrow$$
$$\pm i\ in\ Spec(A) \Rightarrow$$
$$TrA = x+i-i = x$$
$$detA = x\cdot(i)\cdot(-i) = x \Rightarrow detA = TrA$$
$$TrA^* = x\cdot i + x\cdot(-i) + i\cdot(-i) = 1$$
$$detA = \prod_i \lambda_i \text{ and } TrA = \sum_i \lambda_i$$
$$TrA^* = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$
$$Spec(A) \text{ is set of values of } A$$
$$(\Leftarrow)\ \ x = TrA = detA \text{ and } TrA^* = 1$$
$$P(\lambda) = \lambda^3 - TrA\cdot\lambda^2 + TrA^*\cdot\lambda - detA = 0$$
$$P(\lambda) = \lambda^3 - x\cdot\lambda^2 + \lambda - x = 0$$
$$P(i) = -i + x + i - x = 0$$
$$P(-i) = i + x - i - x = 0 \Rightarrow det(A - iI_3) = 0 \text{ and } det(A + iI_3) = 0 \text{ then}$$
$$det(A^2 + I_3) = 0$$

19. If $A, B \in M_n(\mathbb{R}), n \in \mathbb{N}, n \geq 2, A + B = AB, det(AB) \neq 0$ then:

$$det\left((I_n - A^3 - B^3 + (AB)^3)(I_n - A^5 - B^5 + (AB)^5)(I_n - A^7 - B^7 + (AB)^7)\right) \geq 0$$

Proposed by Daniel Sitaru-Romania

Solution by Marian Ursărescu-Romania

$$A + B = AB \Leftrightarrow A + B - AB = O_4 \Leftrightarrow AB - A - B + I_n = I_n$$
$$\Leftrightarrow A(B - I_n) - (B - I_n) = I_n \Leftrightarrow (A - I_n)(B - I_n) = I_n \text{ that mean}$$
$$XY = I_n \Leftrightarrow Y = X^{-1} \Rightarrow YX = I_n$$
$$(A - I_n)(B - I_n) = I_n \Rightarrow BA - B - A + I_n = I_n \Rightarrow BA = A + B \Rightarrow AB = BA$$
$$\text{and } (I_n - A)(I_n - B) = I_n;\quad (1)$$
$$I_n - A^3 - B^3 + (AB)^3 = I_n - A^3 - B^3 + A^3B^3 = I_n - A^3 - B^3(I_n - A^3)$$
$$= (I_n - A^3)(I_n - B^3) = (I_n - A)(I_n - B)(I_n + A + A^2)(I_n + B + B^2)$$

$$\overset{(1)}{\cong}(I_n+A+A^2)(I_n+B+B^2);\quad (2)$$

$$I_n-A^5-B^5+(AB)^5=I_n-A^5-B^5+A^5B^5=I_n-A^5-B^5(I_n-A^5)$$

$$=(I_n-A^5)(I_n-B^5)$$

$$=(I_n-A)(I_n-B)(I_n+A+A^2+A^3+A^4)(I_n+B+B^2+B^3+B^4)$$

$$\overset{(1)}{\cong}(I_n+A+A^2+A^3+A^4)(I_n+B+B^2+B^3+B^4);\quad (3)$$

$$I_n-A^7-B^7+(AB)^7=I_n-A^7-B^7+A^7B^7=I_n-A^7-B^7(I_n-A^7)$$

$$=(I_n-A^7)(I_n-B^7)$$

$$=(I_n-A)(I_n-B)\left(I_n+A+A^2+A^3+A^4+A^5+A^6\right)\left(I_n+B+B^2+B^3+B^4+B^5++B^6\right)$$

$$\overset{(1)}{\cong}\left(I_n+A+A^2+A^3+A^4+A^5+A^6\right)\left(I_n+B+B^2+B^3+B^4+B^5+B^6\right);\quad (4)$$

From (2)+(3)+(4) we must show:

$$det(I_n+A+A^2)det(I_n+A+A^2+A^3+A^4)det\left(I_n+A+A^2+A^3+A^4+A^5+A^6\right)\cdot$$

$$\cdot det\left(I_n+B+B^2\right)det\left(I_n+B+B^2+B^3+B^4\right)det\left(I_n+B+B^2+B^3+B^4+B^5+B^6\right)\geq 0$$

true because

$det(I_n+X+X^2+\cdots+X^{2n})\geq 0$ ***(article-R.M.M.-22)***

http://www.ssmrmh.ro/2019/01/24/old-rmm-22/

20. $A,B\in M_4(\mathbb{C}), B^3=I_4, A^3=AB^2+BA^2, C=\begin{pmatrix}28 & 18 & 36 & 723\\ 120 & 121 & 45 & 891\\ 330 & 27 & 151 & 210\\ 450 & 150 & 180 & 181\end{pmatrix}$

Prove that:

$$det\left((CA-CB)(A^2-B^2)\right)\neq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$(CA-CB)(A^2-B^2)=CA^3-CAB^2-CBA^2+CB^3$$

$$=CA^3-C(AB^2+BA^2)+CB^3=CA^3-CA^3+CB^3=O_4+CI_4=C$$

We must show that: $det(C)\neq 0$

We determine the last digit of $det(C)$:

$$U\left(28\cdot\begin{vmatrix}121 & 45 & 891\\ 27 & 151 & 210\\ 150 & 180 & 181\end{vmatrix}\right)=8$$

$$U\left(18\cdot\begin{vmatrix}120 & 45 & 891\\ 330 & 151 & 210\\ 450 & 180 & 181\end{vmatrix}\right)=0$$

$$U\left(36\cdot\begin{vmatrix}120 & 121 & 891\\ 330 & 27 & 210\\ 450 & 150 & 181\end{vmatrix}\right)=0$$

$$U\left(723\cdot\begin{vmatrix}120 & 121 & 45\\ 330 & 151 & 210\\ 450 & 150 & 180\end{vmatrix}\right)=0$$

$$U(det(C))=8\Rightarrow det(C)\neq 0$$

Solution 2 by Florentin Vişescu-Romania

$$det\big((CA-CB)(A^2-B^2)\big)=det(CA^3-CAB^2-CBA^2+CB^3)$$

$$=det\big(CA^3-C(AB^2+BA^2)+CB^3\big)=det\big(CA^3-CA^3+CB^3\big)=det(O_4+CI_4)=det(C)$$

I was working in $mod10$

$$\widehat{C}=\begin{pmatrix}\widehat{8} & \widehat{8} & \widehat{6} & \widehat{3}\\ \widehat{0} & \widehat{1} & \widehat{5} & \widehat{1}\\ \widehat{0} & \widehat{7} & \widehat{1} & \widehat{0}\\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{1}\end{pmatrix}$$

$$det(\widehat{C})=\widehat{8}\cdot\begin{vmatrix}\widehat{1} & \widehat{5} & \widehat{1}\\ \widehat{7} & \widehat{1} & \widehat{0}\\ \widehat{0} & \widehat{0} & \widehat{1}\end{vmatrix}=\widehat{8}\cdot\begin{vmatrix}\widehat{1} & \widehat{5}\\ \widehat{7} & \widehat{1}\end{vmatrix}=\widehat{6}$$

$$U(det(C))=8\Rightarrow det(C)\neq 0$$

21. If $a,b\geq 0, n\in\mathbb{N}, n\geq 2, A\in M_n(\mathbb{R}), A^2=O_n$ then:

$$det\left(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n\right)\geq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vişescu-Romania

If $a+b=0\Rightarrow a=0, b=0\Rightarrow det\left(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n\right)=detO_n=0$

If $a+b>0\Rightarrow det(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n)=$

$$=\sqrt{3}^n(a+b)^n det\left(A+\frac{a^2+ab+b^2}{\sqrt{3}(a+b)}\cdot I_n\right)$$

$$=(-1)^n\sqrt{3}^n(a+b)^n det\left(-A-\frac{a^2+ab+b^2}{\sqrt{3}(a+b)}\cdot I_n\right)$$

$=(-1)^n\sqrt{3}^n(a+b)^n P_A\left(-\frac{a^2+ab+b^2}{\sqrt{3}(a+b)}\right)$ *where* P_A *–is the characteristic polynomial of* A.

How $A^2=O_n\Rightarrow m_A\ /\ x^2$ **(***where* m_A *–minimal polynomial of* A**)**

So, $m_A=x^k, k=1;2$

By Frobenius m_A, P_A *–they have the same irreductible factors, then* $P_A=x^n$

$$det(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n)=$$

$$=\sqrt{3}^n(a+b)^n(-1)^n\left(-\frac{a^2+ab+b^2}{\sqrt{3}(a+b)}\right)^n=(-1)^{2n}(a^2+ab+b^2)^n>0$$

Solution 2 by Marian Ursărescu-Romania

$$det(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n)=$$

$$=det(A^2+\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n)$$

$$=det\left[\left(A+\frac{\sqrt{3}(a+b)+i(a-b)}{2}I_n\right)\left(A+\frac{\sqrt{3}(a+b)-i(a-b)}{2}I_n\right)\right]$$

$$=det\left[\left(A+\left(\frac{\sqrt{3}(a+b)}{2}+i\frac{a-b}{2}\right)I_n\right)\overline{\left(A+\left(\frac{\sqrt{3}(a+b)}{2}+\frac{\imath(a-b)}{2}\right)I_n\right)}\right]\geq 0$$

Because $det(X\overline{X})\geq 0, \forall X\in M_n(\mathbb{R})$

22. F_n: n^{th} Fibonacci Number

$$F_{2n+1}=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!\,(2k)!}$$

Proposed by Alpaslan Ceran-Turkey

Solution by Jovica Mikic-Sarajevo-Bosnia

Let $S(n)=\sum_{k=0}^{n}\binom{n+k}{2k}$*. We call* $S(n)$ *the main sum.*

Let $P(n)=\sum_{k=0}^{n}\binom{n+k}{2k-1}$*, where* P *is called auxiliary sum.*

We use the well-known Pascal's identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ *(*)*

Let us find the recurrence between S *and* P*! Let* $n\in\mathbb{N}$

$$S(n)\overset{(*)}{=}\sum_{k=0}^{n}\left[\binom{n+k-1}{2k}+\binom{n+k-1}{2k-1}\right]=\sum_{k=0}^{n}\binom{n-1+k}{2k}+\sum_{k=0}^{n}\binom{n-1+k}{2k-1}$$

$$=\sum_{k=0}^{n}\binom{n-1+k}{2k}+\sum_{k=0}^{n}\binom{n-1+k}{2k-1}=S(n-1)+\sum_{k=0}^{n-1}\binom{n-1+k}{2k-1}+\binom{2n-1}{2n-1}\quad (**)$$

By (**) $\Rightarrow S(n)=S(n-1)+P(n-1)+1$ *(***)*

On the other side: $n\in\mathbb{N}$

$$P(n)=\sum_{k=0}^{n}\left|\begin{matrix}n+k\\2k-1\end{matrix}\right|\overset{(*)}{=}\sum_{k=0}^{n}\left|\begin{matrix}n-1+k\\2k-1\end{matrix}\right|+\sum_{k=0}^{n}\left|\begin{matrix}n-1+k\\2k-2\end{matrix}\right|=\sum_{k=0}^{n-1}\left|\begin{matrix}n-1+k\\2k-1\end{matrix}\right|+\left|\begin{matrix}2n-1\\2n-1\end{matrix}\right|+$$

$$+\sum_{k=1}^{n}\binom{n+(k-1)}{2(k-1)};t=k-1$$

$$=P(n-1)+1+\sum_{t=0}^{n-1}\left|\begin{matrix}n+t\\2t\end{matrix}\right|=P(n)+\binom{2n}{2n}+\sum_{t=0}^{n-1}\left|\begin{matrix}n+t\\2t\end{matrix}\right|\Rightarrow P(n)=P(n-1)+S(n)\ (****)$$

By (***)$\Rightarrow P(n-1)=S(n)-S(n-1)-1;n\in\mathbb{N}$ *(*****)*

By (*****)$\Rightarrow P(n)=S(n+1)-S(n)-1$ *(******)*

By (****)$\Rightarrow P(n)-P(n-1)=S(n)$

By (*****) and (******)$\Rightarrow S(n+1)-S(n)-1-(S(n)-S(n-1)-1)=S(n)$

$S(n+1)-2S(n)+S(n-1)=S(n)$*, so, we obtain:*

$$S(n+1)=3S(n)-S(n-1);n\in\mathbb{N}\Rightarrow\lambda^2=3\lambda-1\Leftrightarrow\lambda^2-3\lambda+1=0$$

$$\lambda_1=\frac{3+\sqrt{5}}{2}\wedge\lambda_2=\frac{3-\sqrt{5}}{2}$$

$$S(n)=c_1\left(\frac{3+\sqrt{5}}{2}\right)^n+c_2\left(\frac{3-\sqrt{5}}{2}\right)^n\quad (VII)$$

Obviously, $S(0)=1$ *and* $S(1)=2$

We have: $c_1+c_2=1$ *and* $c_1\left(\frac{3+\sqrt{5}}{2}\right)+c_2\left(\frac{3-\sqrt{5}}{2}\right)=2$

$$c_2=1-c_1$$

$$c_1\left(\frac{3+\sqrt{5}}{2}\right)+(1-c_1)\left(\frac{3-\sqrt{5}}{2}\right)=2$$

$$c_1\left(\frac{3+\sqrt{5}}{2}-\frac{3-\sqrt{5}}{2}\right)+\frac{3-\sqrt{5}}{2}=2$$

$$c_1\sqrt{5}=2-\frac{3-\sqrt{5}}{2}$$

$$c_1\sqrt{5}=\frac{1+\sqrt{5}}{2}\Leftrightarrow c_1=\frac{1+\sqrt{5}}{2\sqrt{5}}$$

$c_2=1-c_1$; so, $c_2=\frac{2\sqrt{5}-1-\sqrt{5}}{2\sqrt{5}}$

$$c_2=\frac{\sqrt{5}-1}{2\sqrt{5}}=-\frac{1-\sqrt{5}}{2\sqrt{5}}$$

Now, $S(n)=c_1\left(\frac{3+\sqrt{5}}{2}\right)^n+c_2\left(\frac{3-\sqrt{5}}{2}\right)^n$

$$=\frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n+(-1)\cdot\frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$S(n)=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^2\right)^n-\frac{1-\sqrt{5}}{2}\left(\left(\left(\frac{1-\sqrt{5}}{2}\right)^2\right)^n\right)$$

$$S(n)=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right)$$

By Binet's formula $\Rightarrow S(n)=F_{2n+1}$

Moreover, by (******): $P(n)=S(n+1)-S(n)-1$

$$P(n)=F_{2(n+1)+1}-F_{2n+1}-1$$

$$P(n)=F_{2n+3}-F_{2n+1}-1$$

$$P(n)=F_{2n+2}-1$$

Therefore,

$P(n)=\sum_{k=0}^{n}\binom{n+k}{2k-1}=F_{2n+2}-1$ *and*

$$S(n)=\sum_{k=0}^{n}\binom{n+k}{2k}=F_{2n+1}$$

23.

$$x^9+1=(x+1)(x^2+ax+1)(x^2+bx+1)(x^2+cx+1)(x^2+dx+1)$$

$\forall x\in\mathbb{C}$. **Find:** $\Omega=a^6+b^6+c^6+d^6$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Sergio Esteban-Argentina

Buscamos ias 9 raices de x^9+1

$\Rightarrow \cos\left(\frac{k\pi}{9}\right)+i\sin\left(\frac{k\pi}{9}\right)$, ***para*** $k=\pm1,\pm3,\pm5,\pm7,9$

Para encontrar factores con coeficientes reaies, multiplicamos ios pares conjugados

$$\left(x-\cos\left(\frac{k\pi}{9}\right)-i\sin\left(\frac{k\pi}{9}\right)\right)\left(x-\cos\left(-\frac{k\pi}{9}\right)-i\sin\left(-\frac{k\pi}{9}\right)\right)=$$

$$=\left(x-\cos\left(\frac{k\pi}{9}\right)\right)^2-\left(i\sin\left(\frac{k\pi}{9}\right)\right)^2=x^2-2\cos\left(\frac{k\pi}{9}\right)x+1$$

Entonces

$$x^9+1=(x+1)\left(x^2-2\cos\left(\frac{\pi}{9}\right)x+1\right)\left(x^2-2\cos\left(\frac{3\pi}{9}\right)x+1\right)\cdot$$

$$\cdot\left(x^2-2\cos\left(\frac{5\pi}{9}\right)x+1\right)\left(x^2-2\cos\left(\frac{7\pi}{9}\right)x+1\right)$$

$$\Omega=\left(-2\cos\left(\frac{\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{3\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{5\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{7\pi}{9}\right)\right)^6$$

$$\Omega=2^6(\cos^6(20^\circ)+\cos^6(100^\circ)+\cos^6(140^\circ)+\cos^6(60^\circ))$$

Sabemos que si:

i) $a+b+c=0$, ***se cumple:***

$$a^6+b^6+c^6=3\left(\frac{a^3+b^3+c^3}{3}\right)^2+2\left(\frac{a^2+b^2+c^2}{2}\right)^3$$

ii) $\cos^2 x+\cos^2(120^\circ+x)+\cos^2(120^\circ-x)=\frac{3}{2}$

iii) $\cos^3 x+\cos^3(120^\circ+x)+\cos^3(120^\circ-x)=\frac{3}{4}\cos 3x$

Entonces como

$\cos x+\cos(120^\circ-x)+\cos(120^\circ+x)=0$, ***se cumple i)***

Reemplazando ii) y iii) en i)

$$\cos^6 x + \cos^6(120° - x) + \cos^6(120° + x) = 3\left(\frac{3}{4}\cdot\frac{\cos 3x}{3}\right)^2 + 2\left(\frac{3}{4}\right)^3$$

Si $x = 20°$

$$\cos^6(20°) + \cos^6(100°) + \cos^6(140°) = \frac{3}{16}\cdot\cos^2(60°) + \frac{27}{32}$$

Finalmente

$$\Omega = 2^6\left[\cos^6(60°) + \frac{3}{16}\cdot\cos^2(60°) + \frac{27}{32}\right]$$

$$\Omega = 64\left[\frac{1}{64} + \frac{3}{16}\cdot\frac{1}{4} + \frac{27}{32}\right]$$

$$\Omega = 1 + 3 + 54 = 58$$

24. Let $\begin{cases} x_0 = 2020 \\ x_{n+1} = 2x_n - n^2 + 2n + 2019\cdot 2020^n, n = 0,1,2,\dots \end{cases}$

Find:

$$\Omega = x_{2030}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Șerban George Florin-Romania

Using the formulas: $S_1 = x + 2^2x^2 + \dots + nx^n = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$,

$$S_2 = x + 2^2x^2 + \dots + n^2x^n = \frac{n^2x^{n+3} + x^{n+2}(-2n^2 - 2n + 1) + (n+1)^2x^{n+1} - x^2 - x}{(x-1)^3}$$

We denote $\alpha = 2019\cdot 2020, b_n = -n^2 + 2n + \alpha^n, x_{n+1} = 2x_n + b_n$

$x_1 = 2x_0 + b_0, x_2 = 2x_1 + b_1 = 2^2x_0 + 2b_0 + b_1, x_3 = x_2 + b_2 = 2^3x_0 + 2^2b_0 + x_n =$

$$= 2^nx_0 + \sum_{k=0}^{n-1} 2^{n-k-1}b_k = 2^nx_0 + \sum_{k=0}^{n-1} 2^{n-k-1}\left(-k^2 + 2k + \alpha^k\right)$$

$$x_n = 2^nx_0 + 2^{n-1}\cdot\sum_{k=0}^{n-1}\frac{-k^2 + 2k + \alpha^k}{2^k} =$$

$$= 2^nx_0 + 2^{n-1}\cdot\left[-\sum_{k=0}^{n-1}k^2\left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{n-1}k\cdot\left(\frac{1}{2}\right)^k + \sum_{k=0}^{n-1}\left(\frac{\alpha}{2}\right)^k\right]$$

$$\Omega = x_{2030} = 2^{2030}x_0 + 2^{2029}\left[-\sum_{k=0}^{2029} k^2\left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k + \sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k\right]$$

$$\sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k = 1 + \left(\frac{\alpha}{2}\right) + \cdots \left(\frac{\alpha}{2}\right)^{2029} = \frac{\left(\frac{\alpha}{2}\right)^{2030} - 1}{\frac{\alpha}{2} - 1} = \frac{\alpha^{2030} - 2^{2030}}{2^{2029}(\alpha - 2)}$$

$$\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + \cdots + 2029\left(\frac{1}{2}\right)^{2029} = \frac{2029\left(\frac{1}{2}\right)^{2031} - 2030\left(\frac{1}{2}\right)^{2030} + \frac{1}{2}}{\frac{1}{4}} =$$

$$= \frac{\frac{2029}{2^{2031}} - \frac{2030}{2^{2030}} + \frac{1}{2}}{\frac{1}{4}} = \frac{2029 - 4060 + 2^{2030}}{2^{2029}} = \frac{2^{2030} - 2031}{2^{2029}}$$

$$\sum_{k=0}^{2029} k^2\left(\frac{1}{2}\right)^k = \frac{2029\left(\frac{1}{2}\right)^{2032} + (-2\cdot 2029^2 - 2\cdot 2029 + 1)\left(\frac{1}{2}\right)^{2031} + 2030^2\left(\frac{1}{2}\right)^{2030} - \frac{3}{4}}{-\frac{1}{8}} =$$

$$= \left(\frac{2029}{2^{2032}} - \frac{8237739}{2^{2031}} + \frac{4120900}{2^{2030}} - \frac{3}{4}\right)\cdot\left(-\frac{8}{1}\right) =$$

$$= \frac{2029 - 16475478 + 16483600 - 3\cdot 2^{2030}}{2^{2032}}\cdot\left(-\frac{8}{1}\right) = \frac{3\cdot 2^{2030} - 10\cdot 151}{2^{2029}}$$

$$\Omega = 2^{2030}\cdot 2020 + 2^{2029}\left[\frac{3\cdot 2^{2030} - 10151}{2^{2029}} + \frac{2^{2030} - 2031}{2^{2028}} + \frac{\alpha^{2030} - 2^{2030}}{2^{2029}(\alpha - 2)}\right] =$$

$$= 2^{2030}\cdot 2020 - 3\cdot 2^{2030} + 10151 + 2^{2031} - 4062 + \frac{\alpha^{2030} - 2^{2030}}{\alpha - 2} =$$

$$= 2^{2030}(2020 - 3 + 2) + 6089 + \frac{\alpha^{2030} - 2^{2030}}{\alpha - 2} =$$

$$\Omega = 2019\cdot 2^{2030} + 6089 + \frac{20192020^{2030} - 2^{2030}}{20192018}$$

$$\Omega = \frac{40767684341\cdot 2^{2030} + 122949197602 + 20192020^{2030}}{20192018}$$

25.

$$x^{2n+1} + 1 = (x+1)\prod_{k=1}^{n}(x^2 + a_i x + 1),\ \forall x \in \mathbb{C}$$

Find:

$$\Omega = \sum_{i=1}^{n} a_i^4 \,, a_i \in \mathbb{C}, i \in \overline{1,n}$$

Proposed by Rahim Shahbazov-Baju-Azerbaijan

Solution 1 by Gabriel Ruddy Cruz Mendez-Lima-Peru

$$8\cos^4 x = 3 + 4\cos(2x) + \cos(4x)$$

$$a_1^4 = \left(-2\cos\left(\frac{\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{\pi}{2n+1}\right) = 6 + 8\cos\left(\frac{2\pi}{2n+1}\right) + 2\cos\left(\frac{4\pi}{2n+1}\right)$$

$$a_2^4 = \left(-2\cos\left(\frac{3\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{3\pi}{2n+1}\right) = 6 + 8\cos\left(\frac{6\pi}{2n+1}\right) + 2\cos\left(\frac{12\pi}{2n+1}\right)$$

$$a_3^4 = \left(-2\cos\left(\frac{5\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{5\pi}{2n+1}\right) = 6 + 8\cos\left(\frac{10\pi}{2n+1}\right) + 2\cos\left(\frac{20\pi}{2n+1}\right)$$

$$a_{n-1}^4 = \left(-2\cos\left(\frac{(2n-3)\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{(2n-3)\pi}{2n+1}\right) =$$

$$= 6 + 8\cos\left(\frac{2(2n-3)\pi}{2n+1}\right) + 2\cos\left(\frac{4(2n-3)\pi}{2n+1}\right)$$

$$a_n^4 = \left(-2\cos\left(\frac{(2n-1)\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{(2n-1)\pi}{2n+1}\right) =$$

$$= 6 + 8\cos\left(\frac{2(2n-1)\pi}{2n+1}\right) + 2\cos\left(\frac{4(2n-1)\pi}{2n+1}\right)$$

Sumando verticalmente $a_n^4 = 6n + 8\left[\frac{2\cos\left(\frac{2n\pi}{2n+1}\right)\sin\left(\frac{2n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right] + 2\left[\frac{2\cos\left(\frac{4n\pi}{2n+1}\right)\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right]$

$$a_n^4 = 6n + 8\left[\frac{\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right] + 2\left[\frac{\sin\left(\frac{8n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right] \to a_n^4 = 6n + 4(-1) + (-1)$$

$$\therefore a_n^4 = 6n - 5$$

Solution 2 by Le Van-Ho Chi Minh-Vietnam

Lemma I: According to Section 7 of Le Van(2019)

$$x^{2s+1}+1=\prod_{j=0}^{2s}\left[x-exp\left(\frac{(2j+1)i\pi}{2s+1}\right)\right]=\prod_{j=0}^{2s}(x-\omega_j)$$

$$=(x+1)\prod_{j=0}^{s-1}\left[(x-\omega_j)(x-\overline{\omega_j})\right]$$

$$=(x+1)\prod_{j=0}^{s-1}\left[x^2-2Re(\omega_j)x+1\right]$$

Which result in

$$a_j=-2Re(\omega_j)=-2cos\left(\frac{(2j+1)\pi}{2s+1}\right);(j=\overline{1,s-1})$$

Applying Vieta's theorem to the expansion

$x^{2s+1}+1=(x+1)(x^{2s}-x^{2s-1}+\cdots-x+1), we\ get$

$$\sum_{j=0,j\neq s}^{2s-1}\omega_j=2\sum_{j=1}^{s-1}\omega_j=\sum_{j=0}^{s-1}Re(\omega_j)=-\sum_{j=0}^{s-1}a_j=1$$

Lemma 2: Replacing $\psi_j=\frac{(2j+1)\pi}{2s+1}$***, the following function is bijective***

$$f:X=\{cos\psi_j\}_{j=\overline{0,s-1}}\to Y=\{cos2\psi_j\}_{j=\overline{0,s-1}}\equiv -X=\{-cos|\pi-\psi_j|\}_{j=\overline{1,s-1}}$$

Firstly, we shall prove that f is injective function. Indeed, for two distinct numbers K and l: $|\pi-\psi_l|=|\pi-\psi_k|\Leftrightarrow\begin{cases}\psi_l=\psi_k\\ \psi_l+\psi_k=2\pi\end{cases}$

$$\Leftrightarrow\begin{cases}2s+1-2(2k+1)=2s+1-2(2l+1)\\ 2(2k+1)+2(2l+1)=2\end{cases}\Leftrightarrow\begin{cases}k=l\\ 2(k+l+1)=1\end{cases}$$

The above result is self-conflicted as $k\neq l\ and\ k,l=\overline{0,s-1}$***.Thus,*** $k\neq l$ ***result in*** $f(k)\neq f(l)$ ***which implies that f is an injection.***

Secondly, we shall prove that f is a surjective function.For each given number $\lambda_0=\overline{0,s-1}$, ***it is supposed to find number*** $j_0=\overline{0,s-1}$ ***such that***

$$\left|\pi-\frac{2\pi(2j_0+1)}{2s+1}\right|=\frac{(2\lambda_0+1)\pi}{2s+1}$$

$$\Leftrightarrow|2s+1-2(2j_0+1)|=2\lambda_0+1\Leftrightarrow\begin{cases}2s+1-2(2j_0+1)=2\lambda_0+1\\ 2s+1-2(2j_0+1)=-2\lambda_0-1\end{cases}$$

$$\Leftrightarrow \begin{cases} 2(2j_0+1) = 2s - 2\lambda_0 \\ 2(2j_0+1) = 2s + 2\lambda_0 + 2 \end{cases} \Leftrightarrow \begin{cases} 2j_0 + 1 = s - \lambda_0 \\ 2j_0 + 1 = s + \lambda_0 + 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} j_0 = \mu = \dfrac{s - \lambda_0 - 1}{2} \\ j_0 = v = \dfrac{s + \lambda_0}{2} \end{cases}$$

The above implies that for every element λ_0, we are able to find j_0 such that

$$cos\psi_{j_0} = -cos\psi_{\lambda_0}$$

Exepli gratia, for $s = 3$, the surjective property of function j is illustrated as follow:

j_0	$cos\psi_j$	$cos2\psi_j$	$-cos\psi_j$	λ_0	***illustration***
0	$cos\frac{\pi}{7}$	$cos\frac{2\pi}{7}$	$-cos\frac{5\pi}{7}$	***2***	$j_1 = \frac{s-\lambda_1-1}{2} = \frac{3-2-1}{2} = 0$
1	$cos\frac{3\pi}{7}$	$cos\frac{6\pi}{7}$	$-cos\frac{\pi}{7}$	***0***	$j_2 = \frac{s-\lambda_2-1}{2} = \frac{3-0-1}{2} = 1$
2	$cos\frac{5\pi}{7}$	$cos\frac{10\pi}{7}$	$-cos\frac{3\pi}{7}$	***1***	$j_3 = \frac{s-\lambda_3}{2} = \frac{3+1}{2} = 2$

As either μ or v is a natural number.Thus, f is a surjection, and consequently a bijection.

Lemma III:

Lemma II results in

$$\sum_{j=0}^{s-1} cos2\psi_j = -\sum_{j=0}^{s-1} cos\psi_j = -\frac{1}{2}\sum_{j=0}^{s-1} Re(\omega_j) = -\frac{1}{2}$$

And therefore

$$\sum_{j=0}^{s-1}(cos\psi_j)^2 = \frac{1}{2}\sum_{j=0}^{s-1}(1 + cos2\psi_j) = \frac{1}{2}\sum_{j=0}^{s-1} 1 + \frac{1}{2}\sum_{j=0}^{s-1} cos2\psi_j = \frac{2s-1}{4}$$

For $a_j = -2cos\psi_j$, we get

$$\sum_{j=1}^{s-1} a_j = 1; \sum_{j=1}^{s-1} a_j^2 = 2s - 1$$

The function f also result in the generalized cased(of which w is a natural number) as follow:

$$\sum_{j=1}^{s-1} cos2^w\psi_j = \frac{(-1)^w}{2}; \sum_{j=1}^{s-1}(cos2^w\psi_j)^2 = \frac{2s-1}{4}$$

Solution: Applying lemma III:

$$\Omega = \sum_{j=0}^{s-1} a_j^4 = \sum_{j=0}^{s-1}\left[\left(-2cos\psi_j\right)^2\right]^2 = 4\sum_{j=0}^{s-1}\left(1 + cos2\psi_j\right)^2$$

$$= 4\sum_{j=0}^{s-1}\left[1 + 2cos2\psi_j + (cos2\psi_j)^2\right]$$

Following the expansion

$$\Omega = 4\sum_{j=1}^{s-1} 1 + 8\sum_{j=0}^{s-1} cos2\psi_j + 4\sum_{j=0}^{s-1}\left(cos2\psi_j\right)^2 = 6s - 5$$

26. $p \in \mathbb{R}, p \neq 1, A(z_1 + pz_2 + p^2z_3), B(z_2 + pz_3 + p^2z_1),$
$C(z_3 + pz_1 + p^2z_2)$

$M(z_1), N(z_2), P(z_3), z_1, z_2, z_3 \in \mathbb{C}$. **Prove that:**

$$AB = BC = CA \Rightarrow MN = NP = PM$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

Let $\omega_1 = z_1 + bz_2 + p^2z_3, w_2 = z_2 + pz_3 + p^2z_1, \omega_3 = z_3 + pz_1 + p^2z_2$

Let $a = z_2 - z_3, b = z_3 - z_1, c = z_1 - z_2$

ΔABC **is equilateral** $\Leftrightarrow (\omega_2 - \omega_3)^2 + (\omega_3 - \omega_1)^2 - (\omega_1 - \omega_2)^2 = 0$. ***Now,***

$$\omega_2 - \omega_3 = a + pb + p^2c, \omega_3 - \omega_1 = b + pc + p^2a, \omega_1 - \omega_2 = c + pa + p^2b$$

$$\Rightarrow (\omega_2 - \omega_3)^2 + (\omega_3 - \omega_1)^2 + (\omega_1 - \omega_2)^2$$

$$= a^2 + p^2b^2 + p^4c^2 + 2pab + 2p^2ac + 2p^3bc +$$

$$+b^2 + p^2c^2 + p^4a^2 + 2pbc + 2p^2ab + 2p^3ac +$$

$$+c^2 + p^2a^2 + p^4b^2 + 2pca + 2p^2bc + 2p^3ab$$

$$= (a^2 + b^2 + c^2)(1 + p^2 + p^4) + (2p + 2p^2 + 2p^3)(ab + bc + ca) \quad \textbf{(1)}$$

As $a+b+c=0$

$$\Rightarrow a^2+b^2+c^2=-2(bc+ca+ab)\ \textit{(2)}$$

$\therefore$ ***From (1), (2), we get:***

$$(\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2+(\omega_1-\omega_2)^2=(a^2+b^2+c^2)(1+p^2+p^4)-$$
$$-p(1+p+p^2)(a^2+b^2+c^2)=$$
$$=(a^2+b^2+c^2)(1+p^2+p^4-p-p^2-p^3)=(a^2+b^2+c^2)(1-p+p^4-p^3)$$
$$=(a^2+b^2+c^2)[1-p-p^3(1-p)]=(a^2+b^2+c^2)(1-p)(1-p)(1+p+p^2)$$
$$=(a^2+b^2+c^2)(1-p)^2(1+p+p^2)$$

Now, ΔABC ***is equilateral*** $\Leftrightarrow (\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2+(\omega_1-\omega_2)^2=0$

$$\Rightarrow (a^2+b^2+c^2)(1-p)^2(1+p+p^2)=0$$

As $p\neq 1$ ***and*** $1+p+p^2\neq 0$***, we get*** $a^2+b^2+c^2=0$

$\Rightarrow (z_2-z_3)^2+(z_3-z_1)^2+(z_1-z_2)^2=0 \Leftrightarrow \Delta MNP$ ***is equilateral.***

27. $\Omega_1=|z_1+3i|+|z_2-i|+|z_3-2i|,\ z_1,z_2,z_3\in\mathbb{C}$

$$\Omega_2=|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|$$

Prove that: $\Omega_1\le\Omega_2$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$2\Omega_2=(|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|)+$$
$$+(|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|)+$$
$$+(|z_1+z_2-z_3+4i|+|-z_1+z_2+z_3-6i|)\ge$$
$$\ge|z_1+z_2-z_3+4i+z_1-z_2+z_3+2i|+|z_1-z_2+z_3+2i-z_1+z_2+z_3-6i|+$$
$$+|z_1+z_2-z_3+4i-z_1+z_2+z_3-6i|=$$
$$=|2z_1+6i|+|2z_3-4i|+|2z_2-2i|=2\Omega_1$$
$$\Omega_1\le\Omega_2$$

Solution 2 by Remus Florin Stanca-Romania

$$|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|\ge|z_1+z_2-z_3+4i+z_1-z_2+z_3+2i|\ge|2z_1+6i|$$
$$|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|\ge|z_1-z_2+z_3+2i-z_1+z_2+z_3-6i|\ge|2z_3-4i|$$

$$|z_1+z_2-z_3+4i|+|-z_1+z_2+z_3-6i| \geq |z_1+z_2-z_3+4i-z_1+z_2+z_3-6i| \geq |2z_2-2i|$$

By adding: $\Omega_1 \leq \Omega_2$

28. $z_1, z_2, z_3 \in \mathbb{C}-\{0\}$ **different in pairs**

$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$$

$$\frac{z_1(z_2+z_3)^2}{|z_2-z_3|^2} + \frac{z_2(z_3+z_1)^2}{|z_3-z_1|^2} + \frac{z_3(z_1+z_2)^2}{|z_1-z_2|^2} = z_1z_2z_3$$

Prove that: $AB = BC = CA$.

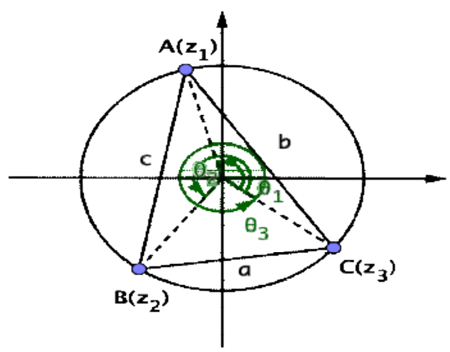
Proposed by Marian Ursărescu – Romania

Solution by Khaled Abd Imouti-Damascus-Syria

If: $|z_1| = |z_2| = |z_3| = 1$

$$A(z_1), B(z_2), C(z_3)$$

$$\frac{z_1(z_2+z_3)^2}{|z_2-z_3|^2} + \frac{z_2(z_3+z_1)^2}{|z_3-z_1|^2} + \frac{z_3(z_1+z_2)^2}{|z_1-z_2|^2} = z_1z_2z_3$$



$$\frac{1}{a^2}z_1(z_2+z_3)^2 + \frac{1}{b^2}z_2(z_3+z_1)^2 + \frac{1}{c^2}z_3(z_1+z_2)^2 = z_1z_2z_3$$

$$\frac{1}{a^2}\frac{z_1(z_2^2+z_3^2+2z_2z_3)}{z_1z_2z_3} + \frac{1}{b^2}\frac{z_2(z_3^2+z_1^2+2z_3z_1)}{z_1z_2z_3} + \frac{1}{c^2}\frac{z_3(z_1^2+z_2^2+2z_1z_2)}{z_1z_2z_3} = 1$$

$$\frac{1}{a^2}\cdot\left(\frac{z_2}{z_3}+\frac{z_3}{z_2}+2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1}+\frac{z_1}{z_3}+2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2}+\frac{z_2}{z_1}+2\right) = 1$$

$$\frac{1}{a^2}\left(\frac{z_2}{z_3}+\frac{1}{\left(\frac{z_2}{z_3}\right)}+2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1}+\frac{1}{\left(\frac{z_3}{z_1}\right)}+2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2}+\frac{1}{\left(\frac{z_1}{z_2}\right)}+2\right) = 1$$

$$\left|\frac{z_2}{z_3}\right| = 1 \Rightarrow \frac{z_2}{z_3}\cdot\overline{\left(\frac{z_2}{z_3}\right)} = 1 \Rightarrow \frac{1}{\left(\frac{z_2}{z_3}\right)} = \overline{\left(\frac{z_2}{z_3}\right)}$$

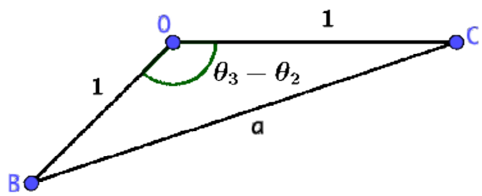
$$\frac{1}{a^2}\left(\frac{z_2}{z_3} + \overline{\left(\frac{z_2}{z_3}\right)} + 2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1} + \overline{\left(\frac{z_3}{z_1}\right)} + 2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)} + 2\right) = 1$$

$$\frac{1}{a^2}\left(2\cos\left(\arg\left(\frac{z_2}{z_3}\right)\right) + 2\right) + \frac{1}{b^2}\left(2\cos\left(\arg\left(\frac{z_3}{z_1}\right)\right) + 2\right) + \frac{1}{c^2}\left(2\cos\left(\arg\left(\frac{z_1}{z_2}\right)\right) + 2\right) = 1$$

$$\frac{2}{a^2}(\cos(\theta_2-\theta_3)+1) + \frac{2}{b^2}(\cos(\theta_3-\theta_1)+1) + \frac{2}{c^2}(\cos(\theta_1-\theta_2)+1) = 1$$

$$\frac{2}{a^2}(\cos(\theta_3-\theta_2)+1) + \frac{2}{b^2}(\cos(\theta_3-\theta_1)+1) + \frac{2}{c^2}(\cos(\theta_2-\theta_1)+1) = 1$$

$$\cos(\theta_3-\theta_2) = ?$$



$$a^2 = 1 + 1 - 2(1)(1)\cos(\theta_3-\theta_2), \qquad a^2 = 2 - 2\cos(\theta_3-\theta_2)$$

$$2\cos(\theta_3-\theta_2) = 2 - a^2, \qquad \cos(\theta_3-\theta_2) = 1 - \frac{a^2}{2}$$

$$\frac{2}{a^2}\left(2-\frac{a^2}{2}\right) + \frac{2}{b^2}\left(2-\frac{b^2}{2}\right) + \frac{3}{c^2}\left(2-\frac{c^2}{2}\right) = 1$$

$$\frac{4}{a^2} - 1 + \frac{4}{b^2} - 1 + \frac{4}{c^2} - 1 = 1, \qquad \frac{4}{a^2}+\frac{4}{b^2}+\frac{4}{c^2} = 4 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = 1$$

$$\frac{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}{3} \ge \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{3}, \qquad \left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2 \ge \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

$$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \le 1 \Rightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \le 3$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \sqrt{3}, \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \ge \sqrt[3]{\frac{1}{abc}} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3\sqrt[3]{\frac{1}{abc}}$$

$$\sqrt{3}\cdot 3\sqrt[3]{\frac{1}{abc}} \le \sqrt{3} \Rightarrow \frac{1}{\sqrt[3]{abc}} \le \frac{1}{\sqrt{3}} \Rightarrow \sqrt[3]{abc} \ge \sqrt{3}$$

$$abc \geq 3\sqrt{3} \Rightarrow 2R\sin A \cdot 2\sin B \cdot 2R\sin C \geq 3\sqrt{3}, R = 1$$

$$8\sin A\sin B\sin C \geq 3\sqrt{3}, \qquad \sin A \cdot \sin B \cdot \sin C \geq \frac{3\sqrt{3}}{8}$$

But as we know: $\sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8}$***. So:*** $\sin A \cdot \sin B \cdot \sin C = \frac{3\sqrt{3}}{8}$

Equality holds when $A = B = C = 60^\circ$

So: ΔABC ***is equilateral.***

29. $\Omega_1 = |z_1 + z_2 + z_3|, z_1, z_2, z_3 \in \mathbb{C}$

$$\Omega_2 = |z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i|$$

Prove that: $\Omega_1 \leq \Omega_2$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega_1 = |z_1 + z_2 + z_3|$$

$$\Omega_2 = |z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i|$$

$$\geq |z_1 + z_2 - z_3 + 4i + z_1 - z_2 + z_3 + 2i - z_1 + z_2 + z_3 - 6i| = |z_1 + z_2 + z_3| = \Omega_1$$

$$\therefore \Omega_2 \geq \Omega_1$$

Solution 2 by Remus Florin Stanca-Romania

$$|z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i| \geq$$

$$\geq |z_1 + z_2 - z_3 + z_1 - z_2 + z_3 - z_1 + z_2 + z_3| = |z_1 + z_2 + z_3| \text{ (Q.E.D.)}$$

Let's prove that $|a| + |b| + |c| \geq |a + b + c|; \forall a, b, c \in \mathbb{C}$***, let*** $a = x_1 + y_1 i$,

$b = x_2 + y_2 i$ ***and*** $c = x_3 + y_3 i \Leftrightarrow$

$$\Leftrightarrow \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} + \sqrt{x_3^2 + y_3^2} \geq \sqrt{(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2} \text{ (?)}$$

From Minkowski's inequality we have that

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \geq \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}| + \sqrt{x_3^2 + y_3^2} \Rightarrow$$

$$\Rightarrow \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} + \sqrt{x_3^2 + y_3^2} \geq \sqrt{(x_1 + x_2)^2 + (y_2 + y_1)^2} + \sqrt{x_3^2 + y_3^2} \geq$$

$$\geq \sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2} \quad \textit{(Q.E.D.)}$$

Solution 3 by Florentin Vișescu-Romania

$$\Omega_1 = |z_1+z_2+z_3| = |2z_1+2z_2+2z_3-z_1-z_2-z_3+4i+2i-6i|$$
$$= |z_1+z_2-z_3+4i+z_1-z_2+z_3+2i-z_1+z_2+z_3-6i|$$
$$\leq |z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i| = \Omega_2$$

30. If $z \in \mathbb{C}-\{0\}$ then:

$$|z-1|^4+\left|z+\frac{1-i\sqrt{3}}{2}\right|^4+\left|z+\frac{1+i\sqrt{3}}{2}\right|^4 \geq 3(1+2|z|^2+|z|^4)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin-Romania

Let be $\varepsilon = \frac{-1+i\sqrt{3}}{2}$ *the cubic root of the unity,* $\varepsilon^3 = 1$

$$\varepsilon^2+\varepsilon+1=0$$

$$|z-1|^4+\left|z+\frac{1-i\sqrt{3}}{2}\right|^4+\left|z+\frac{1+i\sqrt{3}}{2}\right|^4 =$$

$$= |z-1|^4+|z-\varepsilon|^4+\left|z-\varepsilon^2\right|^4 \overset{C.B.S.}{\geq} \frac{\left(|z-1|^2+|z-\varepsilon|^2+\left|z-\varepsilon^2\right|\right)^2}{3} =$$

$$= \frac{[(z-1)(\bar{z}-1)+(z-\varepsilon)(\bar{z}-\bar{\varepsilon})+(z-\varepsilon^2)(\bar{z}-\bar{\varepsilon}^2)]^2}{3} =$$

$$= \frac{(z\cdot\bar{z}-z-\bar{z}+1+z\cdot\bar{z}-z\cdot\bar{\varepsilon}-\varepsilon\cdot\bar{z}+1+z\cdot\bar{z}-z\cdot\bar{\varepsilon}^2-\bar{z}\varepsilon^2+1)^2}{3}$$

$$= \frac{[3|z|^2+3-z(1+\varepsilon+\varepsilon^2)-\bar{z}(1+\varepsilon+\varepsilon^2)]^2}{3} =$$

$$= \frac{3^2\left(|z|^2+1\right)^2}{3} = 3(|z|^4+2|z|^2+1) \quad \textit{true.}$$

Solution 2 by Remus Florin Stanca-Romania

Let $z = a+bi \Rightarrow$ *the inequality can be written as:*

$$((a-1)^2+b^2)^2+\left(\left(a+\frac{1}{2}\right)^2+\left(b-\frac{\sqrt{3}}{2}\right)^2\right)^2+\left(\left(1+\frac{1}{2}\right)^2+\left(b+\frac{\sqrt{3}}{2}\right)^2\right)^2\geq 3(a^2+b^2+1)^2\Leftrightarrow$$

$$\Leftrightarrow\sqrt{\frac{((a-1)^2+b^2)^2+\left(\left(a+\frac{1}{2}\right)^2+\left(b-\frac{\sqrt{3}}{2}\right)^2\right)^2+\left(\left(a+\frac{1}{2}\right)^2+\left(b+\frac{\sqrt{3}}{2}\right)^2\right)^2}{3}}\geq a^2+b^2+1\quad (1)$$

We know that $\sqrt{\frac{((a-1)^2+b^2)^2+\left(\left(a+\frac{1}{2}\right)^2+\left(b-\frac{\sqrt{3}}{2}\right)^2\right)^2+\left(\left(a+\frac{1}{2}\right)^2+\left(b+\frac{\sqrt{3}}{2}\right)^2\right)^2}{3}}\geq$

$$\geq\frac{a^2-2a+1+b^2+a^2+\frac{1}{4}+a+b^2-\frac{3}{4}-b\sqrt{3}+a^2+\frac{1}{4}+a+b^2+\frac{3}{4}+b\sqrt{3}}{3}=$$

$=\frac{3a^2+3b^2+3}{3}=a^2+b^2+1\Rightarrow$ ***(1) is true*** $\Rightarrow$ ***the given inequality is true*** $\Rightarrow$

$$\Rightarrow|z-1|^4+\left|z+\frac{1-i\sqrt{3}}{2}\right|^4+\left|z+\frac{1+i\sqrt{3}}{2}\right|^4\geq 3(1+2|z|^2+|z|^4)\ \textit{(Q.E.D.)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

Let $z=x+iy$ ***(*** x,y ***being reals)***

$$\therefore |z-1|=|x+iy-1|=|x-1+iy|=\sqrt{(x-1)^2+y^2}$$

$$\Rightarrow |z-1|^4\overset{(1)}{=}\{(x-1)^2+y^2\}^2$$

Also, $\left|z+\frac{1-i\sqrt{3}}{2}\right|^4=\left|x+iy+\frac{1-i\sqrt{3}}{2}\right|^4=$

$$=\left|\left(x+\frac{1}{2}\right)+i\left(y-\frac{\sqrt{3}}{2}\right)\right|^4\overset{(2)}{=}\left\{\left(x+\frac{1}{2}\right)^2+\left(y-\frac{\sqrt{3}}{2}\right)^2\right\}^2$$

Moreover, $\left|z+\frac{1+i\sqrt{3}}{2}\right|^4=\left|x+iy+\frac{1+i\sqrt{3}}{2}\right|^4=$

$$=\left|\left(x+\frac{1}{2}\right)+i\left(y+\frac{\sqrt{3}}{2}\right)\right|^4\overset{(3)}{=}\left\{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{\sqrt{3}}{2}\right)^2\right\}^2$$

(1)+(2)+(3)⇒

$$\Rightarrow LHS\geq\frac{1}{3}\left[(x-1)^2+y^2+\left(x+\frac{1}{2}\right)^2+\left(y-\frac{\sqrt{3}}{2}\right)^2+\left(x+\frac{1}{2}\right)^2+\left(y+\frac{\sqrt{3}}{2}\right)^2\right]^2$$

($\because m^2+n^2+p^2\geq\frac{1}{3}(m+n+p)^2$ ***where*** m,n,p ***are any real numbers)***

$$= \frac{1}{3}(3x^2 + 3y^2 + 3)^2 = 3(x^2 + y^2 + 1)^2 =$$

$$= 3\{1 + 2(x^2 + y^2) + (x^2 + y^2)^2\} = 3\{1 + 2|z|^2 + |z|^4\}$$

$$\left(\because |z| = \sqrt{x^2 + y^2}\right) (Proved)$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\textit{Let } a = |z - 1|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}) + 1$$

$$b = \left|z + \frac{1 + \sqrt{3}i}{2}\right|^2 = \left|z - w^2\right|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}\omega^2) + 1$$

$$c = \left|z + \frac{1 - \sqrt{3}i}{2}\right|^2 = |z - \omega|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}\omega) + 1$$

$$a + b + c = 3|z|^2 - 2\,\mathrm{Re}\left(\bar{z}(1 + \omega + \omega^2)\right) + 3 = 3|z|^2 + 3 = 3(|z|^2 + 1)$$

$$\textit{Now, } \frac{a^2+b^2+c^2}{3} \ge \left(\frac{a+b+c}{3}\right)^2 = (|z|^2 + 1)^2 \models |z - 1|^4 + |z - \omega|^4 + \left|z - \omega^2\right|^4$$

$$\ge 3(|z|^2 + 1)^2 = 3(|z|^4 + 2|z|^2 + 1)$$

31. $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, **different in pairs,** $|z_1| = |z_2| = |z_3| = 1$

$A(z_1), B(z_2), C(z_3)$. **Prove that:**

$$\frac{1}{z_1 z_2 z_3} + \sum_{cyc} \frac{z_1}{(z_2 - z_3)^2} = 0 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution by Khaled Abd Imouti-Damascus-Syria

$$\frac{1}{z_1 z_2 z_3} + \frac{z_1}{(z_2 - z_3)^2} + \frac{z_2}{(z_1 - z_3)^2} + \frac{z_3}{(z_1 - z_2)^2} = 0;\ z_1 z_2 z_3 \neq 0$$

$$1 + \frac{z_1^2 z_2 z_3}{(z_2 - z_3)^2} + \frac{z_2^2 z_1 z_3}{(z_1 - z_3)^2} + \frac{z_3^2 z_1 z_2}{(z_1 - z_2)^2} = 0$$

$$1 + \frac{z_1^2 z_2 z_3}{z_2^2 - 2z_2 z_3 + z_3^2} + \frac{z_2^2 z_1 z_3}{z_1^2 - 2z_1 z_3 + z_3^2} + \frac{z_3^2 z_1 z_2}{z_1^2 - 2z_1 z_2 + z_2^2} = 0$$

$$1+\frac{z_1^2}{\frac{z_2}{z_3}+\frac{z_3}{z_2}-2}+\frac{z_2^2}{\frac{z_1}{z_3}+\frac{z_3}{z_1}-2}+\frac{z_3^2}{\frac{z_1}{z_2}+\frac{z_2}{z_1}-2}=0$$

$$\begin{cases}|z_1|=1\\|z_2|=1\\|z_3|=1\end{cases}\Rightarrow\begin{cases}z_1\overline{z_1}=1\\z_2\overline{z_2}=1\\z_3\overline{z_3}=1\end{cases}\Rightarrow\frac{z_2}{z_3}\cdot\frac{\overline{z_2}}{\overline{z_3}}=1\Rightarrow\frac{z_2}{z_3}=\frac{\overline{z_3}}{\overline{z_2}}=\overline{\left(\frac{z_3}{z_2}\right)}$$

$$\left(\frac{z_2}{z_3}\right)\overline{\left(\frac{z_2}{z_3}\right)}=1\Leftrightarrow\left|\frac{z_2}{z_3}\right|=1$$

$$1+\frac{z_1^2}{\overline{\left(\frac{z_3}{z_2}\right)}+\frac{z_3}{z_2}-2}+\frac{z_2^2}{\overline{\left(\frac{z_3}{z_1}\right)}+\frac{z_3}{z_1}-2}+\frac{z_3^2}{\overline{\left(\frac{z_1}{z_2}\right)}+\frac{z_2}{z_1}-2}=0$$

$$\begin{cases}\overline{\left(\frac{z_3}{z_2}\right)}+\frac{z_3}{z_2}-2=2\cos\theta_1\\\overline{\left(\frac{z_3}{z_1}\right)}+\frac{z_3}{z_1}-2=2\cos\theta_2\\\overline{\left(\frac{z_1}{z_2}\right)}+\frac{z_2}{z_3}-2=2\cos\theta_3\end{cases}\Rightarrow\begin{cases}\theta_1=\arg(z_3-z_2)\\\theta_2=\arg(z_3-z_1)\\\theta_3=\arg(z_2-z_1)\end{cases}$$

$$1+\frac{z_1^2}{2(\cos\theta_1-1)}+\frac{z_2^2}{2(\cos\theta_2-1)}+\frac{z_3^2}{2(\cos\theta_3-1)}=0\ldots\text{(i)}$$

$$a^2=1+1-2\cdot1\cdot1\cdot\cos\theta_1$$

$$-a^2=2\cos\theta_1-2,\text{ analogs }-b^2=2\cos\theta_2-2;-c^2=2\cos\theta_3-2$$

Subbtituted in relation (i), we obtain:

$$1+\frac{z_1^2}{-a^2}+\frac{z_2^2}{-b^2}+\frac{z_3^2}{-c^2}=0\ldots\text{(ii)}$$

$$\frac{z_1^2}{a^2}+\frac{z_2^2}{b^2}+\frac{z_3^2}{c^2}=1$$

$$1=\left|\frac{z_1^2}{a^2}+\frac{z_2^2}{b^2}+\frac{z_3^2}{c^2}\right|\le\frac{|z_1^2|}{a^2}+\frac{|z_2^2|}{b^2}+\frac{|z_3^2|}{c^2}$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge1$$

$$a^2b^2+b^2c^2+c^2a^2\ge a^2b^2c^2$$

$$a=2R\sin A;R=1;\ a^2=4\sin^2A$$

$$\Rightarrow16(\sin^2A\sin^2B+\sin^2B\sin^2C+\sin^2C\sin^2A)\ge64\sin^2A\sin^2B\sin^2C$$

$$\sin^2A\sin^2B+\sin^2B\sin^2C+\sin^2C\sin^2A\ge4\sin^2A\sin^2B\sin^2C\ldots\text{(iii)}$$

$$A=B=C=\frac{\pi}{3}$$

The triangle ABC is equilateral.

32. $z_1, z_2, z_3 \in \mathbb{C}^*$, **are different in pairs,** $|z_1|=|z_2|=|z_3|=1$

$$A(z_1), B(z_2), C(z_3), \sum_{cyc}\frac{1}{8-\frac{z_1}{z_2}-\frac{z_2}{z_1}-\frac{z_1}{z_3}-\frac{z_3}{z_1}}=\frac{3}{10}$$

Prove that: $AB=BC=CA$.

Proposed by Marian Ursărescu-Romania

Solution by George Florin Şerban-Romania

$$z_1=\cos\theta_1+i\sin\theta_1; z_2=\cos\theta_2+i\sin\theta_2; z_3=\cos\theta_3+i\sin\theta_3$$

We have $\alpha=\frac{z_1}{z_2}+\frac{z_2}{z_1}+\frac{z_1}{z_3}+\frac{z_3}{z_1}$

$$=\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)+\cos(\theta_2-\theta_1)+i\sin(\theta_2-\theta_1)+\cos(\theta_1-\theta_3)$$
$$+i\sin(\theta_1-\theta_3)+\cos(\theta_3-\theta_1)+i\sin(\theta_3-\theta_1)$$
$$=\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)+\cos(\theta_1-\theta_2)-i\sin(\theta_1-\theta_2)+\cos(\theta_1-\theta_3)$$
$$+i\sin(\theta_1-\theta_3)+\cos(\theta_1-\theta_3)-i\sin(\theta_1-\theta_3)$$
$$=2\cos(\theta_1-\theta_2)+2\cos(\theta_1-\theta_3)$$

So: $8-\alpha=8-2\cos(\theta_1-\theta_2)-2\cos(\theta_1-\theta_3)$

$$=3+2\cos\theta_1\cos\theta_2+2\sin\theta_1\sin\theta_2+2\cos\theta_2\cos\theta_3+2\sin\theta_2\sin\theta_3+2\cos\theta_1\cos\theta_3+2\sin\theta_1\sin\theta_3$$
$$=(\cos\theta_1+\cos\theta_2+\cos\theta_3)^2+(\sin\theta_1+\sin\theta_2+\sin\theta_3)^2\geq 0$$

Let: $x=\cos(\theta_1-\theta_2); y=\cos(\theta_2-\theta_3); z=\cos(\theta_1-\theta_3)$

$$3+2x+2y+2z\geq 0\Rightarrow x+y+z\geq-\frac{3}{2}$$

$$\sum_{cyc}\frac{1}{8-\frac{z_1}{z_2}-\frac{z_2}{z_1}-\frac{z_1}{z_3}-\frac{z_3}{z_1}}=\sum_{cyc}\frac{1}{8-2x-2z}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{\sum_{cyc}8-2x-2z}$$

$$=\frac{9}{24-4\sum x}\geq\frac{3}{10}$$

Equality for: $8-2x-2y=8-2x-2z=8-2y-2z\Leftrightarrow x=y=z$

$$AB = |z_1 - z_2| = |(cos\theta_1 - cos\theta_2) + i(sin\theta_1 - sin\theta_2)|$$

$$= \sqrt{(cos\theta_1 - cos\theta_2)^2 + (sin\theta_1 - sin\theta_2)^2} = \sqrt{2 - 2cos\theta_1 cos\theta_2 - 2sin\theta_1 sin\theta_2}$$

$$= \sqrt{2 - 2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2)} = \sqrt{2 - 2cos(\theta_1 - \theta_2)} = \sqrt{2 - 2x}$$

Analogous: $BC = \sqrt{2 - 2y}$; $AC = \sqrt{2 - 2z}$

33. $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$ **different in pairs,**

$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$.

If $z_1 z_2(z_1 + z_2) + z_2 z_3(z_2 + z_3) + z_3 z_1(z_3 + z_1) + 2z_1 z_2 z_3 = 0$ **then:**

$$\prod_{cyc}(AB^2 + AC^2 - BC^2) = 0$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Florentin Vișescu-Romania

$$(z_1 + z_2)z_1 z_2 + (z_1 + z_3)(z_1 z_3) + (z_2 + z_3)z_2 z_3 + 2z_1 z_2 z_3 = 0$$

$$\frac{z_1}{z_3} + \frac{z_2}{z_3} + \frac{z_1}{z_2} + \frac{z_3}{z_2} + \frac{z_2}{z_1} + \frac{z_3}{z_1} + 2 = 0 \quad \textit{(1)}$$

Let be $z_n = \cos(\alpha_k + i \sin \alpha_k)$; $k = \overline{1,3}$

$$\frac{z_1}{z_2} + \frac{z_2}{z_1} = \cos(\alpha_1 - \alpha_2) + i \sin(\alpha_1 - \alpha_2) + \cos(\alpha_2 - \alpha_1) + i \sin(\alpha_2 - \alpha_1)$$

$$= 2\cos(\alpha_1 - \alpha_2)$$

$$\textit{(1)}\ 2\cos(\alpha_1 - \alpha_2) + 2\cos(\alpha_2 - \alpha_3) + 2\cos(\alpha_3 - \alpha_1) + 2 = 0$$

$$\cos(\alpha_1 - \alpha_2) + 2\cos(\alpha_2 - \alpha_3) + \cos(\alpha_3 - \alpha_1) + 1 = 0$$

$$2\cos\frac{\alpha_1 - \alpha_3}{2}\cos\frac{2\alpha_2 - \alpha_1 - \alpha_3}{2} + 2\cos\frac{\alpha_3 - \alpha_1}{2}\cos\frac{\alpha_3 - \alpha_1}{2} = 0$$

$$\Rightarrow 2\cos\frac{\alpha_1 - \alpha_3}{2}\left(\cos\frac{2\alpha_2 - \alpha_1 - \alpha_3}{2} + \cos\frac{\alpha_3 - \alpha_1}{2}\right) = 0$$

$$2\cos\frac{\alpha_1 - \alpha_3}{2} \cdot 2\cos\frac{\alpha_2 - \alpha_1}{2}\cos\frac{\alpha_2 - \alpha_3}{2} = 0$$

$$\cos\frac{\alpha_1 - \alpha_3}{2}\cos\frac{\alpha_2 - \alpha_1}{2}\cos\frac{\alpha_2 - \alpha_3}{2} = 0$$

$$\Rightarrow \cos\frac{\alpha_1 - \alpha_3}{2} = 0 \Rightarrow \alpha_1 - \alpha_0 = \pm\pi \Rightarrow \begin{matrix} 1)\ \alpha_1 = \alpha_3 + \pi \\ 2)\ \alpha_3 = \alpha_1 + \pi \end{matrix}$$

$\Rightarrow A$ ***and*** C ***are diametrically opposite on the circle*** $\Rightarrow B = \frac{\pi}{2}$ ***or the analogs***

$$\Rightarrow BA^2 + BC^2 - AC^2 = 0 \text{ or the analogs} \Rightarrow \prod_{cyc}(AB^2 + AC^2 - BC^2) = 0$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $z = z_1 + z_2 + z_3$

$$z_1z_2(z_1 + z_2) + z_2z_3(z_2 + z_3) + z_3z_1(z_3 + z_1) + 2z_1z_2z_3 = 0$$

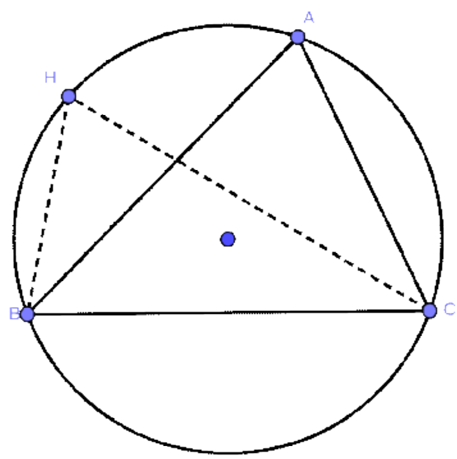
$$\Rightarrow z_1z_2(z - z_3) + z_2z_3(z - z_1) + z_3z_1(z - z_2) + 2z_1z_2z_3 = 0$$

$$\Rightarrow z(z_1z_2 + z_2z_3 + z_3z_1) = z_1z_2z_3$$

$$\Rightarrow z\left(\frac{1}{z_3} + \frac{1}{z_2} + \frac{1}{z_1}\right) = 1 \Rightarrow z(\overline{z}_1 + \overline{z}_2 + \overline{z_3}) = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$$

$\Rightarrow H(z)$ ***lie on the unit circle*** $|z| = 1$

Suppose H ***lie on the arc joining*** A ***and*** B***.***



As H ***is the orthocenter of the triangle*** ABC, $BH \perp BC$, $CH \perp BC \Rightarrow$

$\Rightarrow \angle BHC$ ***is right angle*** $\Rightarrow BC$ ***is a diameter of the circle***

$$\Rightarrow \angle BAC = 90^\circ \therefore AB^2 + AC^2 - BC^2 = 0 \Rightarrow \prod_{cyc}(AB^2 + AC^2 - BC^2) = 0$$

34. Solve for natural numbers:

$$\cos^4\left(\frac{\pi}{2n+1}\right) + \cos^4\left(\frac{2\pi}{2n+1}\right) + \cos^4\left(\frac{3\pi}{2n+1}\right) + \cdots + \cos^4\left(\frac{n\pi}{2n+1}\right) = \frac{55}{16}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution 1 by Gabriel Ruddy Cruz Mendez-Lima-Peru

Tenemos presente: $8\cos^4 x = 3 + 4\cos 2x + \cos 4x$

$$\left.\begin{aligned} 8\cos^4\left(\frac{\pi}{2n+1}\right) &= 3 + 4\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) \\ 8\cos^4\left(\frac{2\pi}{2n+1}\right) &= 3 + 4\cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{8\pi}{2n+1}\right) \\ 8\cos^4\left(\frac{3\pi}{2n+1}\right) &= 3 + 4\cos\left(\frac{6\pi}{2n+1}\right) + \cos\left(\frac{12\pi}{2n+1}\right) \\ 8\cos^4\left(\frac{n\pi}{2n+1}\right) &= 3 + 4\cos\left(\frac{2n\pi}{2n+1}\right) + \cos\left(\frac{4n\pi}{2n+1}\right) \end{aligned}\right\}$$

Sumando verticalmente:

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{\cos\left(\frac{(2n+2)\pi}{2n+1}\right)\sin\left(\frac{2n\pi}{2n+1}\right)}{\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{-2\cos\left(\frac{\pi}{2n+1}\right)\sin\left(\frac{\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{-\sin\left(\frac{2\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\therefore \frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \to n = 10$$

Solution 2 by Naren Bhandari-Bajura-Nepal

From Euler formula we have: $e^{ix} = \cos x + i\sin x$ ***and hence***

$$e^{4ix} = (\cos x + i\sin x)^4$$

$$\Rightarrow \mathcal{R}(e^{ix}) = \cos^4 x - 6\sin^2 x\cos^2 x + 3$$

$$\cos 4x = 8\cos^4 x - 4\cos 2x + 1$$

$$\cos 4x = \frac{\cos 4x + 4\cos 2x + 3}{8}$$

Setting $x = \frac{k\pi}{2n+1}$ ***and perform***

$$\sum_{k=1}^{n}\cos\left(\frac{k^{\pi}}{2n+1}\right) = \frac{1}{8}\sum_{k=1}^{n}\left(3 + \cos\left(\frac{4k\pi}{2n+1}\right) + 4\cos\left(\frac{2k\pi}{2n+1}\right)\right)$$

$$\frac{55}{2}=3n+\sum_{k=1}^{n}\left(\cos\left(\frac{2k\pi}{2n+1}\right)+4\cos\left(\frac{4k\pi}{2n+1}\right)\right)$$

$$\frac{55}{2}-3n=5\sum_{k=1}^{n}\cos\left(\frac{2k\pi}{2n+1}\right)$$

$$-\sum_{k=0}^{n-1}\cos\left(\frac{(2k+1)\pi}{2n+1}\right)=\frac{55-6n}{2}=A$$

$$A=\mathcal{R}\left(\sum_{k=0}^{n-1}e^{\frac{\pi(2k-1)i}{2n-1}}\right)=\mathcal{R}\left(\frac{e^{\frac{\pi i}{2n+1}}\left(e^{\frac{2\pi i(n-1)}{2n+1}}-1\right)}{e^{\frac{2\pi i}{2n+1}}-1}\right)$$

$$A=-\mathcal{R}\left(\frac{1+e^{\frac{\pi i}{2n+1}}}{e^{\frac{2\pi i}{2n+1}}-1}\right)=-\mathcal{R}\left(\frac{1}{1-e^{\frac{\pi i}{2n+1}}}\right)$$

$$A=-\mathcal{R}\left(\frac{\cos\frac{\pi}{2n+1}-1+i\sin\frac{\pi}{2n+1}}{\cos^2\frac{\pi}{2n+1}-2\cos\frac{\pi}{2n+1}+\sin^2\frac{\pi}{2n+1}}\right)=-\frac{1}{2}$$

and hence $-\frac{5}{2}=\frac{55-6n}{2}\Rightarrow n=10$

35. Solve for real numbers:

$$(6x^2+1)^6+2(3x^2+1)^3+3(2x^2+1)^2=6(6x^2+1)(3x^2+1)(2x^2+1)$$

Proposed by Jalil Hajimir-Canada

Solution by Bedri Hajrizi-Mitrovica-Kosovo

Let $a=6x^2+1,\ b=3x^2+1,\ c=2x^2+1$

$$\frac{a^6}{6}+\frac{b^3}{3}+\frac{c^2}{2}=abc$$

$$\frac{a^6}{6}+\frac{b^3}{6}+\frac{b^3}{6}+\frac{c^2}{6}+\frac{c^2}{6}+\frac{c^2}{6}=abc$$

AM-GM

$$abc \geq 6\sqrt[6]{\frac{a^6}{6}\cdot\frac{b^6}{6}\cdot\frac{c^6}{6}} = abc$$

So, $a = b = c$

$$6x^2 + 1 = 3x^2 + 1 = 2x^2 + 1$$

$$\underbrace{3x^2 = 0 \wedge x^2 = 0}_{\substack{\Downarrow \\ x=0}}$$

36. Solve for real numbers:

$$3^{\sin^2 x+\sin x} + 3^{\sin^2 y+\sin y} + 3^{\sin^2 z+\sin z} = 3^{\sin x+\sin y+\sin z}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow 3^{\sin^2 x+\sin x} + 3^{\sin^2 y+\sin y} + 3^{\sin^2 z+\sin z} \overset{AM-GM}{\geq} 3\left(3^{\sum\sin^2 x+\sum\sin x}\right)^{\frac{1}{3}}$$

$$\Rightarrow (3^{\sin x+\sin y+\sin z})^3 \geq 27\cdot 3^{\sum\sin^2 x+\sum\sin x}$$

$$\Rightarrow 3^{3(\sin x+\sin y+\sin z)} \geq 3^{3+\sum\sin^2 x+\sum\sin x}$$

$$\Rightarrow 3\sum\sin x \geq 3 + \sum\sin^2 x + \sum\sin x$$

$$\Rightarrow \sum\sin^2 x - 2\sum\sin x + 1 \leq 0$$

$$\Rightarrow (\sin x - 1)^2 + (\sin y - 1)^2 + (\sin z - 1)^2 \leq 0$$

But for any real $x, y, z \Rightarrow (\sin x - 1)^2 + (\sin y - 1)^2 + (\sin z - 1)^2 \geq 0$

So, this is possible if and only if

$$\Leftrightarrow (\sin x - 1)^2 + (\sin y - 1)^2 + (\sin z - 1)^2 = 0$$

$$\Rightarrow (\sin x - 1)^2 = (\sin y - 1)^2 = (\sin z - 1)^2 = 0 \Rightarrow \sin x = \sin y = \sin z = 1$$

$$\Leftrightarrow x = y = z = (4n+1)\frac{\pi}{2}\,[n \in \mathbb{Z}]$$ ***(Answer)***

Solution 2 by Florentin Vișescu-Romania

$$3^{\sin^2 x+\sin x} + 3^{\sin^2 y+\sin y} + 3^{\sin^2 z+\sin z} = 3^{\sin x+\sin y+\sin z}$$

Consider $f(t) = 3^{t^2+t}; f:\mathbb{R}\to\mathbb{R}$

$$f'(t) = 3t^2 + t\ln 3\cdot(2t+1)$$

$$f''(t)=3^{t^2+t}\ln^2 3\,(2t+1)+3^{t^2+t}\ln 3\cdot 2=$$

$$=3^{t^2+t}\ln 3\,[\ln 3\,(2t+1)+2]>0\Rightarrow f\ \textit{convexe}\Rightarrow$$

$$3^{\sin^2 x+\sin x}+3^{\sin^2 y+\sin y}+3^{\sin^2 z+\sin z}\geq 3\cdot 3^{\left(\frac{\sin x+\sin y+\sin z}{3}\right)^2+\frac{\sin x+\sin y+\sin z}{3}}$$

$$\Rightarrow 3^{\sin x+\sin y+\sin z}\geq 3^{\left(\frac{\sin x+\sin y+\sin z}{3}\right)^2+\frac{\sin x+\sin y+\sin z}{3}}$$

$$3^s\geq 3^{\frac{s^2}{9}+\frac{s}{3}+1}\Rightarrow\frac{s^2}{9}+\frac{s}{3}+1\leq s;\ s^2-6s+9\leq 0$$

$$(s-3)^2\leq 0\Rightarrow s-3=0;\ s=3\Rightarrow$$

$$\sin x+\sin y+\sin z=3\Rightarrow\sin x=\sin y=\sin z\Rightarrow x=2k\pi+\frac{\pi}{2}$$

$$y=2p\pi+\frac{\pi}{2};\ z=2q\pi+\frac{\pi}{2};\ k,p,q\in\mathbb{Z}$$

37. Solve for real numbers:

$$(x^4-3x^2+1)\sqrt{x+2}=1$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by proposer

It is clear that $x>-2$ ***and*** $x<2$. ***We can make the substitution:*** $x=2\cos t$, ***here*** $t\in(0;\pi)$. ***We have:*** $(16\cos^4 t-12\cos^2 t+1)\sqrt{2(1+\cot t)}=1$

$$16\cos^4 t-12\cos^2 t+1=16\sin^4 t-20\sin^2 t+5=\frac{\sin 5t}{\sin t}$$

We have: $\frac{\sin 5t}{\sin t}\cdot 2\cos\frac{t}{2}=1\Rightarrow\sin 5t=\sin\frac{t}{2}$ ***or*** $5t=\frac{t}{2}+2\pi k\Rightarrow$

$t=\frac{4\pi k}{9}$***; if 1)*** $k=0;\ t=0$ ***and*** $x=1.$

$x=1$ ***not root***

2) $k=1;\ x=2\cos\frac{4\pi}{9}$

3) $k=2;\ x=2\cos\frac{8\pi}{9}$

— — — — — — — — — — —

$$\sin 5t=\sin\frac{t}{2}\Rightarrow t=\frac{2\pi}{11}+\frac{4\pi k}{11}$$

$$x = 2\cos\frac{2\pi}{11}, x = 2\cos\frac{6\pi}{11}$$

$$\textit{root}\left\{2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{2\pi}{11}, 2\cos\frac{6\pi}{11}\right\}$$

38. Solve for real numbers:

$$a^{3x} + a^{2x} + b^{3x} + b^{2x} = a^x b^x (a^x + b^x + 2), a, b > 0$$

Proposed by Marin Chirciu-Romania

Solution 1 by Serban George Florin-Romania

We denote: $a^x = u > 0, b^x = v > 0 \Rightarrow u^3 + u^2 + v^3 + v^2 = uv = (u + v + 2)$

But $u^3 + v^2 \geq av(u + v), (\forall)u > v > 0$

$(u + v)(u^2 - uv + v^2) - uv(u + v) \geq 0, (u + v)(u - v)^2 \geq 0$ *(true)*

Equality for $u = v$*. And* $u^2 + v^2 \geq 2uv, (\forall)u, v > 0, (u - v)^2 \geq 0$

Equality for $u = v$*. Adding the two inequalities* $\Rightarrow (u^3 + v^3) + (u^2 + v^2) \geq$

$\geq uv(u + v) + 2uv = uv(u + v + 2) \Rightarrow$ *Equality holds for* $u = v \Rightarrow a^x = b^x$

$$\Rightarrow \left(\frac{b}{a}\right)^x = \left(\frac{b}{a}\right)^0 \Rightarrow x = 0, S = \{0\}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$a^{3x} + a^{2x} + b^{3x} + b^{2x} = a^x b^x (a^x + b^x + 2) \quad (*), a, b > 0$$

$$(a^x)^3 + (a^x)^2 + (b^x)^3 + (b^x)^2 = a^x b^x (a^x + b^x + 2)$$

Suppose: $a^x = y, b^x = z$ *then:*

$$y^3 + y^2 + z^3 + z^2 = y \cdot z(y + z + 2)$$

$$(x + z)(y^2 + z^2) + (y + z)(-yz) + y^2 + z^2 = yz \cdot (y + z) + 2yz$$

$$(y + z)(y^2 + z^2) - (yz)(y + z) + y^2 + z^2 = yz(y + z) + 2yz$$

$$(y + z)[y^2 + z^2 - 2yz] = -y^2 - z^2 + 2yz$$

$$(y + z)(y - z)^2 = -(y - z)^2$$

$$(y + z)(y - z)^2 + (y - z)^2 = 0$$

$$(y - z)^2 \cdot \underbrace{[y + z + 1]}_{>0} = 0$$

$$(y - z)^2 = 0 \Rightarrow y = z \Rightarrow a^x = b^x$$

$$x\ln(a) = x\ln(b) \Rightarrow x[\ln(a) - \ln(b)] = 0$$

$$\begin{cases} x = 0, S = \{0\} \\ \ln(a) - \ln(b) = 0 \Rightarrow a = b \end{cases}$$

In this case equation (*) is written as: $a^{3x} + a^{2x} + a^{3x} + a^{2x} = a^{2x}(2a^x + 2)$

$2a^{3x} + 2a^{2x} = 2a^{3x} + 2a^{2x}$ *(This satisfying)*

If $x \neq 0$*, equation (*) becomes identity-has infinite solutions.*

39. Solve for real numbers:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \log x & \log(ex) & \log(e^2x) & \log(e^3x) \\ \log^2 x & \log^2(ex) & \log^2(e^2x) & \log^2(e^3x) \\ \log^3 x & \log^3(ex) & \log^3(e^2x) & \log^3(e^3x) \end{vmatrix} = 7 + 2^{x-10} + \log_{12} x$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

4th grade Vandermonde Determinant

$$(\ln ex - \ln x) \cdot (\ln e^2x - \ln x)(\ln e^3x - \ln x)(\ln e^2x - \ln ex)(\ln e^3x - \ln ex) \cdot$$

$$\cdot (\ln e^3x - \ln e^2x) = 7 + 2^{x-10} + \log_{12} x$$

$$\ln e \cdot \ln e^2 \cdot \ln e^3 \cdot \ln e \cdot \ln e^2 \cdot \ln e = 7 + 2^{x-10} + \log_{12} x$$

$$12 - 7 - 2^{x-10} = \log_{12} x, x > 0;\ 5 - 2^{x-10} = \log_{12} x$$

So, $x = 12 \Rightarrow 5 - 4 = \log_{12} 12; 1 = 1 \Rightarrow x = 12$ *is a solution*

$f: (0, \infty) \to \mathbb{R}, f(x) = \log_{12} x \nearrow$ *on* $(0, \infty)$

$g: (0, \infty) \to \mathbb{R}, g(x) = 5 - 2^{x-10} \searrow$ *on* $(0, \infty)$

$\Rightarrow$ *Equation* $f(x) = g(x)$ *has an unique solution* $x = 12. S = \{12\}$

Solution 2 by Adrian Popa-Romania

$$x > 0$$

Δ *–Vandermonde's determinant* $\Rightarrow$

$$\Rightarrow \Delta = (\ln(ex) - \ln x)(\ln(e^2x) - \ln x)(\ln(e^3x) - \ln x)(\ln(e^2x) - \ln(ex)) \cdot$$

$$\cdot (\ln(e^x) - \ln(ex))(\ln(e^3x) - \ln(e^2x)) = 7 + 2^{x-10} + \log_{12} x \Rightarrow$$

$$\Rightarrow 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 7 + 2^{x-10} + \log_{12} x \Rightarrow$$

$$\Rightarrow 12 = 7 + 2^{x-10} + \log_{12} x \Rightarrow 2^{x-10} = 5 - \log_{12} x$$

$$2^{x-10} = \log_{12} 12^5 - \log_{12} x$$

$$2^{x-10} = \log_{12} \frac{12^5}{x}$$

$$\left.\begin{array}{r} f(x) = 2^{x-10} \nearrow \\ g(x) = \log_{12} \frac{12^5}{x} \searrow \end{array}\right\} \Rightarrow f(x) = g(x)$$ ***has at most one solution*** $\Rightarrow$ ***we can notice that***

$x = 12$ ***verify*** $f(x) = g(x): 2^{12-10} = 2^2 = 4$

$$\log_{12} \frac{12^5}{12} = \log_{12} 12^4 = 4$$

So, $x = 12.$

Solution 3 by Ravi Prakash-New Delhi-India

Given determinant $= [\log(ex) - \log x][\log(e^2x) - \log x][\log(e^2x) - \log(ex)]$

$$[\log(e^3x) - \log x][\log(e^x x) - \log(ex)]$$

$$[\log(e^3x) - \log(e^2x)]$$

[Vandermode Determinant]

$$= (1)(2)(1)(3)(2)(1) = 12 \therefore 12 = 7 + 2^{x-10} + \log_{12} x \Rightarrow x = 12$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

Suppose $y = \log(x)$

$$I = \begin{vmatrix} 1 & 1 & 1 & 1 \\ y & 1+y & 2+y & 3+y \\ y^2 & (1+y)^2 & (2+y)^2 & (3+y)^2 \\ y^3 & (1+y)^3 & (2+y)^3 & (3+y)^3 \end{vmatrix}$$

$$I = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1+2y & 4+4y & 9+6y \\ 0 & 3y^2+3y+1 & 6y^2+12y+8 & 9y^2+27y+27 \end{vmatrix}$$

$$I = \begin{vmatrix} 1 & 2 & 3 \\ 1+2y & 4+4y & 9+6y \\ 3y^2+3y+1 & 6y^2+12y+8 & 9y^2+27y+27 \end{vmatrix}$$

$$I = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 6y+6 & 18y+24 \end{vmatrix}$$

$$I = 36 + 48 - 36 - 36 = 12;\ 12 = 7 + 2^{x-10} + \log_{12} x$$

$$2^{x-10}+\log_{12}x=5 \quad (*)$$

$$f(x)=2^{x-10}+\log_{12}x=e^{(x-10)\ln(2)}+\frac{\log(x)}{\log(12)}$$

$$\lim_{\substack{x\to 0\\ >}}[f(x)]=-\infty,\ \lim_{x\to+\infty}[f(x)]=+\infty$$

$$f'(x)=\ln(2)\cdot e^{(x-10)\ln(2)}+\frac{1}{\log(1)}\cdot\frac{1}{x}>0$$

x	0	α	$+\infty$
$f'(x)$	+ + + + + + + + + + + + + +		
$f(x)$	$-\infty$	0 ↗	$+\infty$

So: equation (*) has a unique solution.

For $x=12:4+1=5$ ***(is true).*** $S=\{12\}$

40. Solve for real numbers:

$$\sqrt[5]{x^2-5x+4}+\sqrt[5]{2+x-x^2}=\sqrt[5]{6-4x}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

Let $\sqrt[5]{x^2-5x+4}=a,\ \sqrt[5]{2+x-x^2}=b$

$$(a+b)^5=a^5+b^5$$

$$5ab(a^3+2a^2b+2ab^2+b^3)=0$$

1) $a=0\vee$ ***2)*** $b=0\ \vee$ ***3)*** $a^3+2a^2b+2ab^2+b^3=0$

1) $x^2-5x+4=0;\ x_1=1,\ x_2=4$

2) $x^2-x-2=0;\ x_3=-1,\ x_4=2$

3) $(a+b)^3-ab(a+b)=0$

$$(a+b)\underbrace{(a^2+ab+b^2)}_{>0}>0\Rightarrow a=-b$$

$$x^2-5x+4=x^2-x-2$$

$$-4x=-6;\ x_5=\frac{3}{2}$$

So, $x\in\{-1;1;1.5;2;4\}$

Solution 2 by proposer

$$a = \sqrt[5]{x^2 - 5x + 4}, b = \sqrt[5]{2 + x - x^2}, a^5 + b^5 = 6 - 4x$$

$$\sqrt[5]{x^2 - 5x + 4} + \sqrt[5]{2 + x - x^2} = \sqrt[5]{6 - 4x}$$

$$a + b = \sqrt[5]{a^5 + b^5} \Rightarrow (a + b)^5 = a^5 + b^5 \Longrightarrow (a + b)^5 - a^5 - b^5 = 0$$

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 - a^5 - b^5 = 0$$

$$5ab(a^3 + 2a^2b + 2ab^2 + b^3) = 0$$

$$5ab(a^2(a + b) + ab(a + b) + b^2(a + b)) = 0$$

$$5ab(a + b)(a^2 + ab + b^2) = 0$$

$$a = 0 \Longrightarrow x^2 - 5x + 4 = 0 \Longrightarrow x_1 = 1, x_2 = 4$$

$$b = 0 \Rightarrow 2 + x - x^2 = 0, x_3 = -1, x_4 = 2$$

$$a + b = 0 \Rightarrow x^2 - 5x + 4 = x^2 - x - 2 \Longrightarrow x_5 = \frac{3}{2}$$

$$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^3 + \frac{3b^2}{4} \neq 0$$

41. **Solve for real numbers:**

$$3\sqrt[3]{e^{3x} - e^x} - 2\sqrt{e^{2x} - e^x} = e^x + 1$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

By Rado's inequality:

$$3\left(\frac{a + b + c}{3} - \sqrt[3]{abc}\right) \geq 2\left(\frac{a + b}{2} - \sqrt{ab}\right)$$

$$3\sqrt[3]{abc} - 2\sqrt{ab} - c \leq 0, equality\ for\ a = b = c$$

$$a = e^x, b = e^x - 1, c = e^x + 1$$

$$3\sqrt[3]{e^{3x} - e^x} - 2\sqrt{e^{2x} - e^x} - (e^x + 1) = 0 \leftrightarrow$$

$$\leftrightarrow 3\sqrt[3]{e^x(e^x - 1)(e^x + 1)} - 2\sqrt{e^x(e^x - 1)} - (e^x + 1) = 0 \leftrightarrow$$

$\leftrightarrow e^x = e^x - 1 = e^x + 1$. ***No solutions.***

42. Find $x, y, z \in \left(0, \frac{\pi}{2}\right]$ such that:

$$\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{z} = 3 + \frac{8-4\pi}{\pi^2}(x^2 + y^2 + z^2)$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Visescu-Romania

$$Let\ be\ f:\left(0, \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \sin x - x - \frac{8-4\pi}{\pi^3}x^3$$

$$f'(x) = \cos x - 1 - 3\frac{8-4\pi}{\pi^3}x^2;\ f''(x) = -\sin x - 6\frac{8-4\pi}{\pi^3}x$$

$$f'''(x) = -\cos x - 6\frac{8-4\pi}{\pi^3};\ f^{IV} = \sin x > 0, \forall x \in \left(0, \frac{\pi}{2}\right]$$

x	0 r_1 r_2 r_3 $\frac{\pi}{2}$
f^{IV}	+ + + + + + + +
f'''	- - 0 + + + +
f''	0 - - $f''(?)$ - 0 + + +
f'	0 - - - - - 0 + +
f	0 ↘ ↘ — ↗ 0

$$f'''\left(\frac{\pi}{2}\right) = 6\frac{4\pi-8}{\pi^3};\ \lim_{\substack{x\to 0\\ x>0}} f'''(x) = -1 + 6\frac{4\pi-8}{\pi^3} = \frac{24\pi - 48 - \pi^3}{\pi^3} < 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f''(x) = 0;\quad f''\left(\frac{\pi}{2}\right) = -1 - 6^3\frac{8-4\pi}{\pi^3}\cdot\frac{\pi}{2} = \frac{-\pi^2 - 24 + 12\pi}{\pi^2} > 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f'(x) = 0;\quad f'\left(\frac{\pi}{2}\right) = -3\frac{8-4\pi}{\pi^3}\cdot\frac{\pi^2}{4} = -3\frac{2-\pi}{\pi} = \frac{-6+3\pi}{\pi} > 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = 0;\quad f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} - \frac{8-4\pi}{\pi^3}\cdot\frac{\pi^3}{8} = 1 - \frac{\pi}{2} - \frac{2-\pi}{2} = \frac{2-\pi+\pi-2}{2} = 0$$

$$So\ f(x) = 0. \forall x \in \left(0, \frac{\pi}{2}\right]\ equality\ just\ for\ x = \frac{\pi}{2} \Rightarrow \frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{3} \le 3 + \frac{8-4\pi}{\pi^3}(x^2 + y^2 + z^2)$$

$$\forall\ x, y, z \in \left(0, \frac{\pi}{2}\right]\ equality\ just\ for\ x = y = z = \frac{\pi}{2}$$

43. **Solve for real numbers:**

$$32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Orlando Irahola Ortega-La Paz-Bolivia

$$32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0 \Leftrightarrow$$

$$(2x)^6 - 6(2x)^4 + 36(2x)^2 - 4 - 2\sqrt{3} = 0$$

$$Let\ 2x = \sqrt{t+2} \Rightarrow x = \frac{1}{2}\sqrt{t+2} \Rightarrow t^3 + \underbrace{6t}_{p} + \underbrace{(16 - 2\sqrt{3})}_{q} = 0$$

$$\textit{How } \Delta = \frac{q^2}{4} + \frac{p^3}{27} = 75 - 16\sqrt{3} > 0 \Rightarrow \exists t_1 \in \mathbb{R}, t_2, t_3 \in \mathbb{C}$$

Applying Cardano Theorem:

$$t = \sqrt[3]{\sqrt{3} - 8 + \sqrt{75 - 16\sqrt{3}}} + \sqrt[3]{\sqrt{3} - 8 - \sqrt{75 - 16\sqrt{3}}}$$

$$\textit{How } t = \pm\frac{1}{2}\sqrt{t+2} \Rightarrow x = \pm\sqrt[3]{\sqrt[3]{\sqrt{3} - 8 + \sqrt{75 - 16\sqrt{3}}} + \sqrt[3]{\sqrt{3} - 8 - \sqrt{75 - 16\sqrt{3}}} + 2}$$

Solution 2 by Muhindo Visangi Martin-Congo

$$32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0 \overset{x^2=t}{\Leftrightarrow} 32t^3 - 48t^2 + 36t - 2 - \sqrt{3} = 0$$

Using Cardan method

$$\textit{Let } t = y + b \Rightarrow 32(y+b)^3 - 48(y+b)^2 + 36(y+b) - 2 - \sqrt{3} = 0$$

$$32y^3 + 96y^2b + 96yb^2 + 32b^3 - 48y^2 - 96yb - 48b^2 + 36y + 36b - 2 - \sqrt{3} = 0$$

$$32y^3 + (96b - 48)y^2 + (96b^2 - 96b + 36)y + 32b^3 - 48b^2 + 36b - 2 - \sqrt{3} = 0$$

$$\textit{Let } 96b - 48 = 0 \Rightarrow b = \frac{1}{2}$$

$$32y^3 + \left(\frac{96}{4} - \frac{96}{2} + 36\right)y + 32\cdot\frac{1}{8} - 48\cdot\frac{1}{4} + 36\cdot\frac{1}{2} - 2 - \sqrt{3} = 0$$

$$32y^3 + 12y + 12 - \sqrt{2} = 0$$

$$\textit{Let } y = u + v$$

$$32(u+v)^3 + 12(u+v) + 12 - \sqrt{3} = 0$$

$$32u^3+96u^2v+96uv^2+32v^3+12u+12v+12-\sqrt{3}=0$$

$$32(u^3+v^3)+96uv(u+v)+12(u+v)+12-\sqrt{3}=0$$

$$32(u^3+v^3)+(u+v)(96uv+12)+12-\sqrt{3}=0$$

Let $\begin{cases}32(u^3+v^3)+12-\sqrt{3}=0\\(u+v)(96uv+12)=0\end{cases}$

$$\begin{cases}u^3+v^3=\dfrac{-12+\sqrt{3}}{32}\\u\cdot v=-\dfrac{12}{96}\end{cases};\ \begin{cases}u^3+v^3=\dfrac{-12+\sqrt{3}}{32}\\u^3\cdot v^3=-\dfrac{1}{512}\end{cases}$$

We see that u^3 and v^3 are solutions of second degree equation:

$$x^2-\frac{-12+\sqrt{3}}{32}x-\frac{1}{512}=0;\ \Delta=\frac{155-24\sqrt{3}}{1024}$$

$$x_1=u^3=\frac{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}{64}$$

$$x_2=v^3=\frac{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}{64}$$

We know that:

$$y=u+v=\sqrt[3]{\frac{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}{64}}+\sqrt[3]{\frac{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}{64}}$$

$$=\frac{1}{4}\left(\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)$$

If ω is a complexe number, the cubics racine are given by

$$\sqrt[3]{|\omega|},j\sqrt[3]{|\omega|},j^2\sqrt[3]{|\omega|},\ with\ j=-\frac{1}{2}+i\frac{\sqrt{3}}{2};\ (j^3=1,j^2+j+1=0)$$

We know that $t=y+b;(2b=1)$

$$t_1=\frac{1}{2}+\frac{1}{4}\left(\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)$$

$$t_2=\frac{1}{2}+\frac{1}{4}\left(j^2\cdot\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+j\cdot\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)$$

$$t_3=\frac{1}{2}+\frac{1}{4}\left(j\cdot\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+j^2\cdot\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)$$

We know that $x^2=t\Rightarrow x=\pm t$

$$x_{1,2}=\sqrt{\frac{1}{2}+\frac{1}{4}\left(\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)}$$

$$x_{3,4}=\sqrt{\frac{1}{2}+\frac{1}{4}\left(j^2\cdot\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+j\cdot\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)}$$

$$x_{5,6}=\sqrt{\frac{1}{2}+\frac{1}{4}\left(j\cdot\sqrt[3]{-12+\sqrt{3}+\sqrt{155-24\sqrt{3}}}+j^2\cdot\sqrt[3]{-12+\sqrt{3}-\sqrt{155-24\sqrt{3}}}\right)}$$

44. Solve for real numbers:

$$\begin{vmatrix}1 & 1 & 1 & 1\\ e^x & e^{2x} & e^{3x} & 2\\ e^{3x} & e^{6x} & e^{9x} & 8\\ e^{4x} & e^{8x} & e^{12x} & 16\end{vmatrix}=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$Let: a=e^x;\ b=e^{2x};\ c=e^{3x}\ and\ \Delta=\begin{vmatrix}1 & 1 & 1 & 1\\ a & b & c & 2\\ a^3 & b^3 & c^3 & 8\\ a^4 & b^4 & c^4 & 16\end{vmatrix}$$

$$Using\ c_1\to c_1-c_4;\ c_2\to c_2-c_4;\ c_3\to c_3-c_4$$

$$We\ get: \Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-2 & b-2 & c-2 & 2 \\ a^3-8 & b^3-8 & c^3-8 & 8 \\ a^4-16 & b^4-16 & c^4-16 & 16 \end{vmatrix}$$

$= (a-2)(b-2)(c-2)\Delta_1$ **, where**

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2+2a+4 & b^2+2b+4 & c^2+2c+4 \\ a^3+2a^2+4a+8 & b^3+2b^2+4b+8 & c^3+2c^2+4c+8 \end{vmatrix}$$

$$Using\ R_3 \to R_3 - 2R_2;\ R_2 \to R_2 - 4R_1$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2+2a & b^2+2b & c^2+2c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$Using\ C_1 \to C_1 - 2C_2;\ C_2 \to C_2 - C_3$$

We get: $\Delta_1 = (a-b)(b-c)\Delta_2$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & 1 \\ a+b+2 & b+c+2 & c^2+2c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Using $C_1 \to C_1 - C_2$ **,we get:** $\Delta_2 = \begin{vmatrix} a-c & b+c+2 \\ (a-c)(a+b+c) & b^2+bc+c^2 \end{vmatrix}$

$= -(a-c)(ab+bc+ca+2a+2b+2c)$. **Thus,**

$$\Delta = (a-2)(b-2)(c-2)(a-b)(b-c)(c-a)(ab+bc+ca+2a+2b+2c)$$

$$a = 2\ or\ b = 2\ or\ c = 2\ or\ a = b\ or\ c = a$$

$$x \in \left\{log2, \frac{1}{2}log2, \frac{1}{3}log3, 1\right\}$$

Solution 2 by Costel Florea-Romania

$$Let: a = e^x; b = e^{2x}; c = e^{3x}, d = 2\ and\ \Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

$$Using\ c_1 \to c_1 - c_4;\ c_2 \to c_2 - c_4;\ c_3 \to c_3 - c_4$$

$$We\ get: \Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-d & b-d & c-d & d \\ a^3-d^3 & b^3-d^3 & c^3-d^3 & d^3 \\ a^4-d^4 & b^4-d^4 & c^4-d^4 & d^4 \end{vmatrix}$$

Let: $A_1 = a^2+ab+b^2; A_2 = b^2+bc+c^2;\ A_3 = d^2+ad+a^2$

$$B_1 = a^3+a^2b+ab^2+b^3;\ B_2 = c^3+c^2a+ca^2+a^3;\ B_3 = a^3+a^2d+ad^2+d^3$$

$$\Delta = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 1 & 1 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 \\ A_1 & A_2-A_1 & A_3-A_1 \\ B_1 & B_2-B_1 & B_3-B_1 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} A_2-A_1 & A_3-A_1 \\ B_2-B_1 & B_3-B_1 \end{vmatrix}$$

$$= \underbrace{(b-a)(c-a)(d-a)(c-b)(d-b)}_{A}\begin{vmatrix} C_1 & C_2-C_1 \\ D_1 & D_2-D_1 \end{vmatrix} =$$

When $C_1 = a+b+c;\ C_2 = a+b+d;$

$$D_1 = a^2+b^2+c^2+ab+bc+ca;\ D_2 = a^2+b^2+d^2+ab+bd+da$$

$$= A(d-c)\begin{vmatrix} C_1 & 1 \\ D_1 & a+b+c+d \end{vmatrix}$$

$$= A(d-c)[(a+b+c)(a+b+c+d)-(a^2+b^2+c^2+ab+bc+ca)]$$

$$= (a-2)(b-2)(c-2)(a-b)(b-c)(c-a)(ab+bc+ca+2a+2b+2c)$$

$$a = 2 \text{ or } b = 2 \text{ or } c = 2 \text{ or } a = b \text{ or } c = a$$

$$x \in \left\{log2, \frac{1}{2}log2, \frac{1}{3}log3, 1\right\}$$

45. Solve for real numbers:

$$sin2x = (\sqrt{2}-1)(sinx+cosx+1)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Carlos Suarez-Quito-Ecuador

$$sin2x = (\sqrt{2}-1)(sinx+cosx+1)$$

$$(2sinxcosx)^2 = (sinx+cosx+1)^2$$

$$4sin^2xcos^2x = (2-2\sqrt{2}+1)(sin^2x+cos^2x+1+2sinxcosx+2sinx+2cosx))$$

$$4(1-cos^2x)cos^2x = (3-2\sqrt{2})(2+2sinxcosx+2sinx+2cosx)$$

$$4(cos^2x-cos^4x) = 2(3-2\sqrt{2})(1+sinxcosx+sinx+cosx)$$

$$2(cos^2x-cos^4x) = (3-2\sqrt{2})(1+sinxcosx+sinx+cosx)$$

$$\frac{2(\cos^2x-\cos^4x)}{3-2\sqrt{2}}=\left(\frac{2\sin x\cos x}{\sqrt{2}-1}+\sin x\cos x\right)$$

$$\frac{2(\cos^2x-\cos^4x)}{3-2\sqrt{2}}=\sin x\cos x\left(\frac{2}{\sqrt{2}-1}+1\right)$$

$$\frac{2(\cos^2x-\cos^4x)}{\sin x\cos x}=\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{3-2\sqrt{2}}{2}\right)$$

$$\frac{\cos x-\cos^3x}{\sin x}=\frac{(3+2\sqrt{2})(3-2\sqrt{2})}{2}$$

$$\frac{\cos x-\cos^3x}{\sin x}=\frac{1}{2}$$

$$[2(\cos x-\cos^3x)]^2=\sin^2x$$

$$4\cos^6x-8\cos^4x+5\cos^2x-1=0\xRightarrow{v=\cos^2x}$$

$$4v^3-8v^2+5v-1=0\Leftrightarrow(v-1)(2v-1)^2=0$$

$$v_1=1,v_2=\frac{1}{2}\Rightarrow x_1=\pi n,x_2=\frac{\pi}{4}(2n-1)$$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$\sin 2x=(\sqrt{2}-1)(\sin x+\cos x+1)$$

$$2\sin x\cos x=(\sqrt{2}-1)\left(2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}\right)$$

$$2\sin\frac{x}{2}\cos\frac{x}{2}\cos x=(\sqrt{2}-1)\cos\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)$$

$$\Rightarrow x_1=\pi+2k\pi,k\in\mathbb{Z}$$

$$2\sin\frac{x}{2}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)=(\sqrt{2}-1)\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)$$

$$2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)=(\sqrt{2}-1)\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)$$

$$\sin\frac{x}{2}+\cos\frac{x}{2}=0\Rightarrow x_2=\frac{\pi}{2}+4m\pi,m\in\mathbb{Z}$$

$$2\sin\frac{x}{2}\cos\frac{x}{2}-2\sin^2\frac{x}{2}=\sqrt{2}-1$$

$$\sin x+\cos x=\sqrt{2}$$

$$\cos\left(x-\frac{\pi}{4}\right)=1\Rightarrow x_3=\frac{\pi}{4}+2n\pi,n\in\mathbb{Z}$$

Solution 3 by Samuel Vargas-Peru

$$\sin 2x = (\sqrt{2}-1)(\sin x+\cos x+1)$$

$$(\sin x+\cos x+1)(\sin x+\cos x-1) = (\sqrt{2}-1)(\sin x+\cos x+1)$$

$$i)\ \sin x+\cos x+1 = 0$$

$$\sin x+\cos x = -1;\ \sin\left(x+\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2};\ x_1 = n\pi+(-1)^n\left(-\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in\mathbb{Z}$$

$$ii)\ \sin x+\cos x-1 = \sqrt{2}-1$$

$$\sin x+\cos x = \sqrt{2},\ \sin\left(x+\frac{\pi}{4}\right) = 1;\ x_2 = 2n\pi+\frac{\pi}{4}, n\in\mathbb{Z}$$

Solution 4 by Cao Mai Thanh Tam-Vietnam

$$\sin 2x = (\sqrt{2}-1)(\sin x+\cos x+1);\ 2\sin x\cos x = (\sqrt{2}-1)(\sin x+\cos x+1)$$

$$Let: t = \tan\frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}$$

$$-2(t+1)\left[(\sqrt{2}+1)t^2-2t+\sqrt{2}-1\right] = 0$$

$$\begin{cases} t = -1 \\ t = \sqrt{2}-1 \end{cases} \Rightarrow \begin{cases} \tan\frac{x}{2} = -1 \\ \tan\frac{x}{2} = \sqrt{2}-1 \end{cases} \Rightarrow \begin{cases} x = -\frac{\pi}{2}+2k\pi \\ x = \frac{\pi}{4}+2k\pi \end{cases}$$

Solution 5 by Chuluu Batzaya-Mongolia

$$\sin 2x = (\sqrt{2}-1)(\sin x+\cos x+1)$$

$$\sin x+\cos x = t \Rightarrow \sin 2x = t^2-1$$

$$t^2-1 = (\sqrt{2}-1)(t+1)$$

$$t^2-(\sqrt{2}-1)t-\sqrt{2} = 0 \Rightarrow t_1 = \sqrt{2};\ t_2 = -1$$

$$i)\ \sin x+\cos x = \sqrt{2}$$

$$\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x = 1$$

$$\sin\left(x+\frac{\pi}{4}\right) = 1 \Rightarrow x = \frac{\pi}{4}+2k\pi, k\in\mathbb{Z}$$

$$ii)\ \sin x+\cos x = -1$$

$$\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$$

$$sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}\Rightarrow x=(-1)^{n+1}\frac{\pi}{4}-\frac{\pi}{4},n\in\mathbb{Z}$$

$$x\in\left\{x=\frac{\pi}{4}+2k\pi,(-1)^{n+1}\frac{\pi}{4}-\frac{\frac{\pi}{4}}{n},k,n\in\mathbb{Z}\right\}$$

Solution 6 by Gabriel Ruddy Mendez-Lima-Peru

$$sin2x=(\sqrt{2}-1)(sinx+cosx+1)$$

$$4sin\frac{x}{2}cos\frac{x}{2}cosx=(\sqrt{2}-1)\left(2sin\frac{x}{2}cos\frac{x}{2}+2cos^2\frac{x}{2}\right)$$

$$i)\ cos\frac{x}{2}=0\Rightarrow x=2n\pi+\pi$$

$$2sin\frac{x}{2}\left(sin\frac{x}{2}+cos\frac{x}{2}\right)\left(sin\frac{x}{2}-cos\frac{x}{2}\right)=(\sqrt{2}-1)\left(sin\frac{x}{2}+cos\frac{x}{2}\right)$$

$$ii)\ sin\frac{x}{2}+cos\frac{x}{2}=0$$

$$cos\frac{x}{2}=-sin\frac{x}{2}$$

$$tan\frac{x}{2}=-1\Rightarrow x=2n\pi-\frac{\pi}{4}$$

$$2sin\frac{x}{2}\left(cos\frac{x}{2}-sin\frac{x}{2}\right)=\sqrt{2}-1$$

$$sinx-(1-cosx)=\sqrt{2}-1$$

$$sin\left(x+\frac{\pi}{4}\right)=1\Rightarrow x=2n\pi+\frac{\pi}{4}$$

Solution 7 by Kamel Benaicha-Algiers-Algerie

$$sin2x=(\sqrt{2}-1)(sinx+cosx+1)\ldots(E)$$

$$(E)\Leftrightarrow cos\left(2\left(x-\frac{\pi}{4}\right)\right)=(2-\sqrt{2})cos\left(x-\frac{\pi}{4}\right)+\sqrt{2}-1$$

$$2cos^2\left(x-\frac{\pi}{4}\right)=(2-\sqrt{2})cos\left(x-\frac{\pi}{4}\right)+\sqrt{2}$$

$$Let: t=cos\left(x-\frac{\pi}{4}\right), then:$$

$$(E)\Leftrightarrow\begin{cases}2t^2-(2-\sqrt{2})t-\sqrt{2}=0\\ t=cos\left(x-\frac{\pi}{4}\right)\end{cases}$$

$$\begin{cases} 2t^2 - \left(1 - \frac{1}{\sqrt{2}}\right)t - \frac{1}{\sqrt{2}} = 0 \\ t = cos\left(x - \frac{\pi}{4}\right) \end{cases}$$

$$E' \Leftrightarrow t = -\frac{1}{\sqrt{2}} \ and \ t = 1$$

$$\therefore \ cos\left(x - \frac{\pi}{4}\right) = 1 \ and \ cos\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$x \in \left\{2k\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{4} \pm \frac{3\pi}{4}\right\}$$

$$S(E) = \left\{2k\pi + \frac{\pi}{4}; (2k+1)\pi; 2k\pi - \frac{\pi}{2} / k \in \mathbb{Z}\right\}$$

Solution 8 Ravi Prakash-New Delhi-India

$$sin2x = (\sqrt{2} - 1)(sinx + cosx + 1) \dots (1)$$

$$Put \ sinx + cosx = t$$

$$sinx = t^2 - 1$$

Now, (1) becomes: $t^2 - 1 = (\sqrt{2} - 1)(t + 1)$

$$t + 1 = 0 \ or \ t - 1 = \sqrt{2} - 1$$

$$t = -1 \ or \ t = \sqrt{2}$$

$$sinx + cosx = -1 \ or \ sinx + cosx = \sqrt{2}$$

$$cos\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \ or \ cos\left(x - \frac{\pi}{4}\right) = 1$$

$$x - \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4} \ or \ x - \frac{\pi}{4} = 2m\pi; \ m, n \in \mathbb{Z}$$

$$x \in \left\{2n\pi + \frac{\pi}{4}; (2k+1)\pi; 2m\pi - \frac{\pi}{2} / n, k, m \in \mathbb{Z}\right\}$$

46. Solve for real numbers:

$$(a-1)x + 2 = a + a^{\frac{x^2-1}{3}}, a > 1$$

Proposed by Marin Chirciu-Romania

Solution by Khaled Abd Imouti-Damascus-Syria

$$(a-1)x+2=a+a^{\frac{x^2-1}{3}}, a>1\ldots(*)$$

$$(a-1)x+2-a-a^{\frac{x^2-1}{3}}=0, a>1$$

$$a+a^{\frac{x^2-1}{3}}-(a-1)x-2=0$$

Let be the function: $f(x)=a+e^{\frac{1}{3}log(a)\cdot(x^2-1)}-(a-1)x-2$

$$\lim_{x\to-\infty}f(x)=+\infty$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left[x\left(\frac{a}{x}+\frac{e^{\frac{1}{3}log(a)\cdot(x^2-1)}}{x}-(a-1)-\frac{2}{x}\right)\right]=+\infty$$

$$f'(x)=\frac{1}{3}log(a)\cdot 2x\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}-(a-1)$$

$$f''(x)=\frac{2}{3}log(a)\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}+\frac{1}{3}log(a)\cdot 2x\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}\cdot\frac{1}{3}log(a)\cdot 2x$$

$$f''(x)=\left(\frac{2}{3}log(a)+\frac{4}{9}log^2(a)\cdot x^2\right)e^{\frac{1}{3}log(a)\cdot(x^2-1)}>0$$

x	$-\infty$ 1 α 2 ∞
f''(x)	++++++++++++++++++++++++++++++
f'(x)	$-\infty$ ↗ ↗ ↗ ↗ ∞

Us prove $1\overset{?}{<}\alpha\overset{?}{<}2$

$f'(1)=\frac{2}{3}log(a)-(a-1)\overset{?}{<}0$, *let us prove that:*

Let be the function $g(x)=\frac{2}{3}log(x)-(x-1)$

$$g'(x)=\frac{2}{3}\cdot\frac{1}{x}-1=\frac{2-3x}{3x}$$

$$g'(x)=0\Leftrightarrow x=\frac{2}{3}$$

$$g\left(\frac{2}{3}\right)=\frac{2}{3}log\left(\frac{2}{3}\right)+\frac{1}{3}<0$$

x	0 $\qquad\qquad \frac{2}{3} \qquad\qquad +\infty$
$g'(x)$	$+ + + + + 0 - - - - -$
$g(x)$	$-\infty \nearrow\nearrow\nearrow \frac{2}{3}log\left(\frac{2}{3}\right)+\frac{1}{3} \searrow\searrow\searrow -\infty$

$\forall x \geq 1, g(x) \leq 0, so: g(1) = 1$

Let us prove: $f'(2) = \frac{4}{3}alog(a) - (a-1) \overset{?}{>} 0$

Let be the function: $h(x) = \frac{4}{3}xlog(x) - x + 1$

$$h'(x) = \frac{1}{3}(4log(x) + 1);\ \ h'(x) = 0 \Leftrightarrow x = \frac{1}{\sqrt[4]{e}}; h\left(\frac{1}{\sqrt[4]{e}}\right) = \frac{2}{3\sqrt[4]{e}} + 1$$

x	0 $\qquad\qquad \frac{1}{\sqrt[4]{e}} \qquad\qquad +\infty$
$h'(x)$	$- - - - - 0 + + + + +$
$h(x)$	$\searrow\searrow\searrow \quad \frac{2}{3\sqrt[4]{e}} + 1 \quad \nearrow\nearrow\nearrow$

So, $h(x) > 0, \forall x > 0$

$$\frac{4}{3}alog(a) - (a-1) > 0, \forall x > 1 \text{ and hence } 1 < \alpha < 2$$

x	$-\infty$	1	α	2	$+\infty$
$f'(x)$	$- - - - - -$		0	$+ + + + +$	
$f(x)$	$+\infty \searrow\searrow$	$0 \searrow\searrow$	$f(\alpha) \nearrow\nearrow$	$0 \nearrow\nearrow$	$+\infty$

Equation (*) has two solution: $x = 1, x = 2$.

47. Solve for real numbers:

$$5^{2x+1} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

Proposed by Daniel Sitaru-Romania

Solution by Bedri Hajrizi-Mitrovica-Kosovo

$$5^{2x+1} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

$$5\cdot 5^{2x}+20x^2+29x+6=11\cdot 5^x+x\cdot 5^{x+2}$$

$$5\cdot 5^{-2x}-(25x+11)\cdot 5^x+20x^2+29x+6=0$$

$$5^x=\frac{25x+11\pm\sqrt{(25x+11)^2-20(20x^2+29x+6)}}{10}$$

$$=\frac{25x+11\pm\sqrt{625x^2+550x+121-400x^2-580x-120}}{10}$$

$$=\frac{25x+11\pm(15x-1)}{10}$$

$$5^x=4x+1\ or\ 5^x=x+2$$

$$i)\ 5^x=4x+1\Leftrightarrow x\in\{0,1\}$$

$$ii)\ 5^x=x+2\Rightarrow x=-1\ one\ solution$$

Let $f(x)=)\ 5^x-x-2$

$$f(0)=1-1\cdot 2<0\ and\ f(1)=5-1-1\cdot 2>0\Rightarrow\exists\alpha\in(0,1)\ such\ that\ f(\alpha)=0$$

So: $x\in\{-1,0,\alpha,1\}$

48. Solve for real numbers:

$$\begin{vmatrix}1 & 1 & 1 & 1\\ \sqrt{x} & \sqrt[3]{x} & \sqrt[4]{x} & 2\\ x & \sqrt[3]{x^2} & \sqrt{x} & 4\\ x^2 & x\sqrt[3]{x} & x & 16\end{vmatrix}=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$Let\ \sqrt{x}=a,\sqrt[3]{x}=b,\sqrt[4]{x}=c$$

$$\Delta=\begin{vmatrix}1 & 1 & 1 & 1\\ a & b & c & 2\\ a^2 & b^2 & c^2 & 4\\ a^4 & b^4 & c^4 & 16\end{vmatrix}$$

Using $c_1\to c_1-c_4, c_2\to c_2-c_4, c_3\to c_4$

$$\Delta=\begin{vmatrix}0 & 0 & 0 & 1\\ a-2 & b-2 & c-2 & 2\\ a^2-4 & b^2-4 & c^2-4 & 4\\ a^4-16 & b^4-16 & c^4-16 & 16\end{vmatrix}=-(a-2(b-2)(c-2)\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a+2 & b+2 & c+2 \\ a^3+2a^2+4a+8 & b^3+2b^2+4b+8 & c^3+2c^2+4c+8 \end{vmatrix}$$

Using $c_1 \to c_1 - c_3, c_2 \to c_2 - c_3, c_2 \to c_3$ **we get**

$$\Delta_1 = (a-c)(b-c)\Delta_2$$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c+2 \\ a^2+c^2+ac+2(a+c)+4 & b^2+c^2+bc+2(b+c)+4 & c^3+2c^2+4c+8 \end{vmatrix}$$

$$\Delta_1 = (a-c)(b-c)[b^2+c^2+bc+2(b+c)+4-a^2-c^2-ac-2(a+c)-4]$$

$$= (a-c)(b-c)(b-a)(a+b+c+2)$$

$$\Delta = -(a-2(b-2)(c-2)(a-c)(b-c)(b-a)(a+b+c+2)$$

$$\Delta = 0 \Rightarrow a = 2 \text{ or } b = 2 \text{ or } c = 2 \text{ or } a = b \text{ or } b = c \text{ or } c = a$$

$$\Rightarrow x = 4 \text{ or } x = 8 \text{ or } x = 0 \text{ or } x = 1$$

$$So, x \in \{0,1,4,8\}$$

Solution 2 by Costel Florea-Romania

$$Let: \sqrt{x} = a, \sqrt[3]{x} = b, \sqrt[4]{x} = c, d = 2 \text{ and } \Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

$$Using\ c_1 \to c_1 - c_4;\ c_2 \to c_2 - c_4;\ c_3 \to c_3 - c_4$$

$$We\ get: \Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-d & b-d & c-d & d \\ a^2-d^2 & b^2-d^2 & c^2-d^2 & d^2 \\ a^4-d^4 & b^4-d^4 & c^4-d^4 & d^4 \end{vmatrix}$$

Let: $A_1 = a+b;\ A_2 = a+c;\ A_3 = a+d$

$$B_1 = a^3+a^2b+ab^2+b^3;\ B_2 = c^3+c^2a+ca^2+a^3;\ B_3 = a^3+a^2d+ad^2+d^3$$

$$\Delta = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 1 & 1 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 \\ A_1 & A_2-A_1 & A_3-A_1 \\ B_1 & B_2-B_1 & B_3-B_1 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} A_2-A_1 & A_3-A_1 \\ B_2-B_1 & B_3-B_1 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\Delta_1$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 \\ a^2+b^2+c^2+ab+bc+ca & a^2+b^2+d^2+ab+bd+da \end{vmatrix}$$

$$\Delta = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)(a+b+c+d)$$

$$\Delta = 0 \Rightarrow a = 2 \text{ or } b = 2 \text{ or } c = 2 \text{ or } a = b \text{ or } b = c \text{ or } c = a$$

$$\Rightarrow x = 4 \text{ or } x = 8 \text{ or } x = 0 \text{ or } x = 1$$

$$So, x \in \{0, 1, 4, 8\}$$

49. Solve for complex numbers:

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

Proposed by Daniel Sitaru-Romania

Solution by Lety Sauceda-Mexico City-Mexico

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

$$x^6 - 3x^5 + 6x^4 - 7x^3 + 5x^2 - 2x + \frac{1}{3} = 0$$

For: $(x^2 - x + a)(x^2 - x + b)(x^2 - x + c) = 0$ ***we have:***

$$x^6 - 3x^5 + (a+b+c+3)x^4 - (2a+2b+2c+1)x^3 -$$

$$-(a+b+c+ab+bc+ca)x^2 + (ab+bc+ca)x + abc = 0$$

$$\Rightarrow \begin{cases} 3abc = 1 \\ a+b+c = 3 \\ ab+bc+ca = 2 \end{cases}$$

$$3a^3 - 9a^2 + 6a - 1 = 0 \Leftrightarrow 3(a-1)^3 - 3(a-1) - 1 = 0$$

$$\overset{a-1=w}{\Longrightarrow} 3w^3 - 3w - 1 = 0 \Leftrightarrow w^3 - w + \frac{1}{3} = 0 \overset{w=s+r}{\Longrightarrow}$$

$$\begin{cases} s^3 + r^3 + (3sr+1)w + \frac{1}{3} = 0 \\ r^3 = \frac{1}{27s^3} \\ s^3 + r^3 - \frac{1}{3} = 0 \end{cases}$$

$$27s^6 - 9s^3 + 1 = 0 \Rightarrow s^3 = \frac{3 + \sqrt{-3}}{18} \overset{w=s+r}{\Longrightarrow}$$

$$\begin{cases} w_1 = \sqrt[3]{\frac{3+\sqrt{-3}}{18}} + + \sqrt[3]{\frac{3-\sqrt{-3}}{18}} \\ w_2 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} \\ w_3 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} \end{cases}$$

$$\begin{cases} a_1 = w_1 + 1 \\ a_2 = w_2 + 1 \\ a_3 = w_3 + 1 \end{cases}$$

Let: $a_1 = a;\ a_2 = b;\ a_3 = c \Rightarrow$

$$(x^2 - x + w_1 + 1)(x^2 - x + w_2 + 1)(x^2 - x + w_3 + 1) = 0$$

$$\begin{cases} a = 2.1371580426\ldots \\ b = 0.25777280103\ldots \\ c = 0.60506915636\ldots \end{cases}$$

$$(x^2 - x + 2.1371580426\ldots)(x^2 - x + .25777280103\ldots)(x^2 - x + c = .60506915636\ldots) = 0$$

50. Solve for real numbers:

$$x^3 - x^2 - 17x + 25 = 2(x-1)^2\sqrt{2x-3}$$

Proposed by Orlando Irahola Ortega-La Paz-Bolivia

Solution by Miguel Velasquez Culque-Cajamarca-Peru

$$x^2(x-1) - 17(x-1) + 8 = 2(x-1)^2\sqrt{2(x-1)-1}$$

Change of variable: $x - 1 = a \to x = a + 1$

$$(a+1)^2a - 17a + 8 = 2a^2\sqrt{2a-1}$$

$$(a^2 + 2a + 1)a - 17a + 8 - 2a^2\sqrt{2a-1} = 0$$

$$a^3 + 2a^2 + a - 17a + 8 - 2a^2\sqrt{2a-1} = 0$$

$$a^3 + 2a^2 - 8(2a-1) - 2a^2\sqrt{2a-1} = 0$$

$$a^3 + 2a^2 - 8\sqrt{(2a-1)^2} - 2a^2\sqrt{2a-1} = 0;\ \sqrt{2a-1} = m$$

$$a^3 + 2a^2 - 8m^2 - 2a^2m = 0$$

$$a^2(a-2m) + 2(a+2m)(a-2m) = 0$$

$$(-2m)\left(a^2+2(a+2m)\right)=0$$

$a=2m$ ***solution and*** $a^2=-2(a+2m)$ ***not solution.***

But $\sqrt{2a-1}=m$

$$a=2m\to a=2\sqrt{2a-1}\to a^2=4(2a-1)\to a^2-8a+4=0$$

$$a=\frac{8\pm\sqrt{48}}{2}=\frac{8\pm4\sqrt{3}}{2}=4\pm\sqrt{3}$$

We know that: $x=a+1$ ***then*** $x_{1,2}=5\pm2\sqrt{3}$ ***are solutions.***

51. Solve:

$$\frac{[x]}{[x]+1}+\frac{8[2x]}{[2x]+8}=\frac{[x][2x]+8[2x]+[x]+8}{[2x]+[x]+9}$$

$[x]$ **–greatest integer function.**

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by George Florin Șerban-Romania

$$[x]=a,[2x]=b,a,b\in\mathbb{Z}\Rightarrow\frac{a}{a+1}+\frac{8b}{b+8}=\frac{ab+8b+a+8}{a+b+9}$$

$$(8a+8b+9ab)(a+b+9)=(ab+8b+a+8)$$

$$8a^2+8ab+72a+8ab+8b^2+72b+9a^2b+9ab^2+81ab=$$

$$=a^2b^2+8ab^2+a^2b+8ab+8a^2b+64ab+8a^2+64a+ab^2+8b^2+ab+8b$$
$$+8ab+64b+8a+64\Rightarrow$$

$$89ab+9a^2b+9ab^2=a^2b^2+9ab^2+9a^2b+73ab+64$$

$$a^2b^2-16ab+64=0\Rightarrow(ab-8)^2=0\Rightarrow ab=8\Rightarrow a/8\Rightarrow$$

$$a\in\{\pm1,\pm2,\pm4,\pm8\}$$

If $a=1,b=8\Rightarrow[x]=1,[2x]=8\Rightarrow\begin{cases}1\le x<2\\8\le 2x<9\end{cases}\Rightarrow$

$x\in[1,2)\cap\left[4,\frac{9}{2}\right)=\emptyset$, ***absurd.***

If $a=-1,b=-8\Rightarrow[x]=-1,[2x]=-8\Rightarrow\begin{cases}-1\le x<0\\-8\le 2x<-5\end{cases}\Rightarrow$

$x\in[-1,0)\cap\left[-4,-\frac{7}{2}\right)=\emptyset$, ***absurd.***

$$\text{If } a = 2, b = 4 \Rightarrow [x] = 2, [2x] = 4 \Rightarrow \begin{cases} 2 \le x < 3 \\ 4 \le 2x < 5 \end{cases} \Rightarrow x \in \left[2, \frac{5}{2}\right)$$

$$\text{If } a = -2, b = -4 \Rightarrow [x] = -2, [2x] = -4 \Rightarrow \begin{cases} -2 \le x < -1 \\ -8 \le 2x < -7 \end{cases} \Rightarrow x \in \left[-2, -\frac{3}{2}\right)$$

$$\text{If } a = 4, b = 2 \Rightarrow [x] = 4, [2x] = 2 \Rightarrow \begin{cases} 4 \le x < 5 \\ 2 \le 2x < 3 \end{cases} \Rightarrow$$

$$x \in [4, 5) \cap \left[1, \frac{3}{2}\right) = \emptyset, \text{ absurd.}$$

$$\text{If } a = -4, b = -2 \Rightarrow [x] = -4, [2x] = -2 \Rightarrow \begin{cases} -4 \le x < -3 \\ -2 \le 2x < -1 \end{cases} \Rightarrow$$

$$x \in [-4, -3) \cap \left[-1, -\frac{3}{2}\right) = \emptyset, \text{ absurd.}$$

$$\text{If } a = 8, b = 1 \Rightarrow [x] = 8, [2x] = 1 \Rightarrow \begin{cases} 8 \le x < 9 \\ 1 \le 2x < 2 \end{cases} \Rightarrow$$

$$x \in [8, 9) \cap \left[\frac{1}{2}, 1\right) = \emptyset, \text{ absurd.}$$

$$\text{If } a = -8, b = -1 \Rightarrow [x] = -8, [2x] = -1 \Rightarrow \begin{cases} -8 \le x < -7 \\ -1 \le 2x < 0 \end{cases} \Rightarrow$$

$$x \in [-8, -7) \cap \left[-\frac{1}{2}, 0\right) = \emptyset, \text{ absurd.}$$

$$x \in \left[-2, -\frac{3}{2}\right) \cup \left[-2, -\frac{3}{2}\right)$$

Solution 2 by Asish Singh-India

$$\text{Case 1: Let } x = I + f; af < \frac{1}{2}$$

$$\text{Now: } \frac{I}{I+1} + \frac{16I}{2I+8} = \frac{2I^2+16I+I+8}{2I+I+9} \Rightarrow \frac{9I^2+12I}{I^2+3I+4} = \frac{2I^2+17I+8}{3I+9}$$

$$27I^3 + 117I^2 + 108I =$$

$$= 2I^4 + 17I^3 + 68I + 8I^2 + 10I^3 + 85I^2 + 8I^2 + 40I + 32$$

$$\Rightarrow I^4 - 8I^2 + 16 = 0 \Rightarrow (I^2 - 4)^2 = 0 \Rightarrow I = \pm 2$$

$$x \in \left[-2, -\frac{3}{2}\right) \cup \left[-2, -\frac{3}{2}\right)$$

$$\text{Case 2: Let: } x = I + f; \frac{1}{2} < f < 1$$

$$\text{Now: } \frac{I}{2I+1} + \frac{8(2I+1)}{2I+9} = \frac{(2I+1)I+8(2I+1)+I+8}{3I+10}$$

$$4I^4 + 4I^3 - 31I^2 - 16I + 64 = 0$$

$$\text{Case 2.1: If } I \ge 0$$

$$\underbrace{4I^4 + 4I^3 + 64}_{Am-Gm} = 31I^2 + 16I$$

$31I^2 + 16I \geq 32I^2 + 4I^3 \Rightarrow 0 \geq 4I^2 + I - 16 \Rightarrow I = 1$ *only solution, but* $I = 1$ *satisfies conditions. So, no solution in this case.*

Case 2.2: $I < 0$

$0 \geq 4I^3 + I^2 - 16I \Rightarrow I(4I^2 + I - 16) \geq 0$ ***no solutions.***

So, $x \in \left[-2, -\frac{3}{2}\right) \cup \left[-2, -\frac{3}{2}\right)$

52. Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x)f(y) = f(x+y) + \left(\frac{\alpha}{\beta}\right)^2 \cdot xy, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$f(x)f(y) = f(x+y) + \left(\frac{\alpha}{\beta}\right)^2 \cdot xy \quad (*)$$

In (*), we put $x = y = 0 \to (f(0))^2 = f(0) \leftrightarrow \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$

$$f(0) = 0$$

Let $y = 0 \overset{(*)}{\to} f(x) \equiv 0 \to \left(\frac{\alpha}{\beta}\right)^2 xy \equiv 0$ ***(contrary)***

$$f(0) = 1$$

Let $x = \frac{\beta}{\alpha}; y = -\frac{\beta}{\alpha} \overset{(*)}{\to} f\left(\frac{\beta}{\alpha}\right) f\left(-\frac{\beta}{\alpha}\right) = 0 \leftrightarrow \left[\begin{array}{l} f\left(\frac{\beta}{\alpha}\right) = 0 \\ f\left(-\frac{\beta}{\alpha}\right) = 0 \end{array}\right.$

With $f\left(\frac{\beta}{\alpha}\right) = 0$***, let*** $y = \frac{\beta}{\alpha} \overset{(*)}{\to}: f\left(x + \frac{\beta}{\alpha}\right) = -\frac{\alpha}{\beta}x \leftrightarrow f(x) = -\frac{\alpha}{\beta}x + 1$

With $f\left(-\frac{\beta}{\alpha}\right) = 0$***, let*** $y = -\frac{\beta}{\alpha} \overset{(*)}{\to}: f\left(x - \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta}x \leftrightarrow f(x) = \frac{\alpha}{\beta}x + 1$

Solution: $f(x) = -\frac{\alpha}{\beta}x + 1$ ***or*** $f(x) = \frac{\alpha}{\beta}x + 1$

53. Find all functions $f: \mathbb{R} \to \mathbb{R}$ that satisfy:

$$f^2(x) \cdot f^2(y) = f(x+y) + \frac{2019}{2020}xy, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

Let $\alpha^2 = 2019, \beta^2 = 2020$

$$f^2(x)\cdot f^2(y) = f(x+y) + \left(\frac{\alpha}{\beta}\right)^2 \cdot xy \quad (1)$$

In (1) we put: $x = y = 0 \to f^4(0) = f(0) \to \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$

Case: $f(0) = 0$

In (1) we put: $x = 0 \to f(x) \equiv 0 \to \left(\frac{\alpha}{\beta}\right)^2 \cdot xy \equiv 0, \forall x$ (contrary)

Case: $f(0) = 1$

In (1) we put: $x = \frac{\beta}{\alpha}; y = -\frac{\beta}{\alpha} \to f^2\left(\frac{\beta}{\alpha}\right)\cdot f^2\left(-\frac{\beta}{\alpha}\right) = 0 \to \left[\begin{array}{l} f\left(\frac{\beta}{\alpha}\right) = 0 \\ f\left(-\frac{\beta}{\alpha}\right) = 0 \end{array}\right.$

$f\left(\frac{\beta}{\alpha}\right) = 0$. In (1) we put: $y = \frac{\beta}{\alpha} \to f\left(x + \frac{\beta}{\alpha}\right) + \frac{\alpha}{\beta}x = 0 \to$

$$f(x) = \frac{\alpha}{\beta}\left(x - \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta}x - 1 = \sqrt{\frac{2019}{2020}}x - 1$$

$f\left(-\frac{\beta}{\alpha}\right) = 0$. In (1) we put: $y = -\frac{\beta}{\alpha}$

$$\to f\left(x - \frac{\beta}{\alpha}\right) - \frac{\alpha}{\beta}x = 0 \to f(x) = \frac{\alpha}{\beta}\left(x + \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta}x + 1 = \sqrt{\frac{2019}{2020}}x + 1$$

We check two case |: don't satisfy. No solution.

54. Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x)\cdot f(y) = f^2(x+y) + \frac{\alpha}{\beta}\cdot x^\alpha \cdot y^\beta, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$f(x)\cdot f(y) = f^2(x+y) + \frac{\alpha}{\beta}\cdot x^\alpha \cdot y^\beta \quad (*)$$

In (*), we put $y = 0$: $f(x)\cdot f(0) = f^2(x) \leftrightarrow \left[\begin{array}{l} f(x) \equiv 0 \\ f(x) \equiv f(0) \end{array}\right.$

Case: $f(x) \equiv 0$. In (*), we let $y = 1$: $f^2(x+1) = -\frac{\alpha}{\beta}\cdot x^\alpha, \forall x \in \mathbb{R}$

It is false because we choose $x = 1 \rightarrow f^2(2) = -\frac{\alpha}{\beta}$ *(contrary)*→ *No solution.*

Case: $f(x) \equiv f(0) = c$ *(const)*

From (*) we have: $c \cdot c = c^2 + \frac{\alpha}{\beta} \cdot x^\alpha \cdot y^\beta \leftrightarrow \frac{\alpha}{\beta} \cdot x^\alpha \cdot y^\beta \equiv 0$ *(contrary)*→ *No solution.*

Solution 2 by Ahmed Bach-Bouira-Algerie

$$f(x) \cdot f(y) = f^2(x+y) + \frac{\alpha}{\beta} \cdot x^\alpha \cdot y^\beta \quad (*)$$

for $x = 0$ *we find that:* $\forall y \in \mathbb{R}; \big(f(0) - f(y)\big)f(y) = 0 \quad (1)$

From (1) we have $f(1) = 0$ *or* $f(1) = f(0)$

Suppose that $f(1) = 0$ *we find from (*) by letting* $x = y = 1;\ -f^2(2) = \frac{\alpha}{\beta}$

(contradiction). Then we have $f(1) = f(0)$

for $x = y = 1$ *in (*) we find:* $f^2(1) - f^2(2) = \frac{\alpha}{\beta} \Leftrightarrow f^2(0) - f^2(2) = \frac{\alpha}{\beta}$

if $f(2) = f(0)$ *we find* $\frac{\alpha}{\beta} = 0$ *(contradiction). So,* $f(2) = 0$

$x = 2 \wedge y = 1 \Rightarrow 0 = f^2(3) + \frac{\alpha}{\beta} 2^\alpha$ *(contradiction). No solution.*

55. **Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following relationship:**

$$f(\alpha x + \beta y) \cdot f(\beta x + \alpha y) = f(x+y) + \alpha\beta \cdot xy, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$f(\alpha x + \beta y) \cdot f(\beta x + \alpha y) = f(x+y) + \alpha\beta \cdot xy \quad (*)\ \forall x, y \in \mathbb{R}$$

In (*) we put: $x = y = 0 \rightarrow f^2(0) = f(0) \leftrightarrow \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$

Case: $f(0) = 0$

In (*) we let: $y = -\frac{\beta}{\alpha}x \rightarrow 0 = f\left(x - \frac{\beta}{\alpha}x\right) + \alpha\beta \cdot x \cdot \left(-\frac{\beta}{\alpha}x\right) \rightarrow f\left(\frac{\alpha-\beta}{\alpha}x\right) = \beta^2 x^2$

If $\alpha = \beta$ *then* $f(0) = \alpha^2 x^2 \leftrightarrow 0 \equiv \alpha^2 x^2$ *(contrary)*

If $\alpha \neq \beta$ *then* $f\left(\frac{\alpha-\beta}{\alpha}x\right) = \beta^2 x^2 \rightarrow f(x) = \beta^2 \cdot \frac{\alpha^2}{(\alpha-\beta)^2} x^2 = \left(\frac{\alpha\beta}{\alpha-\beta}\right)^2 x^2$

But: Deg of $LHS_{(*)} = 4 > 2 = \deg$ ***of*** $RHS_{(*)} \rightarrow$ ***No solution.***

Case: $f(0) = 1$

In (*) we let: $y = -\frac{\beta}{\alpha}x \rightarrow f\left(\alpha x - \beta \cdot \frac{\beta}{\alpha}x\right) \cdot 1 = f\left(x - \frac{\beta}{\alpha}x\right) + \alpha\beta \cdot x \cdot \left(-\frac{\beta}{\alpha}x\right)$

$$\rightarrow f\left(\frac{\alpha^2 - \beta^2}{\alpha}x\right) = f\left(\frac{\alpha - \beta}{\alpha}x\right) - \beta^2 x^2$$

If $\alpha = \beta$ ***then*** $1 = 1 - \beta^2 x^2 \leftrightarrow 0 \equiv \beta^2 x^2$ ***(contrary)***

If $\alpha \neq \beta$ ***then we put*** $x := \frac{\alpha}{\alpha - \beta}x$

$$\rightarrow f((\alpha + \beta)x) = f(1) - \left(\frac{\alpha\beta}{\alpha - \beta}\right)^2 x^2$$

$$\rightarrow f(x) = f(1) - \left(\frac{\alpha\beta}{\alpha - \beta}\right)^2 \left(\frac{x}{\alpha + \beta}\right)^2 = f(1) - \left(\frac{\alpha\beta}{\alpha^2 - \beta^2}\right)^2 x^2$$

But $f(0) = 1 \rightarrow f(0) = f(1) = 1$

Hence: $f(x) = 1 - \left(\frac{\alpha\beta}{\alpha^2 - \beta^2}\right)^2 x^2 \rightarrow f(1) = 1 - \left(\frac{\alpha\beta}{\alpha^2 - \beta^2}\right)^2 \neq 1 \rightarrow$ ***No solution.***

Answer: No solution.

56. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(xy + x) \cdot f(xy - x) = f(xy) + \frac{2019}{2020}xy, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$f(xy + x) \cdot f(xy - x) = f(xy) + \frac{2019}{2020}xy \quad (*)$$

In (*) we let: $x = y = 0 \rightarrow f^2(0) = f(0) \rightarrow \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$

Case: $f(0) = 0$

In (*) we let: $y = 1 \rightarrow f(2x) \cdot f(0) = f(x) + \frac{2019}{2020}x \rightarrow f(x) = -\frac{2019}{2020}x$

We check in (*): $\frac{2019}{2020}(xy + x) \cdot \frac{2019}{2020} \cdot (xy - x) = -\frac{2019}{2020}xy + \frac{2019}{2020}xy, \forall x, y$

$$\leftrightarrow \left(\frac{2019}{2020}\right)^2 \cdot ((xy)^2 - x^2) = 0, \forall x, y$$

(This is contrary) → No solution

Case: $f(0) = 1$

In (*) we let: $y = 1 \to f(2x) \cdot f(0) = f(x) + \frac{2019}{2020}x \to f(2x) = f(x) + \frac{2019}{2020}x$

$$\to f(2x) = f\left(\frac{x}{2^n}\right) + \frac{2019}{2020}\left(\frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n}\right)$$

$$= f\left(\frac{x}{2^n}\right) + \frac{2019}{2020} \cdot \frac{x}{2}\left(\frac{\left(\frac{x}{2}\right)^n - 1}{\frac{x}{2} - 1}\right) \quad (1)$$

Suppose: f ***– continuous in*** $x = 0$

In (1) we let $n \to +\infty$: $f(2x) = f(0) + \frac{2019}{2020} \cdot \frac{x}{2} \cdot \frac{0-1}{\frac{x}{2}-1} = 1 - \frac{x}{x-2}$

We also check in (*) → contrary → No solution.

57. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f^2(x + y) = f(x) \cdot f(y) + 2019x^2y^2, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Bado Idriss Olivier-Cote D'Ivoire

$$f(f(x)) = f(f(2x - x)) = f(2x)f(-x) + 8076x^4$$

$$\textit{Moreover } f(f(x)) = f(x)f(0)$$

$$f(f(2x)) = f(f(x + x)) = f(x)^2 + 2019x^4 = f(2x)f(0)$$

and $f(f(0)) = f(f(x - x)) = f(x)f(-x) + 2019x^4$***. As*** $f(f(x)) = f(x)f(0)$

hence $f(x)f(0) = f(2x)f(-x) + 8076x^4$ ***multiplying each member by*** $f(0)$ ***we have:***

$$f(x)f(0)^2 = f(2x)f(0)f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = (f(x)^2 + 2019x^4)f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = f(x)f(x)f(-x) + 2019x^4f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = f(x)(f(f(0)) - 2019x^4 + 2019x^4f(-x) + 8076x^4f(0)$$

as $f(f(0)) = f(0)^2$ ***so*** $2019x^4f(x) = 2019x^4f(-x) + 8076x^4f(0), \forall x \in \mathbb{R}$

and $f(x) = f(-x) + 4f(0), \forall x \in \mathbb{R}$ ***we deduce*** $f(-x) = f(x) + 4f(0)$ ***then***

$f(0) = 0$***. Finally:*** $0 = f(f(0)) = f(0)^2 = f(x)^2 + 2019x^4$***. No solution.***

58. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(0) = \frac{1}{4}, f(5x) - f(x) = x, \forall x \in \mathbb{R}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$f(5x) - f(x) = x \Rightarrow f(x) - f\left(\frac{x}{5}\right) = \frac{x}{5} \Rightarrow f\left(\frac{x}{5}\right) - f\left(\frac{x}{5^2}\right) = \frac{x}{5^2} \Rightarrow \cdots$$

$$\Rightarrow \cdots f\left(\frac{x}{5^{n-1}}\right) - f\left(\frac{x}{5^n}\right) = \frac{x}{5^n}, \qquad n \in \mathbb{N} - \{0\}$$

$$f(x) - f\left(\frac{x}{5}\right) + f\left(\frac{x}{5}\right) - f\left(\frac{x}{5^2}\right) + \cdots + f\left(\frac{x}{5^{n-1}}\right) - f\left(\frac{x}{5^n}\right) = \sum_{k=1}^{n} \frac{x}{5^k}$$

$$f(x) - f\left(\frac{x}{5^n}\right) = x \cdot \frac{\frac{1}{5}\left(\frac{1}{5^n} - 1\right)}{\frac{1}{5} - 1}$$

$$\lim_{n\to\infty}\left(f(x) - f\left(\frac{x}{5^n}\right)\right) = \lim_{n\to\infty}\left(x \cdot \frac{\frac{1}{5}\left(\frac{1}{5^n} - 1\right)}{\frac{1}{5} - 1}\right)$$

$$f(x) - f\left(\lim_{n\to\infty}\frac{x}{5^n}\right) = x \cdot \frac{-\frac{1}{5}}{-\frac{4}{5}} \Rightarrow f(x) - f(0) = \frac{x}{4} \Longrightarrow f(x) = \frac{x+1}{4}$$

59. Let $\alpha > 0$. Find all functions $f: [0;1] \to \mathbb{R}$ such that:

$$f(\alpha x + y) = \alpha f^2(x) - 2020 \cdot x^{\alpha} \cdot t^{\sqrt{\alpha}}, \forall x, y \in [0,1]$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tran Hong-Vietnam

$$f(\alpha x + y) = \alpha f^2(x) - 2020 \cdot x^{\alpha} \cdot t^{\sqrt{\alpha}} \ldots (*)$$

In (*) put $x = y = 0 \in [0;1] \Rightarrow f(0) = \alpha f^2(0) \Rightarrow \begin{cases} f(0) = 0 \\ f(0) = \frac{1}{\alpha} \end{cases}$

Case: $f(0) = 0$

In (*) we let:

$$x = 0 \Rightarrow f(y) = \alpha f^2(0) - 2020 \cdot 0^{\alpha} y^{\sqrt{\alpha}} \Rightarrow f(y) = 0 \Rightarrow f(x) = 0, \forall x, y \in [0,1]$$

Checking:

$$0 = \alpha \cdot 0 - 2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} \Rightarrow -2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} = 0;\ \forall x, y \in [0,1]$$

$$\text{if } x = y = 0 \Rightarrow -2020 = 0 \ \textit{which is contrary}.$$

No solution.

$$\textit{Case}: f(0) = \frac{1}{\alpha}$$

$$\textit{In } (*) \textit{ we let}: x = 0 \Rightarrow f(y) = \alpha f^2(0) = \alpha \cdot \frac{1}{\alpha^2} = \frac{1}{\alpha}$$

Checking:

$$\frac{1}{\alpha} = \alpha \cdot \frac{1}{\alpha} - 2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} \Rightarrow -2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} = 0, \forall x, y \in [0,1]$$

$$\textit{if } x = y = 1 \Rightarrow -2020 = 0 \ \textit{which is contrary}.$$

No solution.

60. Let $a > b > 0$. Find all functions $\varphi: \mathbb{R} \to \mathbb{R}$ such that:

$$a \cdot \varphi(bx^2) = b \cdot \varphi^2(ay) + \frac{a}{b} \cdot x^{2019} x^{2020}, \forall x, y \in R$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Tran Hong-Dong-Vietnam

$$a \cdot \varphi(bx^2) = b \cdot \varphi^2(ay) + \frac{a}{b} \cdot x^{2019} x^{2020} \ \ldots\ldots (*)$$

$$\textit{In } (*) \textit{put } x = y = 0 \to a \cdot \varphi(0) = b \cdot \varphi^2(0) \to \begin{cases} \varphi(0) = 0 \\ \varphi(0) = \frac{a}{b} \end{cases}$$

$$(i) \textit{Case}: \varphi(0) = 0, \textit{in } (*) \textit{put } x = 0 \to b \cdot \varphi^2(y) = 0, \forall y \in R \to \varphi(y) = 0, \forall x \in R$$

$$\textit{Check}: a \cdot 0 = b \cdot 0 + \frac{a}{b} x^{2019} \cdot y^{2020} \to \frac{a}{b} \cdot x^{2019} y^{2020} = 0 \ (\textit{contrary}) \to \textit{no solution}$$

$(ii)Case: \varphi(0) = \frac{a}{b}, in\ (*)\ put\ x = 0 \to a \cdot \frac{a}{b} = b \cdot \varphi^2(ay) \to \varphi^2(ay) = \frac{a^2}{b^2} \to$

$$\varphi(ay) = \pm\frac{a}{b} \overset{x=ay}{\Rightarrow} \varphi(x) = \pm\frac{a}{b}.$$ ***Check:***

$$\varphi(x) = \frac{a}{b} \to a \cdot \frac{a}{b} = b \cdot \frac{a^2}{b^2} + \frac{a}{b}x^{2019}x^{2020} \to \frac{a}{b}x^{2019}x^{2020} = 0\ (contrary) \to no\ solution.$$

$$\varphi(x) = -\frac{a}{b} \to -\frac{a^2}{b} = b \cdot \frac{a^2}{b^2} + \frac{a}{b} \cdot x^{2019}x^{2020} \to 2 \cdot \frac{a^2}{b} = \frac{a}{b}x^{2019}x^{2020} \to 2a = x^{2019}x^{2020}$$

contrary...no solution

61. Find all continuous functions $f: [0,\infty) \to (0,\infty)$ such that:

$$f\left(\sqrt{\frac{x^2 + y^2}{2}}\right) = \sqrt{f(x) \cdot f(y)}, \forall x, y \geq 0$$

Proposed by Marian Ursărescu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

First, we need to solve:

***Problem 1**: Find all continous functions* $\varphi: [0; +\infty) \to (0; +\infty)$ *such that*

$$\varphi\left(\sqrt{\frac{u+v}{2}}\right) = \frac{\varphi(\sqrt{u}) + \varphi(\sqrt{v})}{2}, \quad \forall u, v \geq 0$$

Let $\varphi(\sqrt{u}) = g(u),\ u \geq 0 \to g(u)$ ***continous on*** $[0; +\infty)$

$$\to g\left(\frac{u+v}{2}\right) = \frac{g(u) + g(v)}{2}, \quad \forall u, v \geq 0$$

Let $g(u) - g(0) = h(u) \to h(u)$ ***continous on*** $[0; +\infty),\ h(0) = 0,$

$$h\left(\frac{u+v}{2}\right) = \frac{h(u) + h(v)}{2}, \forall u, v \geq 0 \qquad (*)$$

In (*) we put:

- $u = 0 \to h\left(\frac{v}{2}\right) = \frac{h(v)}{2}$
- $v = 0 \to h\left(\frac{u}{2}\right) = \frac{h(u)}{2}$

$$\rightarrow h\left(\frac{u+v}{2}\right)=\frac{h(u)}{2}+\frac{h(v)}{2},\quad \forall u,v\geq 0$$

$$\rightarrow h(u+v)=h(u)+h(v),\quad \forall u,v\geq 0$$

$$\rightarrow h(u)=au+b,\quad (a,b\ const)$$

$$\overset{h(0)=0}{\Rightarrow}\ h(u)=au\rightarrow g(u)=au+g(0)=au+c,\qquad (c=g(0))$$

$$\varphi(\sqrt{u})=g(u)=au+c,\quad \forall u\geq 0\ \rightarrow\ \varphi(u)=au^2+c$$

Now, ***Find all continous functions*** $\varphi:[0;\ +\infty)\ \rightarrow(0;\ +\infty)$ ***such that***

$$f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\sqrt{f(x)f(y)}\qquad (**)$$

If $\exists x_0\geq 0$ ***such that*** $f(x_0)=0$ ***then***

$$f\left(\sqrt{\frac{x_0^2+y^2}{2}}\right)=\sqrt{f(x_0)f(y)}=0,\ \forall y>0$$

$$\rightarrow f(x)\equiv 0,\quad \forall x\geq\frac{x_0}{\sqrt{2}}$$

$$\rightarrow f\left(\frac{x_0}{\sqrt{2}}\right)=0\ \overset{x_0:=\frac{x_0}{\sqrt{2}}}{\Rightarrow}\ f\left(\frac{x_0}{(\sqrt{2})^2}\right)=0\ \dots\dots\rightarrow\ f\left(\frac{x_0}{(\sqrt{2})^n}\right)=0,\qquad \forall n\in\mathbb{N}$$

$$\rightarrow\lim_{n\to\infty}f\left(\frac{x_0}{(\sqrt{2})^n}\right)=f\left(\lim_{n\to\infty}\frac{x_0}{(\sqrt{2})^n}\right)=f(0)=0.$$

Hence,

$$f\left(\sqrt{\frac{x^2+0^2}{2}}\right)=f\left(\frac{x}{\sqrt{2}}\right)=\sqrt{f(x)f(0)}=0,\quad \forall x\geq 0$$

$$\rightarrow f(x)\equiv 0,\quad \forall x\geq 0$$

Suppose: $f(x)\neq 0,\ \forall x\geq 0$**.** **If** $\exists x_1\geq 0$ ***such that*** $f(x_1)<0$ ***then by (**) we have:***

$$f\left(\sqrt{\frac{{x_1}^2+y^2}{2}}\right)=\sqrt{f(x_1)f(y)},\quad \forall y\geq 0$$

$$\rightarrow f(y)<0,\forall y\geq 0\ \ (\,contrary\ with\ f(y)>0,\qquad \forall y\geq 0)$$

So, $f(x)>0\ ,\ \forall x\geq 0$

$$(**) \leftrightarrow \ln f\left(\sqrt{\frac{x^2+y^2}{2}}\right) = \frac{\ln f(x)+\ln f(y)}{2}, \quad \forall x, y \geq 0$$

Let $\mu(x) = \ln f(x) \to \mu(x)$ *continous on* $[0;\ +\infty)$ *and*

$$\mu\left(\sqrt{\frac{x^2 + y^2}{2}}\right) = \frac{\mu(x) + \mu(y)}{2}, \quad \forall x, y \geq 0$$

Let $x = \sqrt{u}, y = \sqrt{v} \quad (\ \forall u, v \geq 0)$. *Using* ***Problem 1*** *we have:*

$$\mu(x) = ax^2 + c, \quad \forall x \geq 0$$

$$\to f(x) = e^{ax^2+c}, \qquad \forall x \geq 0$$

Ans: $f(x) \equiv 0, \quad \forall x \geq 0$ *or* $f(x) = e^{ax^2+c}, \ \forall x \geq 0$

62. Find the last 3 digits of:

$$\Omega = 2019^{\overbrace{201920192019\ldots201953}^{50\ times\ "2019"}}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Ajao Yinka-Nigeria

$$\underbrace{201920192019\ldots2019}_{2019\ \text{times}}53 =$$

$$= 2 \times 10^{201} + 0 \times 10^{200} + 1 \times 10^{199} + 9 \times 10^{198} + \cdots +$$

$$+2 \times 10^5 + 0 \times 10^4 + 1 \times 10^3 + 9 \times 10^2 + 5 \times 10 + 3 \times 10^\circ$$

We can use Euler's quotient function and Euler's theorem:

Since $1000 = 8 \times 125$*. We evaluate* $\Omega(mod\ 1000)$

Evaluating $\Omega(mod\ 8)$*;* $\phi(8) = 4$*;* $2019 = 3(mod\ 8)$

$$20192019\ldots201953 = 53 = 1(mod\ 4)$$

Since all other terms are multiples of 4*.*

So, $\Omega = 3' = 3(mod\ 8)$*. Evaluating* $\Omega(mod\ 125)$

$$\phi(125) = 4 \times 25 = 100; 2019 = 19\ mod\ 125$$

$$20192019\ldots201953 = 53\ (mod\ 100)$$

Since all other terms are multiples of 100*, so* $\Omega = 19^{53}(mod\ 125)$

$$= 19^{52} \times 19 = \left(19^4\right)^{13} \times 19 = 11^{13} \times 19$$

$$= (71^4)^3 \times 71 \times 19 = 56^3 \times 71 \times 19 = 116 \times 71 \times 19 = 109 (mod\ 125)$$

So, $\Omega = 3\ mod\ 8 = 3 + 8k$ *and* $\Omega = 109\ mod\ 125$

$$3 + 8k = 109\ mod\ 125$$

$$k = 107\ mod\ 125$$

So, $\Omega = 859\ mod\ 1000$*. The last three digits are* 859*.*

63. $u_0 = 2, u_1 = 4, u_n + u_{n+2} = 4u_{n+1}, n \in \mathbb{N}$**. Find:**

$$\Omega = \left(\sum_{k=1}^{n} u_k\right) (mod\ 2)$$

Proposed by Surjeet Singhania-India

Solution by George Florin Șerban-Romania

$$x^2 - 4x + 1 = 0, \Delta = 16 - 4 = 12, x_1 = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}, x_2 = 2 - \sqrt{3}$$

$$u_n = \alpha \cdot \left(2 + \sqrt{3}\right)^n + \beta\left(2 - \sqrt{3}\right)^n, \begin{cases} \alpha + \beta = 2 = u_0 \\ 2(\alpha + \beta) + \sqrt{3}(\alpha - \beta) = u_1 = 4 \end{cases}$$

$$\begin{cases} \alpha + \beta = 2 \\ \alpha - \beta = 0 \end{cases} \Rightarrow \alpha = \beta = 1 \Rightarrow u_n = \left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n$$

$$\Omega = \sum_{k=1}^{n} u_k = \frac{\left(2 + \sqrt{3}\right)\left[\left(2 - \sqrt{3}\right)^n - 1\right]}{1 + \sqrt{3}} + \frac{\left(2 - \sqrt{3}\right)\left[\left(2 - \sqrt{3}\right)^n - 1\right]}{1 - \sqrt{3}}$$

$$\frac{2 + \sqrt{3}}{1 + \sqrt{3}} = \frac{\left(2 + \sqrt{3}\right)\left(\sqrt{3} - 1\right)}{3 - 1} = \frac{2\sqrt{3} - 2 + 3 - \sqrt{3}}{2} = \frac{1 + \sqrt{3}}{2},$$

$$\frac{2 - \sqrt{3}}{1 - \sqrt{3}} = \frac{\left(2 - \sqrt{3}\right)\left(1 + \sqrt{3}\right)}{1 - 3} = \frac{2 + 2\sqrt{3} - \sqrt{3} - 3}{-2} = \frac{-1 + \sqrt{3}}{-2} = \frac{1 - \sqrt{3}}{2}$$

$$\Omega = \frac{1 + \sqrt{3}}{2} \cdot \left[\left(2 + \sqrt{3}\right)^n - 1\right] + \frac{1 - \sqrt{3}}{2} \cdot \left[\left(2 - \sqrt{3}\right)^n - 1\right]$$

$$\Omega = \frac{\left(1 + \sqrt{3}\right) \cdot \left(2 + \sqrt{3}\right)^n + \left(1 - \sqrt{3}\right)\left(2 - \sqrt{3}\right)^n - 1 - \sqrt{3} - 1 + \sqrt{3}}{2}$$

$$\left(2 + \sqrt{3}\right)^n = a_n + b_n\sqrt{3} \Rightarrow \left(2 - \sqrt{3}\right)^n = a_n - b_n\sqrt{3}$$

$$\Omega=\frac{(1+\sqrt{3})(a_n+b_n\sqrt{3})+(1-\sqrt{3})(a_n-b_n\sqrt{3})-2}{2}=$$

$$=\frac{a_n+b_n\sqrt{3}+a_n\sqrt{3}+3b_n+a_n-b_n\sqrt{3}-a_n\sqrt{3}-a_n\sqrt{3}+3b_n}{2}$$

$$-1=\frac{2a_n+6b_n}{2}-1=a_n+3b_n-1$$

$$\left(2+\sqrt{3}\right)^n=a_n+b_n\sqrt{3}$$

If $n=$ ***even*** $=2k\Rightarrow$

$$\Rightarrow\left(2+\sqrt{3}\right)^{2k}=C_{2k}^0 2^{2k}+C_{2k}^1 2^{2k-1}\sqrt{3}+C_{2k}^2 2^{2k-2}\sqrt{3}^2+\dots+C_{2k}^{2k-1}2\cdot\sqrt{3}^{2k-1}+$$

$$+C_{2k}^{2k}\sqrt{3}^{2k}=\left(\underbrace{C_{2k}^0 2^{2k}+C_{2k}^2 2^{2-k}\cdot 3+\dots+C_{2k}^{2k-2}2^2\cdot 3^{k-1}}_{even}+\underbrace{C_{2k}^{2k}3^k}_{odd}\right)+$$

$$+\sqrt{3}\left(\underbrace{C_{2k}^1 2^{2k-1}+\dots+C_{2k}^{2k-1}2\cdot 3^{k-1}}_{even}\right)$$

$\Rightarrow a_n$ ***odd and*** b_n ***even*** $\Rightarrow \Omega=a_n+3b_n-1=$ ***even*** $=0\ (mod\ 2)$

If $n=$ ***odd*** $=2k+1$, $\left(2+\sqrt{3}\right)^{2k+1}=C_{2k+1}^0 2^{2k+1}+C_{2k+1}^1 2^{2k}\sqrt{3}+C_{2k+1}^2 2^{2k-1}\sqrt{3}^2+$

$$+\dots+C_{2k+1}^{2k}2\sqrt{3}^{2k}+C_{2k+1}^{2k+1}\cdot\sqrt{3}^{2k+1}=$$

$$=\left(\underbrace{C_{2k+1}^0 2^{2k-1}+C_{2k+1}^2 2^{2n-1}\cdot 3+\dots+C_{2k+1}^{2k}n\cdot 3^k}_{even}\right)+$$

$+\sqrt{3}\cdot\left(\underbrace{C_{2k+1}^1 2^{2k}+\dots+C_{2k+1}^{2k-1}\cdot 2^2\cdot 3^{k-1}}_{even}+\underbrace{C_{2k+1}^{2k+1}\cdot 3^k}_{odd}\right)\Rightarrow a_n=$ ***even and*** $b_n=$***odd***

$\Rightarrow \Omega=a_n+3b_n-1=$ ***even*** $=0\ (mod\ 2)$

In conclusion $\Omega=0\ (mod\ 2)$

64. Solve for real numbers:

$$\begin{cases}(\log(xy)+x)^3=(\log(xy)-x)^3+\left(x-\log\left(\frac{x}{y}\right)\right)^3+\left(x+\log\left(\frac{x}{y}\right)\right)^3\\ x^y+y^z+z^x=5\\ x,y,z>0\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Bedri Hajrizi-Mitrovica-Kosovo

$$(a+b)^3-(a-b)^3=2b(3a^2+b^2) \quad (*)$$

$$(a+b)^3+(a-b)^3=2a(a^2+3b^2) \quad (**)$$

First equation is equivalent with:

$$\underbrace{(\log(xy)+x)^3-(\log(xy)-x)^3}_{(*)}=\underbrace{\left(x-\log\left(\frac{x}{y}\right)\right)^3+\left(x+\log\left(\frac{x}{y}\right)\right)^3}_{(**)}$$

$$2x(3\log^2(xy)+x^2)=2x\left(x^2+3\log^2\left(\frac{x}{y}\right)\right);\ \log^2(xy)=\log^2\left(\frac{x}{y}\right)$$

$xy=\pm\frac{x}{y}$ *(sign "–" not accepted);* $y=\frac{1}{y}\Leftrightarrow y=1$

Second equation: $x+1+z^x=5;\ x+z^x=4$

Let $x=a, a\in(0,4)\Rightarrow z=\sqrt[a]{4-a}$

Conclusion: Solution is $\left(a,1,\sqrt[a]{4-a}\right), a\in(0,4)$

65. Solve for $x,y,z>0$:

$$\begin{cases} x-y+z=\frac{1}{2} \\ 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=2(x+y+z) \\ 8xyz=1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin – Romania

$$3\sum\frac{1}{1+2x+4xy}=2(x+y+z)=3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4\cdot\frac{1}{8x}}+\frac{1}{1+2\cdot\frac{1}{8xy}+4\cdot\frac{1}{8y}}\right)$$

$$=3\left(\frac{1}{1+2x+4xy}+\frac{2x}{2x+4xy+1}+\frac{4xy}{4xy+1+2x}\right)=$$

$$=3\frac{1+2x+4xy}{1+2x+4xy}=3=2(x+y+z)\Rightarrow x+y+z=\frac{3}{2}$$

$$x-y+z=\frac{1}{2}\Rightarrow 2y=1\Rightarrow y=\frac{1}{2}\Rightarrow x+z=1,$$

$$8\cdot\frac{1}{2}xz=1$$

$$\Rightarrow xz=\frac{1}{4}: t^2-t+\frac{1}{4}=0, \left(t-\frac{1}{2}\right)^2=0\Rightarrow t_1=t_2=\frac{1}{2}$$

$$\Rightarrow x=z=\frac{1}{2}.\ \textbf{So, } x=y=z=\frac{1}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Find all $x,y,z>0 \mid x-y+z\overset{(i)}{=}\frac{1}{2}, 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)\overset{(ii)}{=}2\sum x$ ***and***

$$8xyz\overset{(iii)}{=}1$$

$$\because 8xyz=1 \therefore \frac{1}{1+2z+4zx}=\frac{1}{8xyz+2z+4zx}=\frac{1}{2z(1+2x+4xy)}\Rightarrow$$

$$\Rightarrow \frac{1}{1+2z+4zx}\overset{(1)}{=}\frac{1}{2z(1+2x+4xy)} \text{ and } \frac{1}{1+2y+4yz}=\frac{1}{8xyz+2y+4yz}=\frac{1}{2y(1+2z+4zx)}\overset{by\ (1)}{=}\frac{1}{4yz(1+2x+4xy)}$$

$$\therefore \frac{1}{1+2y+4yz}\overset{(2)}{=}\frac{1}{4yz(1+2x+4xy)}$$

$$\therefore (1)+(2)\Rightarrow 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=$$

$$=3\left(\frac{1}{1+2x+4xy}+\frac{1}{4yz(1+2x+4xy)}+\frac{1}{2z(1+2x+4xy)}\right)$$

$$=\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)\overset{\because 1=8xyz}{=}8xyz\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)=$$

$$=\left(\frac{3}{1+2x+4xy}\right)(8xyz+2x+4xy)=\left(\frac{3}{1+2x+4xy}\right)(1+2x+4xy)=3$$

$$\therefore 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=3\overset{by\ (ii)}{=}$$

$$=2\sum x\Rightarrow x+y+z\overset{(iv)}{=}\frac{3}{2}$$

$$\textbf{(iv)+(i)}\Rightarrow 2(z+x)=2\Rightarrow z+x\overset{(v)}{=}1\ \ (iv)\cdot(v)\Rightarrow y=\frac{1}{2}\overset{by\ (iii)}{=}4zx\overset{(vi)}{=}1$$

$$\textbf{(v)}\Rightarrow (z+x)^2=1\Rightarrow (z-x)^2+4zx=1\overset{by\ (iv)}{\Rightarrow}(z-x)^2+4zx=1\overset{by\ (iv)}{\Rightarrow}$$

$$\Rightarrow (z-x)^2+1=1\Rightarrow (z-x)^2=0\Rightarrow z\overset{(vii)}{=}x$$

$$\textbf{(vi), (vii)}\Rightarrow 4x^2=1\Rightarrow (2x+1)(2x-1)=0\Rightarrow x=\frac{1}{2}(\because x>0)\overset{by\ (vii)}{=}x=z=\frac{1}{2}$$

$\therefore x = y = z = \frac{1}{2}$ *(answer)*

66. Solve for integers:

$$\begin{cases} x + y + z = 6 \\ x^2(y - z) + y^2(z - x) + z^2(x - y) = 2 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$x^2(y - z) + y^2(z - x) + z^2(x - y) = xy(x - y) - z(x + y)(x - y) + z^2(x - y)$$

$$= (x - y)\big(x(y - z) - z(y - z)\big) = (x - y)(y - z)(x - z) = 2$$

$\because (x - y), (y - z)$ and $(x - z)$ are integers whose product $= 2$,

$\therefore$ the following are all possible cases :

$\boxed{Case\ (1)}$ $x - y = 1, y - z = 1$ and $x - z = 2 \Rightarrow y = x - 1$ and $z = x - 2$ and

$\because x + y + z = 6, \therefore x + x - 1 + x - 2 = 6 \Rightarrow \boxed{x = 3 \therefore y = 2 \text{ and } z = 1}$

$\boxed{Case\ (2)}$ $x - y = 1, y - z = 2$ and $x - z = 1 \Rightarrow x - y + y - z = 3 \Rightarrow x - z = 3$

$\Rightarrow 3 = 1 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (3)}$ $x - y = 1, y - z = -1$ and $x - z = -2 \Rightarrow x - y + y - z = 0 \Rightarrow x - z$

$= 0 \Rightarrow 0 = -2 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (4)}$ $x - y = 1, y - z = -2$ and $x - z = -1 \Rightarrow y = x - 1$ and z

$= x + 1$ and $\because x + y + z = 6, \therefore x + x - 1 + x + 1 = 6 \Rightarrow \boxed{x = 2 \therefore y = 1 \text{ and } z = 3}$

$\boxed{Case\ (5)}$ $x - y = 2, y - z = 1$ and $x - z = 1 \Rightarrow x - y + y - z = 3 \Rightarrow x - z = 3$

$\Rightarrow 3 = 1 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (6)}$ $x - y = 2, y - z = -1$ and $x - z = -1 \Rightarrow x - y + y - z = 1 \Rightarrow x - z$

$= 1 \Rightarrow 1 = -1 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (7)}$ $x - y = -1, y - z = 2$ and $x - z = -1 \Rightarrow x - y + y - z = 1 \Rightarrow x - z$

$= 1 \Rightarrow 1 = -1 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (8)}$ $x - y = -1, y - z = -2$ and $x - z = 1 \Rightarrow x - y + y - z = -3$

$\Rightarrow x - z = -3 \Rightarrow -3 = 1 \to$ impossible $\Rightarrow$ no solution exists under this case

$\boxed{Case\ (9)}$ $x - y = -2, y - z = 1$ *and* $x - z = -1 \Rightarrow y = x + 2$ *and* $z = x + 1$ *and* $\because x + y + z = 6, \therefore x + x + 2 + x + 1 = 6 \Rightarrow \boxed{x = 1 \therefore y = 3 \text{ and } z = 2}$

$\boxed{Case\ (9)}$ $x - y = -2, y - z = -1$ *and* $x - z = 1 \Rightarrow x - y + y - z = -3$

$\Rightarrow x - z = -3 \Rightarrow -3 = 1 \rightarrow$ *impossible* $\Rightarrow$ *no solution exists under this case*

$\therefore$ *all possible integer triplets of* (x, y, z) *satisfying*

the 2 given equations are $\boxed{\boxed{(3,2,1), (2,1,3) \text{ and } (1,3,2)}}$ *(answer)*

67. Solve for $x, y, z > 0$:

$$\begin{cases} x - y + z = \dfrac{1}{2} \\ 3\left(\dfrac{1}{1 + 2x + 4xy} + \dfrac{1}{1 + 2y + 4yz} + \dfrac{1}{1 + 2z + 4zx}\right) = 2(x + y + z) \\ 8xyz = 1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin-Romania

$$3\sum \frac{1}{1 + 2x + 4xy} = 2(x + y + z) =$$

$$= 3\left(\frac{1}{1 + 2x + 4xy} + \frac{1}{1 + 2y + 4 \cdot \frac{1}{8x}} + \frac{1}{1 + 2 \cdot \frac{1}{8xy} + 4 \cdot \frac{1}{8y}}\right)$$

$$= 3\left(\frac{1}{1 + 2x + 4xy} + \frac{2x}{2x + 4xy + 1} + \frac{4xy}{4xy + 1 + 2x}\right) =$$

$$= 3\frac{1 + 2x + 4xy}{1 + 2x + 4xy} = 3 = 2(x + y + z) \Rightarrow x + y + z = \frac{3}{2}$$

$$x - y + z = \frac{1}{2} \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x + z = 1, 8 \cdot \frac{1}{2}xz = 1$$

$$\Rightarrow xz = \frac{1}{4}; t^2 - t + \frac{1}{4} = 0, \left(t - \frac{1}{2}\right)^2 = 0 \Rightarrow t_1 = t_2 = \frac{1}{2}$$

$$\Rightarrow x = z = \frac{1}{2}. \text{ So, } x = y = z = \frac{1}{2}$$

Solution 2 by Remus Florin Stanca-Romania

$$3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4xz}\right)=$$

$$=3\left(\frac{8xyz}{8xyz+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{8}{8xyz+2z+4xz}\right)=$$

$$=3\left(\frac{8xyz}{8xyz+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{4xy}{4xy+8xyz+2x}\right)=$$

$$=3\left(\frac{4yz}{4yz+1+2y}+\frac{1}{4yz+2y+1}+\frac{2y}{4yz+2y+1}\right)=2(x+y+z)\Rightarrow$$

$$\Rightarrow 3\frac{4yz+2y+1}{4yz+2y+1}=2(x+y+z)\Rightarrow x+y+z=\frac{3}{2}\Rightarrow\frac{1}{2}+2y=\frac{3}{2}\Rightarrow 2y=1\Rightarrow y=\frac{1}{2}\Rightarrow$$

$$\Rightarrow x+z=1,xz=\frac{1}{4}\Rightarrow x(1-x)=\frac{1}{4}\Rightarrow$$

$$\Rightarrow 4x-4x^2-1=0\Rightarrow(2x-1)^2=0\Rightarrow x=\frac{1}{2}\Rightarrow z=\frac{1}{2}\Rightarrow x=y=z=\frac{1}{2}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\because 8xyz=1\therefore\frac{1}{1+2z+4zx}=\frac{1}{8xyz+2z+4zx}=\frac{1}{2z(1+2x+4xy)}\Rightarrow$$

$$\Rightarrow\frac{1}{1+2z+4zx}\overset{(1)}{=}\frac{1}{2z(1+2x+4xy)}$$

And $\frac{1}{1+2y+4yz}=\frac{1}{8xyz+2y+4yz}=\frac{1}{2y(1+2z+4zx)}\overset{by\ (1)}{=}\frac{1}{4yz(1+2x+4xy)}\therefore\frac{1}{1+2y+4yz}\overset{(2)}{=}\frac{1}{4yz(1+2x+4xy)}$

$$\therefore(1)+(2)\Rightarrow\ 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=$$

$$=3\left(\frac{1}{1+2x+4xy}+\frac{1}{4yz(1+2x+4xy)}+\frac{1}{2z(1+2x+4xy)}\right)$$

$$=\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)\overset{\because 1=8xyz}{=}8xyz\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)=$$

$$=\left(\frac{3}{1+2x+4xy}\right)(8xyz+2x+4xy)=\left(\frac{3}{1+2x+4xy}\right)(1+2x+4xy)=3$$

$$\therefore 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=3\overset{by\ (ii)}{=}2\sum x\Rightarrow\boxed{x+y+z\overset{(iv)}{=}\frac{3}{2}}$$

$$(iv)+(i)\Rightarrow 2(z+x)=2\Rightarrow z+x\overset{(v)}{=}1\ (iv)-(v)\Rightarrow \boxed{\boxed{y=\frac{1}{2}}}\overset{by\ (iii)}{\Rightarrow}\ 4zx\overset{(vi)}{=}1$$

$$(v)\Rightarrow(z+x)^2=1\Rightarrow(z-x)^2+4zx=1\overset{by\ (iv)}{\Rightarrow}(z-x)^2+1=1\Rightarrow(z-x)^2=0\Rightarrow\boxed{z\overset{(vii)}{=}x}$$

$$(vi),(vii)\Rightarrow 4x^2=1\Rightarrow\boxed{(2x+1)(2x-1)=0}\Rightarrow x=\frac{1}{2}(\because x>0)\overset{by\ (vii)}{\Rightarrow}\boxed{\boxed{x=z=\frac{1}{2}}}$$

$$\therefore x=y=z=\frac{1}{2}$$

68. Solve for $x,y,z>0$:

$$\begin{cases}4(xy+yz+zx)=3\\ 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=2(x+y+z)\\ 8xyz=1\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Miguel Velasquez Culque-Lima-Peru

$$\frac{x+y+z}{3}\geq\sqrt[3]{xyz}$$

$$x+y+z\geq 3\sqrt[3]{\frac{8xyz}{8}}$$

$$x+y+z\geq 3\sqrt[3]{\frac{1}{8}}$$

$$x+y+z\geq 3\frac{1}{2}$$

$$x+y+z=\frac{3}{2}$$

$$4(xy+yz+xz)=3$$

$$4\left(\frac{y}{8yz}+\frac{z}{8xz}+\frac{x}{8xy}\right)=3$$

$$4\left(\frac{1}{8z}+\frac{1}{8x}+\frac{1}{8y}\right)=3$$

$$4\frac{1}{8}\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)=3$$

$$\frac{1}{z}+\frac{1}{x}+\frac{1}{y}=6$$

$$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\left(x+y+z+\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}\right)+\frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\geq 6\sqrt[6]{xzy\cdot\frac{1}{(4x)(4y)(4z)}}+\frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3+\frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\geq 3+\frac{3}{4}\cdot\frac{9}{x+y+z}\geq 3+\frac{27}{4}\cdot\frac{1}{3/2}=\frac{15}{2}\ \textit{True};\ \frac{3}{2}+6=\frac{15}{2}$$

$$\text{So, } \min x=y=z=\frac{1}{2}$$

Solution 2 by Remus Florin Stanca-Romania

We know that $x+y+z\geq 3\sqrt[3]{xyz}\Leftrightarrow x+y+z\geq\frac{3}{2}$

We also know that: $\frac{3}{1+2x+4xy}=\frac{3}{1+2x+\frac{1}{2z}}=\frac{3}{\frac{1}{1}+\frac{1}{\frac{1}{2x}}+\frac{1}{\frac{1}{2z}}}\leq\frac{1+\frac{1}{2x}+2z}{3}\Rightarrow$

$$\Rightarrow 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2+4yz}+\frac{1}{1+2z+4xz}\right)\leq$$

$$\leq\frac{3+\frac{1}{2x}+\frac{1}{2y}+\frac{1}{2z}+2(x+y+z)}{3}=\frac{3+3+2(x+y+z)}{3}=\frac{6+2(x+y+z)}{3}\ \textbf{(1)}$$

$$x+y+z\geq\frac{3}{2}\Rightarrow 4(x+y+z)\geq 6\Rightarrow 6(x+y+z)\geq 6+2(x+y+z)\Rightarrow$$

$$\Rightarrow\frac{6+2(x+y+z)}{3}\leq 2(x+y+z)\ \textbf{(2)}$$

$\overset{(1);(2)}{\Rightarrow}\ 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4xz}\right)\leq 2(x+y+z)$, *but we know that:*

$$3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4xz}\right)=2(x+y+z)\Rightarrow$$

$$\Rightarrow 1=\frac{1}{2x}=2z=\frac{1}{2y}=2x=\frac{1}{2z}=2y\Rightarrow x=y=z=\frac{1}{2}$$

Solution 3 by Michael Sterghiou-Greece

$$4(xy+yz+zx)=3\ \textbf{(1)}$$

$$3\cdot\left(\sum_{cyc}\frac{1}{1+2x+4xy}\right)=2(x+y+z)\ \textbf{(2)}$$

$$8xyz=1\ \textbf{(3)}$$

From (1): $\frac{3}{4}=\sum_{cyc}xy\overset{AM-GM}{\geq}3\cdot\sqrt[3]{(xyz)^2}\rightarrow xyz\leq\frac{1}{8}$ *with equality when* $x=y=z$.

But from (3) $xyz=\frac{1}{8}$ *hence* $x=y=z=\frac{1}{2}$. *This triad satisfies (2) and therefore is the only solution.*

Solution 4 by Soumava Chakraborty-Kolkata-India

$$4\sum xy \overset{(i)}{\cong} 3, 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right) \overset{(ii)}{\cong} 2\sum x \textit{ and } 8xyz \overset{(iii)}{\cong} 1$$

$$\because 8xyz = 1 \therefore \frac{1}{1+2z+4zx} = \frac{1}{8xyz+2z+4zx} = \frac{1}{2z(1+2x+4xy)} \Rightarrow$$

$$\Rightarrow \frac{1}{1+2z+4zx} \overset{(1)}{\cong} \frac{1}{2z(1+2x+4xy)}$$

$$\textit{and } \frac{1}{1+2y+4yz} = \frac{1}{8xyz+2y+4yz} = \frac{1}{2y(1+2z+4zx)} \overset{\text{by (1)}}{\cong} \frac{1}{4yz(1+2x+4xy)}$$

$$\therefore \frac{1}{1+2y+4yz} \overset{(2)}{\cong} \frac{1}{4yz(1+2x+4xy)}$$

$$\therefore (1)+(2) \Rightarrow 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right) =$$

$$= 3\left(\frac{1}{1+2x+4xy}+\frac{1}{4yz(1+2x+4xy)}+\frac{1}{2z(1+2x+4xy)}\right)$$

$$= \left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right) \overset{\because 1 = 8xyz}{\cong} 8xyz\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right) =$$

$$= \left(\frac{3}{1+2x+4xy}\right)(8xyz+2x+4xy) = \left(\frac{3}{1+2x+4xy}\right)(1+2x+4xy) = 3$$

$$\therefore 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right) = 3 \overset{\text{by (ii)}}{\cong} 2\sum x \Rightarrow$$

$$\Rightarrow 4\left(\sum x\right)^2 = 9 \Rightarrow 4\sum x^2 + 8\sum xy = 9 \overset{\text{by (i)}}{\Rightarrow} 4\sum x^2 + 8\sum xy = 12\sum xy$$

$$\Rightarrow 2\sum x^2 - 2\sum xy = 0 \Rightarrow \sum(x^2+y^2-2xy) = 0 \Rightarrow$$

$$\Rightarrow (x-y)^2+(y-z)^2+(z-x)^2 = 0 \Rightarrow \boxed{x=y=z}$$

$\therefore$ *using* (*iii*) *and the fact that* $x = y = z$, *we have* :

$$8x^3 = 1 \Rightarrow (2x-1)(4x^2-2x+1) = 0 \Rightarrow 2x-1 = 0$$

$$(\because x \textit{ is real as } x > 0) \Rightarrow \boxed{x = \frac{1}{2}} \therefore x = y = z = \frac{1}{2} (\textit{answer})$$

69. **Solve for real numbers:**

$$\begin{cases} x, y, z > 0 \\ x + y + z + xyz \ge 4 \\ \sqrt{x} + \sqrt{y} + \sqrt{z} = x + y + z \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \left(\sqrt{x}\cdot 1 + \sqrt{y}\cdot 1 + \sqrt{z}\cdot 1\right)^2 \overset{CBS}{\le} (1^2 + 1^2 + 1^2)\left(\sum (\sqrt{x})^2\right)$$

$$\Rightarrow \left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^2 \le 3(x + y + z) \Rightarrow (x + y + z)^2 \le 3(x + y + z)$$

$$\Rightarrow (x + y + z) \le 3$$

$$\Leftrightarrow \frac{x + y + z + xyz}{4} \overset{AM-GM}{\ge} (xyz)^{\frac{1}{2}} \Rightarrow \sqrt{xyz} \le \frac{4}{4} \Rightarrow xyz \le 1$$

$$\Leftrightarrow x + y + z + xyz = 4 \Rightarrow x + y + z = 4 - xyz \ge 4 - 1 = 3$$

$$\Rightarrow x + y + z \ge 3$$

$$\Leftrightarrow \textit{It's obvious that} \Rightarrow x + y + z = 3$$

$$\Rightarrow xyz = 4 - (x + y + z) = 1$$

$\Leftrightarrow$ ***Case 1:*** ***for*** $0 < x, y, z < 1$ ***there does not exist any three distinct equal/non-equal proper fraction whose multiplication is equal to or greater than*** 1***; No solution in this bound*** $(0, 1)$

$\Rightarrow$ ***Case 2:*** ***for*** $x, y, z \ge (1, +\infty)$***. If*** $xyz = 1$ ***and*** $x + y + z = 3$***, these can be possible if and only if*** $x = z = 1$***, for*** $x, y, z > 1$ ***those criterias can't be satisfied***

$$\Leftrightarrow x = y = z = 1 \quad \textit{(Answer)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

With $x, y, z > 0$ ***we have:***

$$x + y + z = 1\cdot\sqrt{x} + 1\cdot\sqrt{y} + 1\cdot\sqrt{z} \overset{B.C.S}{\le} \sqrt{3}\cdot\sqrt{x + y + z}$$

$$\Rightarrow \sqrt{x + y + z} \le \sqrt{3} \Rightarrow x + y + z \le 3 \quad (*)$$

$$xyz \le \frac{(x + y + z)^3}{27} \Rightarrow 4 \le xyz + x + y + z \le \frac{(x + y + z)^3}{27} + (x + y + z)$$

$$\Leftrightarrow (x + y + z)^3 + 27(x + y + z) - 108 \ge 0$$

$$\overset{t=x+y+z>0}{\Leftrightarrow} t^3 + 27t - 108 \geq 0 \Leftrightarrow (t-3)(t^2+3t+36) \geq 0$$

$$\Leftrightarrow t \geq 3 \Leftrightarrow x+y+z \geq 3 \quad (**)$$

$$\overset{(*),(**)}{\Rightarrow} x+y+z = 3 \Leftrightarrow x = y = z = 1. \Rightarrow (x;y;z) = (1;1;1) \textit{ (Answer)}$$

Solution 3 by Michael Sterghiou-Greece

Solve for real numbers: $x, y, z > 0$ *(1),* $x + y + z + xyz \geq 4$ *(2)*

$\sum_{cyc} \sqrt{x} = \sum_{cyc} x$ *(3)*

from (1) and (2) by AM-GM $x + y + z \geq 3$ *(4) (if* $x + y + z \leq 3$

then $xyz < 1$ *and (2) false). From (3)* $\sum_{cyc} x \geq \frac{(\sum \sqrt{x})^2}{3}$

or $\sum_{cyc} x \geq \frac{(\sum x)^2}{3} \to x + y + z \leq 3$ *(5). (4)* $\wedge$ *(5)* $\to x + y + z = 3$

and now (2) $\to xyz = 1$ *and* $x = y = z$ *(Equality in AM-GM)*

therefore $x = y = z = 1$ *as only solution. Done!*

70. Solve for $x, y, z, t > 0$:

$$\begin{cases} xt = 4e \\ \dfrac{x^2+y^2}{x+y} + \dfrac{y^2+z^2}{y+z} + \dfrac{z^2+x^2}{z+x} = x + y + z \\ t^{\log y} = 4 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

For all $a, b > 0$ *we have:* $\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}$ *(*)*

$$\Leftrightarrow 2(a^2+b^2) \geq (a+b)^2 \Leftrightarrow a^2 - 2ab + b^2 \geq 0$$

$$\Leftrightarrow (a-b)^2 \geq 0 \quad \textit{(true) Equality} \Leftrightarrow a = b$$

So, $\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} \geq \frac{x+y}{2} + \frac{y+z}{2} + \frac{z+x}{2} = x + y + z$

$$\text{"="} \Leftrightarrow x = y = z$$

$$t^{\log y} = 4 \Rightarrow \log\left(t^{\log y}\right) = \log(4)$$

$$\Rightarrow (\log y)(\log t) = \log(4) \Rightarrow (\log x)(\log t) = \log(4)$$

$$\overset{xt=4e}{\Rightarrow} t=\frac{4e}{x}\Rightarrow(\log x)\left(\log\frac{4e}{x}\right)=\log(4)\Leftrightarrow(\log x)(\log(4e)-\log x)=\log(4)$$

$$\Leftrightarrow(\log x)(1+\log(4)-\log x)=\log(4)\overset{\alpha=\log x}{\Leftrightarrow}\alpha(1+\log(4)-\alpha)-\log(4)=0$$

$$\Leftrightarrow-\alpha^2+\alpha+(\alpha-1)\log(4)=0\Leftrightarrow-\alpha(\alpha-1)+(\alpha-1)\log(4)=0$$

$$\Leftrightarrow(\alpha-1)(\log(4)-\alpha)=0\Leftrightarrow\alpha=1\text{ or }\alpha=\log(4)$$

$$(*)\ \alpha=1\Rightarrow x=y=z=e\Rightarrow t=e$$

$$(*)\ \alpha=\log(4)\Rightarrow x=y=z=4\Rightarrow t=e$$

71. Solve for real numbers:

$$\begin{cases} x\geq 0, y,z,t>0, [*]-\textit{great integer function}\\ [x](x-[x])+y+t=x^2+2z\\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{6}{(y+z)^3+2}+\frac{6}{(z+t)^3+2}+\frac{6}{(t+y)^3+2}\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\therefore(y+z)^3+2=(y+z)^3+1+1\overset{AM-GM}{\geq}3\sqrt[3]{(y+z)^3\cdot1\cdot1}=3(y+z)$$

Similary:

$$(z+t)^3+2\geq3(z+t)\textit{and }(t+y)^3+2\geq3(t+z)$$

$$\Omega=\frac{6}{(y+z)^3+2}+\frac{6}{(z+t)^3+2}+\frac{6}{(t+y)^3+2}$$

$$\leq\frac{2}{y+z}+\frac{2}{z+t}+\frac{2}{t+y}\leq\frac{1}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{t}\right)+\frac{1}{2}\left(\frac{1}{t}+\frac{1}{y}\right)$$

$$=\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\Psi\left(\therefore\textit{We are using}:\frac{4}{\alpha+\beta}\leq\frac{1}{\alpha}+\frac{1}{\beta},\forall\alpha,\beta>0\right)$$

$$\Omega=\Psi\Leftrightarrow\begin{cases}y=z=t\\ y+z=z+t=t+y=1\end{cases}\Leftrightarrow y=z=t=\frac{1}{2}$$

$$[x](x-[x])+y+t=x^2+2z\overset{y=z=t}{\Leftrightarrow}[x](x-[x])=x^2\ldots(1);(x\geq0\Rightarrow[x]\geq0)$$

$$(1)\Leftrightarrow[x]\{x\}=(\{x\}+[x])^2\Leftrightarrow[x]\{x\}=([x])^2+2[x]\{x\}+(\{x\})^2$$

$$\Leftrightarrow([x])^2+[x]\{x\}+(\{x\})^2=0$$

But:

$$[x] \geq 0;\ \{x\} \geq 0, equality\ for\ [x] = \{x\} = 0 \Rightarrow x = 0$$

So,

$$(x, y, z, t) = \left(0; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}\right)$$

72. Solve for real numbers:

$$\begin{cases} x^3 + \log_2 x + \log_4 y = 67 \\ x^{\sqrt{xy}} \cdot y^{\frac{1}{\sqrt{xy}}} + x^{\frac{1}{\sqrt{xy}}} \cdot y^{\sqrt{xy}} = \sqrt{x^{x+y}} \cdot y^{\frac{2}{x+y}} + \sqrt{y^{x+y}} \cdot x^{\frac{2}{x+y}} \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Khaled Abd Imouti-Damascus-Syria

$$x^{\sqrt{xy}} \cdot y^{\frac{1}{\sqrt{xy}}} + x^{\frac{1}{\sqrt{xy}}} \cdot y^{\sqrt{xy}} = x^{\frac{x+y}{2}} \cdot y^{\frac{2}{x+y}} + y^{\frac{x+y}{2}} \cdot x^{\frac{2}{x+y}}$$

$$x^G \cdot y^{\frac{1}{G}} + x^{\frac{1}{G}} \cdot y^G = x^M \cdot y^{\frac{1}{M}} + y^M \cdot x^{\frac{1}{M}}, M = \frac{x+y}{2}, G = \sqrt{xy}, M \geq G$$

$$e^{G\ln(x)+\frac{1}{G}\ln(y)} + e^{\frac{1}{G}\ln(x)+G\ln(y)} = e^{M\ln(x)+\frac{1}{M}\ln(y)} + e^{M\ln(y)+\frac{1}{M}\ln(x)} \quad (*)$$

(*) satisfying when $G = M$

When $G = M = 4$ ***and*** $x = 4, y = 4$ ***then:*** $(u)^3 + \frac{\ln(4)}{\ln(2)} + \frac{\ln(4)}{\ln(4)} = 64 + 2 + 1 = 67$

$$S = \{(x, y) = (4, 4)\}$$

From (*): $e^{G\ln(x)+\frac{1}{G}\ln(y)} - e^{M\ln(x)+\frac{1}{M}\ln(y)} = e^{M\ln(x)+\frac{1}{M}\ln(y)} - e^{G\ln(y)+\frac{1}{G}\ln(x)}$

Suppose: $f(t) = e^{t\ln(x)+\frac{1}{t}\ln(y)}, t > 0$

$$f'(t) = \left(\ln(x) - \frac{1}{t^2}\ln(y)\right) \cdot e^{t\ln(x)+\frac{1}{t}\ln(y)}$$

$\exists c_1 \in]M, G[$ ***such that:***

$$\left.\begin{matrix} f(G) - f(M) = \left(\ln(x) - \frac{1}{C_1^2}\ln(y)\right)(G - M) \\ in\ a\ similarlly\ way: \\ \exists c_2 \in]G, M[\ such\ that: \\ f(M) - f(G) = \left(\ln(x) - \frac{1}{c_2^2}\ln(y)\right)(M - G) \end{matrix}\right\} \Rightarrow$$

$$\left(\ln(x)-\frac{1}{c_1^2}\ln(y)\right)(G-M)+\left(\ln(x)-\frac{1}{c_2^2}\ln(y)\right)(M-G)=0$$

$$\left(\ln(x)-\frac{1}{c_1^2}\ln(y)\right)(G-M)-\left(\ln(x)-\frac{1}{c_2^2}\ln(y)\right)(G-M)=0$$

$$\left[\ln(x)-\frac{1}{c_1^2}\ln(y)-\ln(x)+\frac{1}{c_2^2}\ln(y)\right](G-M)=0$$

$$\underbrace{\left(\frac{1}{c_2^2}-\frac{1}{c_1^2}\right)}_{\neq 0}\cdot\ln(y)(G-M)=0$$

So: $G-M=0\Rightarrow G=M.$

73. Solve for real numbers:

$$\begin{cases} x+y+z+u=4 \\ x^2(x^2-v^2)+y^2(y^2-v^2)+v^4=z^2(v^2-z^2)+u^2(v^2-u^2) \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$x^4+y^4+z^4+u^4+v^4=(x^2+y^2+z^2+u^2)v^2\ \dots(*)$$

$$\therefore x^4+\frac{1}{4}v^4\overset{Am-Gm}{\geq}2\sqrt{x^4\cdot\frac{1}{4}v^4}=x^2v^2\ \therefore y^4+\frac{1}{4}v^4\overset{Am-Gm}{\geq}y^2v^2$$

$$\therefore z^4+\frac{1}{4}v^4\geq z^2v^2\ \therefore u^4+\frac{1}{4}v^4\overset{Am-Gm}{\geq}u^2v^2$$

$$x^4+y^4+z^4+u^4+\frac{4}{4}v^4=(x^2+y^2+z^2+u^2)v^2\Leftrightarrow$$

$$x^4+y^4+z^4+u^4+v^4=(x^2+y^2+z^2+u^2)v^2\Rightarrow(*)\text{true}.$$

$$Equality\Leftrightarrow x^2=y^2=z^2=u^2=\frac{1}{2}v^2=\alpha^2\ \ (\alpha>0)$$

$$\Rightarrow|x|=|y|=|z|=|u|=\alpha;\ |v|=\alpha\sqrt{2}$$

- $If\ x;y;z;u\geq 0\Rightarrow x+y+z+u=4\alpha=4\Rightarrow\alpha=1\Rightarrow x=y=z=u=1; v=\pm\sqrt{2}$

$$\Rightarrow(x;y;z;u)=(1;1;1;1), v=\pm\sqrt{2}$$

- $If\ x;y;z;u<0\Rightarrow x+y+z+u<0<4$

No solution.

$$\bullet\ If \begin{cases} xy < 0 \\ zu < 0 \end{cases} \text{or} \begin{cases} xz < 0 \\ yu < 0 \end{cases} or \begin{cases} xu < 0 \\ yz < 0 \end{cases} then\ x + y + z + u = 0 < 4 \Rightarrow$$

No solution.

$$\bullet\ If\ x \geq 0; y = z = u \leq 0\ (and\ cyclic) then\ x + y + z + u = \alpha - \alpha - \alpha - \alpha = 4 \Rightarrow$$

$\alpha = -2 \Rightarrow$***no solution.***

$$\bullet\ If\ x < 0; y = z = u = 2 (and\ cyclic) then\ x + y + z + u = -\alpha + \alpha + \alpha + \alpha = 2\alpha = 4 \Rightarrow$$

$$\alpha = 2 \Rightarrow x = -2; y = z = u = 2; v = \pm 2\sqrt{2}$$

$$\Rightarrow (x; y; z; u) = (-2; 2; 2; 2) (and\ cyclic), v = \pm 2\sqrt{2}$$

74. Solve for complex numbers:

$$\begin{cases} \dfrac{x^7}{y^{30}} + \dfrac{y^7}{z^{30}} + \dfrac{z^7}{x^{30}} = \dfrac{(x + y + z)^7}{(x^5 + y^5 + z^5)^6} \\ x^4 - 3y^3 - 2z^2 - 3y + 1 = 0 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Petre Daniel Alexandru-Romania

$$\frac{x^7}{y^{30}} + \frac{y^7}{z^{30}} + \frac{z^7}{x^{30}} = \frac{x^7}{(y^5)^6} + \frac{y^7}{(z^5)^6} + \frac{z^7}{(x^5)^6} \overset{Radon}{\geq} \frac{(x + y + z)^7}{(x^5 + y^5 + z^5)^6}$$

Equality for $\frac{x}{y} = \frac{y}{z} = \frac{z}{x} \Leftrightarrow x = y = z$

$$x \in \mathbb{Z}, x^4 - 3y^3 - 2z^2 - 3y + 1 = 0 \Leftrightarrow (x^2 - 4x + 1)(x^2 + x + 1) = 0$$

$$x^2 - 4x + 1 = 0 \Rightarrow x_{1,2} = 2 \pm \sqrt{3}; x^2 + x + 1 = 0 \Leftrightarrow x_{3,4} = \frac{-1 \pm i\sqrt{3}}{2}$$

75. Solve for real numbers:

$$\begin{cases} x + y + z = 11 \\ \dfrac{yz + 36x}{x(y - x)(z - x)} + \dfrac{zx + 36y}{y(x - y)(z - y)} + \dfrac{xy + 36z}{z(x - z)(y - z)} = 1 \\ xyz = 36 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Bedri Hajrizi-Mitrovica-Kosovo

Let $\sum x = 11 \quad (1); \prod x = 36 \quad (3)$

From second questions (+(1),(3)) we get: $-\sum \frac{y^2z^2+36^2}{36(x-y)(z-x)} = 1$

$$-36\prod(x-y) = y^3z^2 - y^2z^3 + 36^2y - 36^2z + z^3x^2 - z^2x^3 + 36^2z - 36^2x +$$

$$+x^3y^2 - x^2y^3 + 36^2x - 36^2y$$

$$-36\prod(x-y) = x^2z^2(z-x) + y^3(z+x)(z-x) - y^2(z^3-x^3)$$

$$-36\prod(x-y) = -(z-x)(y-z)(x-y)(xy+yz+zx)$$

$$-36\prod(x-y) = -\prod(x-y)\left(\sum xy\right)$$

If $x = y$ ***or*** $y = z$ ***or*** $z = x$ ***no real solution!***

System become $\begin{cases} x+y+z = 11 \\ xy+yz+zx = 36 \\ xyz = 36 \end{cases}$; x, y, z ***solution of the equation:***

$$t^3 - 11t^2 + 36t - 36 = 0; \ (t-2)(t-3)(t-6) = 0$$

Solution are $(2,3,6)$ ***and permutations.***

76. Solve for real numbers:

$$\begin{cases} xy+yz+zx = 26 \\ \frac{48+yz(y+z)}{(x-y)(x-z)} + \frac{48+zx(z+x)}{(y-x)(y-z)} + \frac{48+xy(x+y)}{(z-x)(z-y)} = 9 \\ xyz = 24 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$x \neq y; y \neq z; z \neq x, x, y, z \neq 0$$

$$xy+yz+zx = 26; xyz = 24$$

$$\frac{48+yz(y+z)}{(x-y)(x-z)} + \frac{48+zx(z+x)}{(y-x)(y-z)} + \frac{48+xy(x+y)}{(z-x)(z-y)} = 9 \Leftrightarrow$$

$$48\left(\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} + \frac{1}{(z-x)(z-y)}\right) +$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9\Leftrightarrow$$

$$48\left(\frac{-(y-z)-(z-x)-(x-y)}{(x-y)(y-z)(z-x)}\right)+$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9$$

$$\left(\frac{-yz(y+z)(y-z)-zx(z+x)(z-x)-xy(x+y)(x-y)}{(x-y)(y-z)(z-x)}\right)=9$$

$$yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)=-9(x-y)(y-z)(z-x)$$

$$x^3y+y^3z+z^3x-(xy^3+yz^3+zx^3)+9(x^2z+z^2y+y^2x-xz^2-zy^2-yx^2)=0$$

$$(x-y)(y-z)(z-x)(x+y+z-9)=0\xleftrightarrow{x\neq y;y\neq z;z\neq x}x+y+z-9=0$$

$$x+y+z=9$$

So, by Vieta's Theorem: $X^3-9X^2+26X-24=0$

$(x;y;z)=(2;3;4)$ ***and cyclic.***

77. Solve for real numbers:

$$\begin{cases} 6x+3y+2z=18 \\ 108(x+y+z)^{x+y+z}=xy^2z^3\cdot 6^{x+y+z} \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Jalil Hajimir-Toronto-Canada

From second equation $x+y+z>0$,

since $6x+3y+2z>0$ ***we conclude:*** $x,y,z>0$

$$\left(\frac{x+y+z}{6}\right)^{x+y+z}=x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3 \text{ (1)}$$

$x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3\overset{Am-Gm}{\leq}\left(\frac{x+y+z}{6}\right)^6\therefore\ \frac{x+y+z}{6}=1;\ t^{t-1}\geq 1$ ***(2). From (1),(2) we have:***

$$\begin{cases} x=\frac{y}{2}=\frac{z}{3} \\ 6x+3y+2z=18 \end{cases}\Rightarrow x=1;y=2;z=3$$

78. In ΔABC the following relationship holds $(\forall z \in \mathbb{C})$:

$$|z - cosA - isinA| + |z - cosB - isinB| + |z - cosC - isinC| \geq 3(|z| - 1)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let $z = x + yi,\ (x, y \in \mathbb{R})$.

$$\to |z - \cos A - i\sin A| + |z - \cos B - i\sin B| + |z - \cos C - i\sin C|$$

$$= |x + yi - \cos A - i\sin A| + |x + yi - \cos B - i\sin B| + |x + yi - \cos C - i\sin C|$$

$$= |(x - \cos A) + (y - \sin A)i| + |(x - \cos B) + (y - \sin B)i| + |(x - \cos C) + (y - \sin C)i|$$

$$\geq |(x - \cos A) + (y - \sin A)i + (x - \cos B) + (y - \sin B)i + (x - \cos C) + (y - \sin C)i|$$

$$= \left|(3x - (\cos A + \cos B + \cos C)) + (3y - (\sin A + \sin B + \sin C))i\right|$$

$$= \sqrt{(3x - (\cos A + \cos B + \cos C))^2 + (3y - (\sin A + \sin B + \sin C))^2}$$

$$\overset{\sum \sin A \leq \frac{3\sqrt{3}}{2}}{\underset{\sum \cos A \leq \frac{3}{2}}{\geq}} \sqrt{\left(3x - \frac{3}{2}\right)^2 + \left(3y - \frac{3\sqrt{3}}{2}\right)^2} = \sqrt{9\left(x - \frac{1}{2}\right)^2 + 9\left(y - \frac{\sqrt{3}}{2}\right)^2}$$

$$= 3\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2} \overset{(*)}{\geq} 3(|z| - 1) = 3\left(\sqrt{x^2 + y^2} - 1\right);$$

- ***If*** $\sqrt{x^2 + y^2} - 1 < 0$ ***then*** $(*)$ ***is true.***
- ***If*** $\sqrt{x^2 + y^2} - 1 \geq 0$ ***then***

$$(*) \leftrightarrow \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2} \geq \sqrt{x^2 + y^2} - 1$$

$$\leftrightarrow x^2 + y^2 + 1 - (x + \sqrt{3}y) \geq x^2 + y^2 - 2\sqrt{x^2 + y^2} + 1$$

$$\leftrightarrow 2\sqrt{x^2 + y^2} \overset{(**)}{\geq} (x + \sqrt{3}y)$$

If $x + \sqrt{3}y < 0$ ***then*** $(**)$ ***is true***

If $x + \sqrt{3}y \geq 0$ ***then***

$$(**) \leftrightarrow 4(x^2 + y^2) \geq \left(x + \sqrt{3}y\right)^2$$

$$\leftrightarrow 4(x^2 + y^2) \geq x^2 + 2\sqrt{3}xy + 3y^2 \leftrightarrow 3x^2 - 2\sqrt{3}xy + y^2 \geq 0$$

$$\leftrightarrow \left(\sqrt{3}x - y\right)^2 \geq 0 \ \ (true\ for\ all\ x, y\ )$$

Equality if and only if $\begin{cases} y = \sqrt{3}x \\ \sqrt{x^2 + y^2} - 1 = 0 \end{cases} \leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{\sqrt{3}}{2} \end{cases}$

Solution 2 by George Florin Şerban-Romania

$$z = x + iy$$

$$\sum_{cyc} \sqrt{x^2 + y^2 + cos^2A + sin^2A - 2xcosA - 2ysinA} =$$

$$= \sum_{cyc} \sqrt{x^2 + y^2 + 1 - 2(xcosA + ysinA)} \overset{BCS}{\geq}$$

$$\geq \sum_{cyc} \sqrt{x^2 + y^2 + 1 - 2\sqrt{(x^2 + y^2)(cos^2A + sin^2A)}}$$

$$= \sum_{cyc} \sqrt{x^2 + y^2 + 1 - 2\sqrt{(x^2 + y^2)}} = \sum_{cyc} \sqrt{\left(\sqrt{(x^2 + y^2)} - 1\right)^2} =$$

$$= \sum_{cyc} \left|\sqrt{(x^2 + y^2)} - 1\right| \geq \sum_{cyc} (|z| - 1) = 3(|z| - 1),$$

true from $|t| \geq t, \forall t \in \mathbb{R}$

Solution 3 by Ravi Prakash-New Delhi-India

Let z_0 *–be any complex number such that* $|z_0| = 1$*. We have:*

$$|z| = |(z - z_0) + z_0| \leq |z - z_0| + |z_0| \to |z - z_0| \geq |z| - |z_0| \to$$

$$|z - z_0| \geq |z| - 1; (|z_0| = 1)$$

Let: $z_A = cosA + isinA, z_B = cosB + isinB, z_C = cosC + isinC$ *then*

$$|z_A| = |z_B| = |z_C| = 1 \to |z - z_A|, |z - z_B|, |z - z_C| \geq |z| - 1$$

Adding, we get: $|z - z_A| + |z - z_B| + |z - z_C| \geq 3(|z| - 1)$

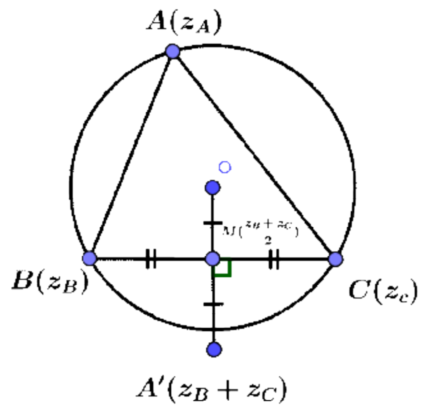
79. $z_A, z_B, z_C \in \mathbb{C}^*$ **- different in pairs,** $|z_A| = |z_B| = |z_C| = 1$**. Prove that:**

$$\sum_{cyc} |z_B(z_A - z_B) - z_C(z_A - z_C)| = 6\sqrt{3} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

Solution by Florentin Vișescu – Romania

$$|z_B(z_A - z_B) - z_C(z_A - z_C)| = |z_B z_A - z_B^2 - z_C z_A + z_C^2| =$$
$$= |z_A(z_B - z_C) - (z_B^2 - z_C^2)| = |z_B - z_C||z_A - (z_B + z_C)| =$$
$$= a|z_A - (z_B + z_C)| = a|z_A - z_A| = a \cdot AA'$$



A' symmetric of O towards M of BC, AM the median in ΔABC

$$AM^2 = \frac{2(AB^2 + AC^2) - BC^2}{4}; AM^2 = \frac{2(AO^2 + AA'^2) - OA'^2}{4}$$
$$\Rightarrow 2AB^2 + 2AC^2 - BC^2 = 2AO^2 + 2AA'^2 - OA'^2$$
$$OM = \sqrt{R^2 - \frac{a^2}{4}} \Rightarrow OA' = 2\sqrt{R^2 - \frac{a^2}{4}} \Rightarrow OA'^2 = 4R^2 - a^2$$
$$\Rightarrow 2c^2 + 2b^2 - a^2 = 2R^2 + 2AA'^2 - 4R^2 + a^2$$
$$\Rightarrow AA'^2 = c^2 + b^2 - a^2 + R^2 \Rightarrow AA' = \sqrt{c^2 + b^2 - a^2 + R^2}$$
$$\Rightarrow \sum_{cyc} a \cdot \sqrt{c^2 + b^2 - a^2 + R^2} = 6\sqrt{3}$$
$$\left(\sum_{cyc} a\sqrt{c^2 + b^2 - a^2 + R^2}\right)^2 \overset{CBS}{\leq} (a^2 + b^2 + c^2)(a^2 + b^2 + c^2 + 3R^2) \leq 9R^2 \cdot 12R^2;$$
$$(a^2 + b^2 + c^2 \leq 9R^2)$$
$$R = 1 \Rightarrow$$

$$\left.\begin{array}{r}\sum_{cyc} a\sqrt{c^2+b^2-a^2+R^2}R \le 108 \\ but \left(\sum_{cyc} a\sqrt{c^2+b^2-a^2+R^2}\right)^2 = 108\end{array}\right\} \Rightarrow \Delta ABC \text{ equilateral.}$$

80. Find $x, y, z > \frac{1}{2}max\{a, b, c\};\ a, b, c > 0$ such that:

$$\frac{(x+a)^2}{y+z-a} + \frac{(y+b)^2}{z+x-b} + \frac{(z+c)^2}{x+y-c} = 3(a+b+c)$$

Proposed by Marin Chirciu-Romania

Solution by Florică Anastase-Romania

$$\frac{(x+a)^2}{y+z-a} + \frac{(y+b)^2}{z+x-b} + \frac{(z+c)^2}{x+y-c} \overset{Bergström}{\ge} \frac{(a+b+c+x+y+z)^2}{2(x+y+z)-(a+b+c)} \Leftrightarrow$$

$$3(a+b+c) \ge \frac{(a+b+c+x+y+z)^2}{2(x+y+z)-(a+b+c)} \Leftrightarrow$$

$$(x+y+z)^2 + 2(x+y+z)(a+b+c) + (a+b+c)^2 \le$$

$$\le 3(a+b+c)[2(x+y+z)-(a+b+c)] \Leftrightarrow$$

$$(x+y+z)^2 - 4(a+b+c)(x+y+z) + 4(a+b+c)^2 \le 0$$

$$[(x+y+z) - 2(a+b+c)^2] \le 0 \Leftrightarrow x+y+z = 2(a+b+c)$$

and how $x+y \ne c; y+z \ne a; z+x \ne b$

$$\Rightarrow x = b+c; y = c+a; z = a+b.$$

81. Solve:

$$\left[\frac{x}{3}\right] - \left[\frac{x}{4}\right] = \frac{x}{7}$$

$[x]$ –is the greatest integer part of x

Proposed by Jalil Hajimir-Toronto-Canada

Solution by George Florin Şerban-Romania

$$\left[\frac{x}{3}\right] - \left[\frac{x}{4}\right] = \frac{x}{7}; \left[\frac{x}{3}\right], \left[\frac{x}{4}\right] \in \mathbb{Z} \to \frac{x}{7} \in \mathbb{Z}$$

$$\frac{x}{3} - \left\{\frac{x}{3}\right\} - \frac{x}{4} + \left\{\frac{x}{4}\right\} = \frac{x}{7}$$

$$\left\{\frac{x}{4}\right\} - \left\{\frac{x}{3}\right\} = \frac{x}{7} + \frac{x}{4} - \frac{x}{3} = \frac{5x}{84} \in (-1, 1) \leftrightarrow -1 < \frac{5x}{84} < 1 \leftrightarrow$$

$$-3 < -\frac{12}{5} < \frac{x}{7} < \frac{12}{5} < 3 \leftrightarrow \frac{x}{7} \in \{-2, -1, 0, 1, 2\}$$

If $\frac{x}{7} = 2 \to x = 14 \to \left[\frac{14}{3}\right] - \left[\frac{14}{4}\right] = 2 \to 4 - 3 = 2$ ***no solution.***

If $\frac{x}{7} = 1 \to x = 7 \to \left[\frac{7}{3}\right] - \left[\frac{7}{4}\right] = 1 \to 2 - 1 = 1$ ***true.***

If $\frac{x}{7} = 0 \to x = 0 \to \left[\frac{0}{3}\right] - \left[\frac{0}{4}\right] = 0$ ***true.***

If $\frac{x}{7} = -1 \to x = -7 \to \left[-\frac{7}{3}\right] - \left[-\frac{7}{4}\right] = -1 \to -3 + 2 = 1$ ***true.***

If $\frac{x}{7} = -2 \to x = -14 \to \left[-\frac{14}{3}\right] - \left[-\frac{14}{4}\right] = -2 \to -5 + 4 = -2$ ***no solution.***

So, $x \in \{-7, 0, 7\}$

82. Solve for real numbers:

$$\frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{1+x^3}} = 2\sqrt[3]{4}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Do Chinh-Vietnam

$$\frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{1+x^3}} = 2\sqrt[3]{4}, x \neq 1; (1) \Leftrightarrow \frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{(1+x)(1-x+x^2)}} = 2\sqrt[3]{4}$$

$$3 + \frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{1-x+x^2}} = 2\sqrt[3]{4(x+1)}.$$ ***Let:*** $f(x) = \frac{x}{\sqrt[3]{1-x+x^2}}, x \in \mathbb{R}$

$$f'(x) = \frac{-x^2 - 1}{3\sqrt[3]{(1-x+x^2)^2}}, f(x) < 0, \forall x \in \mathbb{R}$$

$f(x)$ ***–is decreasing function (2)***

Let: $g(x) = 2\sqrt[3]{4(x+1)} - 3, x \in \mathbb{R}$

$g'(x) = \frac{2\sqrt[3]{4}}{3\sqrt[3]{(x+1)^2}}, g(x) > 0, \forall x \in \mathbb{R}$. $g(x)$ ***–is increasing function (3)***

We have, $f(1) = g(1)$ ***and from*** $(2), (3) \Rightarrow x = 1$ ***is only solution.***

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\frac{3}{\sqrt[3]{1+x}} + \frac{x}{\sqrt[3]{1+x^3}} = 2\sqrt[3]{4} \Leftrightarrow \sqrt[3]{\frac{27}{1+x}} + \sqrt[3]{\frac{x^3}{1+x^3}} = 2\sqrt[3]{4}; \ (*)$$

Suppose: $\begin{cases}\frac{x^3}{1+x^3}=y\\ \frac{27}{1+x}=z\end{cases}\Rightarrow\begin{cases}x^3=\frac{y^3}{1-y^3}\\ x=\frac{27-z^3}{z^3}\end{cases}\Rightarrow\left(\frac{27-z^3}{z^3}\right)^3=\frac{y^3}{1-y^3};(1)$

and from $(*): z+y=2\sqrt[3]{4}$

Now, let be the function: $f(y)=\frac{y^3}{1-y^3}, y\neq 1$

$$f'(y)=\frac{3y^2}{(1-y^3)^2}; f'(y)=0\Rightarrow y=0\Rightarrow f(0)=0$$

y	$-\infty$ $\quad 0 \quad \alpha \quad 1 \quad +\infty$
$f'(y)$	$+\ +\ +\ +\ +\ +\ +$
$f(y)$	$-1\nearrow\nearrow 0\ \nearrow\nearrow +\infty / -\infty \nearrow\nearrow 1$

$$f(\alpha)=1\Rightarrow\frac{y^3}{1-y^3}=1\Rightarrow y^3=\frac{1}{2}\Rightarrow y=\frac{1}{\sqrt[3]{2}}$$

Now, let be the function: $g(z)=\left(\frac{27-z^3}{z^3}\right)^3; g'(z)=3\left(\frac{27-z^3}{z^3}\right)^2\left(\frac{-3z^2\cdot 27}{z^6}\right),$

$$g'(z)=0\Rightarrow z=3, g(z)=0$$

z	$-\infty$ $\quad 0 \quad \beta \quad 3 \quad +\infty$
$g'(z)$	$----/--0---$
$g(z)$	$-1\searrow\searrow\ -\infty/+\infty\searrow\searrow\ 0\ \searrow\searrow -1$

$$g(\beta)=1\Rightarrow\left(\frac{27-z^3}{z^3}\right)^3=1\Rightarrow z^3=\frac{27}{2}\Rightarrow z=\frac{3}{\sqrt[3]{2}}$$

$$\text{So, } y^3=\frac{1}{2}\Rightarrow\frac{x^3}{1+x^3}=\frac{1}{2}\Rightarrow x^3=1\Rightarrow x=1.$$

$$z^3=\frac{27}{2}\Rightarrow\frac{27}{1+x}=\frac{27}{2}\Rightarrow x=1.$$

$$\text{So, } S=\{1\}.$$

83. Solve for real numbers:

$$4(sinx+2cosy)+3(cosx+2siny)=15$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$4(sinx+2cosy)+3(cosx+2siny)=15$$

$$(4sinx + 3cosy)^2 \overset{CBS}{\leq} (4^2 + 3^2)(sin^2x + cos^2x) = 25$$

$$4sinx + 3cosx \leq 5; (1)$$

$$(8cosy + 6siny)^2 \overset{CBS}{\leq} (8^2 + 6^2)(sin^2y + cos^2y) = 100$$

$$8cosy + 6siny \leq 10; (2)$$

From $(1), (2) \Rightarrow 4sinx + 3cosx + 8cosy + 6siny \leq 15$

Equality holds $\frac{4}{sinx} = \frac{3}{cosx} \Rightarrow \frac{4}{3} = \frac{sinx}{cosx} = tanx \Rightarrow x = tan^{-1}\frac{4}{3} + k\pi, k \in \mathbb{Z}$

And $\frac{8}{cosy} = \frac{6}{siny} \Rightarrow \frac{8}{6} = \frac{cosy}{siny} \Rightarrow tany = \frac{3}{4} \Rightarrow y = tan^{-1}\frac{3}{4} + q\pi, q \in \mathbb{Z}$

Solution 2 by Florentin Vişescu-Romania

$$15 = 4(sinx + 2cosy) + 3(cosx + 2siny) \Leftrightarrow$$

$$225 = [4(sinx + 2cosy) + 3(cosx + 2siny)]^2 \leq$$

$$\leq (4^2 + 3^2)((sinx + 2cosy)^2 + (cosx + 2siny)^2)$$

$$= 25\left(sin^2x + cos^2x + 4(sin^2y + cos^2y)\right) = 25(5 + 4sin(x + y)) \Leftrightarrow$$

$$9 \leq 5 + 4sin(x + y) \Leftrightarrow 4 \leq 4sin(x + y) \Leftrightarrow sin(x + y) \geq 1$$

$$\Leftrightarrow sin(x + y) = 1 \Leftrightarrow x + y = 2n\pi + \frac{\pi}{2} \Leftrightarrow y = 2n\pi + \frac{\pi}{2} - x$$

$$15 = 4\left(sinx + 2cos\left(2n\pi + \frac{\pi}{2} - x\right)\right) + 3\left(cosx + 2sin\left(2n\pi + \frac{\pi}{2} - x\right)\right)$$

$$15 = 4(sinx + 2sinx) + 3(cosx + 2cosx) \Leftrightarrow$$

$$5 = 4sinx + 3cosx$$

$$25 = (4sinx + 3cosx)^2 \leq (4^2 + 3^2)(sin^2x + cos^2x) \Leftrightarrow 25 \leq 25$$

$$\frac{sinx}{4} = \frac{cosx}{3} \Leftrightarrow xy = \frac{4}{3} \Rightarrow \begin{cases} x = k\pi + tan^{-1}\frac{4}{3} \\ y = 2p\pi + \frac{\pi}{2} - x \end{cases}$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

Let: $cos\theta = \frac{4}{5}, sin\theta = \frac{3}{5}; (*)$

$$4(sinx + 2cosy) + 3(cosx + 2siny) = 15 \Leftrightarrow$$

$$5\left(\frac{4}{5}sinx + \frac{3}{5}cosx\right) + 10\left(\frac{8}{10}cosy + \frac{6}{10}siny\right) = 15$$

$$5\left(\frac{4}{5}sinx+\frac{3}{5}cosx\right)+10\left(\frac{4}{5}cosy+\frac{3}{5}siny\right)=15$$

$$5(sin\theta sinx+cos\theta cosx)+10(sin\theta cosy+cos\theta siny)=15$$

$$5cos(x-\theta)+10sin(y+\theta)=15$$

$cos(x-\theta)+2sin(y+\theta)=3$ ***is satisfying when***

$$x-\theta=0, y+\theta=\frac{\pi}{2}\Rightarrow\theta=x, sinx=\frac{4}{5}, cosx=\frac{3}{5}\Rightarrow tanx=\frac{4}{3}$$

$$\Rightarrow x=tan^{-1}\frac{4}{3}$$

But $x+y=\frac{\pi}{2}\Rightarrow y=\frac{\pi}{2}-x=\frac{\pi}{2}-tan^{-1}\frac{4}{3}=tan^{-1}\frac{3}{4}$

So, $S=\{x=tan^{-1}\frac{4}{3}+k\pi, y=tan^{-1}\frac{3}{4}+k\pi$

Solution 4 by Bedri Hajrizi-Serbie

$$4(sinx+2cosy)+3(cosx+2siny)=15$$

$$3cosx+4sinx+8cosy+6siny=15$$

$$5\cdot\frac{3cosx+4sinx}{5}+10\cdot\frac{8cosy+6siny}{10}=10$$

$$5cos\left(x-cos^{-1}\frac{3}{5}\right)+10cos\left(y-cos^{-1}\frac{4}{5}\right)=15$$

$$\begin{cases}cos\left(x-cos^{-1}\frac{3}{5}\right)=1\\ cos\left(y-cos^{-1}\frac{4}{5}\right)=1\end{cases}\Rightarrow\begin{cases}x=cos^{-1}\frac{3}{5}+2m\pi\\ y=cos^{-1}\frac{4}{5}+2n\pi\end{cases}; m,n\in\mathbb{Z}$$

84. Solve for real numbers:

$$15cosx\cdot cosy\cdot cosz+4\sum_{cyc}cos5x\cdot cosy\cdot cosz=0, -\frac{\pi}{2}<x,y,z<\frac{\pi}{2}$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Vişescu-Romania

$$15cosx\cdot cosy\cdot cosz+4\sum_{cyc}cos5x\cdot cosy\cdot cosz=0\Leftrightarrow$$

$$\sum_{cyc}(5\cos x\cdot\cos y\cdot\cos z+4\cos 5x\cdot\cos y\cdot\cos z)=0$$

$$\sum_{cyc}\cos y\cdot\cos z(5\cos x+4\cos 5x)=0$$

$$\sum_{cyc}\cos y\cdot\cos z\left(5\cos x+64\cos^5 x-80\cos^3 x+20\cos x\right)=0$$

$$\sum_{cyc}\cos x\cdot\cos y\cdot\cos z(64\cos^4 x-80\cos^2 x+25)=0$$

$$\cos x\cdot\cos y\cdot\cos z\sum_{cyc}(8\cos^2 x-5)^2=0$$

For $-\frac{\pi}{2}<x,y,z<\frac{\pi}{2}$ **we have:** $\cos x\cdot\cos y\cdot\cos z\neq 0$ *then*

$$\sum_{cyc}(8\cos^2 x-5)^2=0$$

$$8\cos^2 x-5=8\cos^2 y-5=8\cos^2 z-5=0;\ -\frac{\pi}{2}<x,y,z<\frac{\pi}{2}$$

Equality holds for:

$$cox=\cos y=\cos z=\sqrt{\frac{5}{8}}$$

85. We consider the equation:

$$x^4-8x^3+12x^2-8x+2=0$$

Show that the equation has two real roots, of which the largest is:

$$a=2+\sqrt[4]{2}+\sqrt[4]{4}+\sqrt[4]{8}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Abner Chinga Bazo-Lima-Peru

$$x^4-8x^3+12x^2-8x+\left(2^2-\sqrt{2^2}\right)=0$$

$$12x^2-4x^2=8x^2=16-8$$

$$\left(x^2-(4+2\sqrt{2})x+(2+\sqrt{2})\right)\left(x^2-(4-2\sqrt{2})x+(2-\sqrt{2})\right)=0 \text{ then}$$

$$x^2 - (4 + 2\sqrt{2})x + (2 + \sqrt{2}) = 0 \text{ or } x^2 - (4 - 2\sqrt{2})x + (2 - \sqrt{2}) = 0$$

$$x = \frac{(4+2\sqrt{2}) \pm \sqrt{(4+2\sqrt{2})^2 - 4(2+\sqrt{2})}}{2} \text{ or } x = \frac{(4-2\sqrt{2}) \pm \sqrt{(4-2\sqrt{2})^2 - 4(2-\sqrt{2})}}{2}$$

$$x = \frac{4+2\sqrt{2} \pm \sqrt{16+12\sqrt{2}}}{2} \text{ or } x = \frac{4-2\sqrt{2} \pm \sqrt{16-12\sqrt{2}}}{2}$$

$$x = \frac{4+2\sqrt{2} \pm 2\sqrt{\sqrt{2}(3+2\sqrt{2})}}{2} \text{ or } x \in \mathbb{C}$$

$$x = 2 + \sqrt{2} \pm \sqrt[4]{2}\sqrt{3 + 2\sqrt{2}}$$

$$x = 2 + \sqrt[4]{2} + \sqrt[4]{4} + \sqrt[4]{8};\ x = 2 + \sqrt[4]{2} - \sqrt[4]{4} - \sqrt[4]{8}$$

Solution 2 by Orlando Irahola Ortega-La Paz-Bolivia

$$x^4 - 8x^3 + 12x^2 - 8x + 2 = 0 \Rightarrow$$

$$(x-2)^4 - 12(x-2)^2 - 24(x-2) - 14 = 0 \xRightarrow{t=x-2}$$

$$t^4 - 12t^2 - 24t - 14 = 0 \Rightarrow t^4 - 4t^2 + 4 = 8t^2 + 24t + 18$$

$$(t^2 - 2)^2 = (2\sqrt{2}t + 3\sqrt{2})^2 \Rightarrow t^2 - 2 = \pm(2\sqrt{2}t + 3\sqrt{2});\ \text{or } t \in \mathbb{C} \Rightarrow$$

$$t^2 - 2\sqrt{2}t - (3\sqrt{2} + 2) = 0 \Rightarrow t = \sqrt{2} \pm \sqrt{4 + 3\sqrt{2}} \Rightarrow$$

$$t = \sqrt{2} \pm \sqrt{\sqrt{2}(2\sqrt{2} + 3)} \Rightarrow$$

$$t = \sqrt{2} \pm \sqrt[4]{2} \cdot \sqrt{(\sqrt{2} + 1)^2} = \sqrt{2} \pm \sqrt[4]{2}(\sqrt{2} + 1)$$

$$t_1 = \sqrt{2} + \sqrt[4]{3} + \sqrt[4]{8} \xRightarrow{t=x-2} x = 2 + \sqrt[4]{2} + \sqrt[4]{4} + \sqrt[4]{8}$$

Solution 3 by Carlos Paiva-Brazil

$$x^4 - 8x^3 + 12x^2 - 8x + 2 = 0 \Rightarrow x^4 - 8x^3 = -12x^2 + 8x - 2$$

$$x^4 - 8x^3 + 16x^2 = 4x^2 + 8x - 2 \Rightarrow$$

$$(x^2 - 4x + \lambda)^2 = 4x^2 + 8x - 2 + 2\lambda(x^2 - 4x) + \lambda^2$$

$$= (2\lambda + 4)x^2 - (8\lambda - 8)x + (\lambda^2 - 2)$$

$$[-(8\lambda - 8)]^2 - 4(2\lambda + 4)(\lambda^2 - 2) = 0$$

$$8\lambda^3 - 48\lambda^2 + 112\lambda - 16 = 0 \Leftrightarrow \lambda^3 - 6\lambda^2 + 14\lambda - 12 = 0 \Leftrightarrow$$

$$(\lambda - 2)(\lambda^2 - 4\lambda + 6) = 0 \Rightarrow \lambda_1 = 2.$$

Then: $(x^2-4x+2)^2=(2\cdot 2+4)x^2-(8\cdot 2-8)x+(2^2-2)$

$$(x^2-4x+2)^2=8x^2-8x+2$$

$$(x^2-4x+2)^2=2(4x^2-4x+1)=2(2x-1)^2$$

$$x^2-4x+2=\pm\sqrt{2}(2x-1)$$

Case 1: $x^2-4x+2=\sqrt{2}(2x-1)$

$$x^2-(2\sqrt{2}+4)x+(2+\sqrt{2})=0\Rightarrow x_{1,2}=2+\sqrt{2}\pm\sqrt{4+2\sqrt{2}}$$

$$x_{1,2}=2+\sqrt{2}\pm\sqrt{\sqrt{2}(2\sqrt{2}+3)}=2+\sqrt{2}\pm\sqrt[4]{2}\left(\sqrt{2\sqrt{2}+3}\right)$$

$$=2+\sqrt{2}\pm\sqrt[4]{2}(\sqrt[4]{4}+\sqrt[4]{1})=2\pm\sqrt[4]{2}+\sqrt[4]{4}\pm\sqrt[4]{8}$$

Case 2: $x^2-4x+2=-\sqrt{2}(2x-1)$

$$x^2-(4-2\sqrt{2})x+(2-\sqrt{2})=0$$

$$x_{3,4}=2-\sqrt[4]{4}\pm i(\sqrt[4]{8}-\sqrt[4]{2})$$

All solutions, including the complex roots:

$$x_{1,2}=2\pm\sqrt[4]{2}+\sqrt[4]{4}\pm\sqrt[4]{8}$$

$$x_{3,4}=2-\sqrt[4]{4}\pm i(\sqrt[4]{8}-\sqrt[4]{2})$$

86. Find all functions $f:\mathbb{R}\to\mathbb{R}$ is such that:

$$f(x)f(y)+3f(y)+3y=f(x+y)+f(-xy)+2f(x)+2x,\forall x,y\in\mathbb{R}$$

Proposed by Rajeev Rastogi-India

Solution 1 by Catinca Alexandru-Romania

We chose $y=0\Rightarrow (x)f(0)+3f(0)=f(x)+f(0)+2f(x)+2x\Leftrightarrow$

$$f(x)(f(0)-3)=-2f(0)+2x$$

Let: $f(0)=a\Rightarrow f(x)(a-3)=-2a+2x$

i) *If* $a\neq 3\Rightarrow f(x)=\frac{2x-2a}{a-3}\overset{x=0}{\Longrightarrow} a=-\frac{2a}{a-3}\Rightarrow a\in\{0,1\}$

i.i) For $\begin{cases}a=0\\ f(0)=0\end{cases}\Rightarrow f(x)=-\frac{2}{3}x,\forall x\in\mathbb{R}$

We check in the condition

$$\left(-\frac{2}{3}x\right)\left(-\frac{2}{3}y\right)+3\left(-\frac{2}{3}y\right)=\left(-\frac{2}{3}(x+y)\right)+\left(-\left(-\frac{2}{3}xy\right)\right)-2\cdot\frac{2}{3}x+2x$$

$$\Leftrightarrow\frac{4xy}{9}-6y=-\frac{2}{3}x-\frac{2}{3}y+\frac{2}{3}xy-\frac{4}{3}x+2x$$

$\Leftrightarrow\frac{-2xy}{9}-\frac{16y}{3}=0$ *which is not true for any* $x,y\in\mathbb{R}$

i.ii) For $\begin{cases}a=1\\ f(x)=-\frac{2x-2}{2}=1-x\end{cases}$

We check the condition

$$(1-x)(1-y)+3(1-y)+3y=(-x-y+1)+(1+xy)+2(1-x)+2x$$

$$\Leftrightarrow 1-x-y+xy+3-3y+3y=-x-y+1+1+xy+2-2x+2x$$

$0=0$ *true.*

So: $f(x)=1-x\Rightarrow f(2020)=-2019$

ii) *If* $a=0\Rightarrow 0=-6+2x\Leftrightarrow x=3$ *for every* $x\in\mathbb{R}$ *impossible.*

Therefore $f(2020)=-2019$

Solution 2 by Sagar Kumar-Kolkata-India

$$f(x)f(y)+3f(y)+3y=f(x+y)+f(-xy)+2f(x)+2x$$

For $x=y=0\Rightarrow f^2(0)-f(0)=0\Rightarrow f(0)\in\{0,1\}$

Put $x=0,y=x\Rightarrow f(x)(f(0)+2)=3f(0)-3x\Rightarrow f(x)=\frac{3(f(0)-x)}{f(0)+2}$

i) $f(0)=0\Rightarrow f(x)=-\frac{3}{2}x$

ii) $f(0)=1\Rightarrow f(x)=1-x$

But $f(x)=-\frac{3}{2}x$ *do not satisfy given equation and* $f(x)=1-x$ *satisfies it*

$$\Rightarrow f(x)=1-x;f(2020)=-2019$$

Note: $x=0,y=1,f(1)=\frac{-2f(0)+2}{f(0)-3}$

If $f(0)=0,f(1)=-\frac{2}{3}$

But $f(x)=-\frac{3}{2}x$ *is not satisfied.*

If $f(0)=1,f(1)=0$ *is satisfied.*

So: $f(x)=1-x;f(2020)=-2019$

87. **Find all continuous functions $f:[0,\infty)\to(0,\infty)$ such that:**

$$f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\sqrt{f(x)\cdot f(y)},\forall x,y\geq 0$$

Proposed by Marian Ursărescu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$f(x)>0,\forall x\in[0,\infty)\Rightarrow f(0)>0$$

Note: $f^{\alpha}(x)=[f(x)]^{\alpha}$

Now, let $y=0\Rightarrow f\left(\sqrt{\frac{x^2+0^2}{2}}\right)=\sqrt{f(x)\cdot f(0)}\Leftrightarrow$

$$f\left(\frac{x}{\sqrt{2}}\right)=\sqrt{f(x)\cdot f(0)}=\sqrt{f(x)}\cdot\sqrt{f(0)}\Rightarrow$$

$$f(x)=\frac{\left[f\left(\frac{x}{\sqrt{2}}\right)\right]^2}{f(0)}=\frac{1}{f(0)}\cdot\left[f\left(\frac{x}{\sqrt{2}}\right)\right]^2=\frac{1}{f(0)}\cdot\left[\frac{1}{f(0)}\cdot f^2\left(\frac{x}{\sqrt{2}^2}\right)\right]^2=\frac{1}{f^3(0)}\cdot\left[f\left(\frac{x}{\sqrt{2}^2}\right)\right]^4$$

$$=\frac{1}{f^3(0)}\cdot\left[\frac{1}{f(0)}\cdot f^2\left(\frac{x}{\sqrt{2}^3}\right)\right]^4$$

$$=\frac{1}{f^7(0)}\cdot f^8\left(\frac{x}{\sqrt{2}^3}\right)=\cdots=\frac{1}{f^{2^n-1}(0)}\cdot f^{2^n}\left(\frac{x}{\sqrt{2}^n}\right),\forall n\in\mathbb{N},n\geq 1$$

$$\xRightarrow[\text{on }[0,\infty)]{f\text{ continuous}}\frac{x}{\sqrt{2}^n}\xrightarrow[n\to\infty]{}0\Rightarrow f^{2^n}\left(\frac{x}{\sqrt{2}^n}\right)\to f^{2^n}(0)$$

$$f(x)=\frac{f^{2^n}(0)}{f^{2^n-1}(0)}=f(0),\forall x\in[0,\infty)\Rightarrow f(x)=f(0)=c,(c>0)$$

88. **Solve for real numbers:**

$$\begin{cases}0\leq x,y,z\leq 2\\ \dfrac{x}{y+z+1}+\dfrac{y}{z+x+1}+\dfrac{z}{x+y+1}+xye^z=\dfrac{6}{5}+4e^2\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Jalil Hajimir-Toronto-Canada

$$\text{Let: } f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z$$

$$f_2(x,y,z)=\frac{y}{z+x+1}+\frac{1}{3}xye^z$$

$$f_3(x,y,z)=\frac{z}{x+y+1}+\frac{1}{3}xye^z$$

$f(x,y,z)=f_1(x,y,z)+f_2(x,y,z)+f_3(x,y,z)$ ***is convex function.***

$A=\{x,y,z\in\mathbb{R},0\le x,y,z\le 2\}$ ***is a closed convex set***

$f(2,2,2)=\frac{6}{5}+4e^2$ ***is the greatest value, then*** $f(x,y,z)\le\frac{6}{5}+4e^2$ ***is sum of there convex functions is convex.***

Let's prove $f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z$ ***is convex.***

$\nabla^2(f)$ ***is a positive semi definite matrix.***

$$\begin{pmatrix} 0 & e^z-\frac{1}{(y+z+1)^2} & ye^z-\frac{1}{(y+z+1)^2} \\ e^z-\frac{1}{(y+z+1)^2} & \frac{2x}{(y+z+1)^3} & xe^z+\frac{2x}{(y+z+1)^3} \\ ye^z-\frac{1}{(y+z+1)^2} & xe^z+\frac{2x}{(y+z+1)^3} & xye^z+\frac{2x}{(y+z+1)^3} \end{pmatrix}$$

Therefore, $x=y=z=2$ ***is only solution for the given equation.***

89. Solve for real numbers:

$$\begin{cases} x+y+[z]=37 \\ x+2[y]+3[z]=47 \end{cases};\ [*]-\text{great integer function.}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Adrian Popa-Romania

$$\begin{cases} x+y+[z]=37 \\ x+2[y]+3[z]=47 \end{cases}\Rightarrow\begin{cases} x+y=37-[z] \\ x=47-2[y]-3[z] \end{cases}\Rightarrow\begin{cases} x+y\in\mathbb{Z} \\ x\in\mathbb{Z} \end{cases}\Rightarrow x,y\in\mathbb{Z}$$

$$\text{Let: } [z]=\alpha\in\mathbb{Z}$$

$$\begin{cases} x+y+\alpha=37 \\ x+2y+\alpha=47 \end{cases}\Rightarrow\begin{cases} x+y=37-\alpha \\ x+2y=47-\alpha \end{cases}\Rightarrow y=10-2\alpha\Rightarrow x=27+\alpha$$

$$(x,y,z)\in\{(27+\alpha,10-\alpha,\beta)/\beta\in(\alpha-1;\alpha];\alpha\in\mathbb{Z},\}$$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

$$\begin{cases}x+y+[z]=37; & (1)\\ x+2[y]+3[z]=47;(2)\end{cases}\Rightarrow\begin{cases}x+y=37-[z]\in\mathbb{Z}\\ x=47-2[y]-3[z]\in\mathbb{Z}\end{cases}\Rightarrow\begin{cases}x=[x]\\ y=[y]\end{cases}\Rightarrow x,y\in\mathbb{Z}$$

$$\textbf{Let: } [z]=k\in\mathbb{Z}\overset{(2)-(1)}{\Longrightarrow}y=10-k, x=k+27$$

$$(x,y,z)\in\{([\alpha]+27,10-2[\alpha],\alpha);\ \alpha\in\mathbb{R}\}$$

90. Solve for real numbers:

$$\begin{cases}x,y\geq 0;[*]-great\ integer\ function\\ (x+2)(y+3)=8\\ \sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{xy}\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$(x+2)(y+3)=8\ldots(1)$$

$$\sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{xy}\ldots(2)$$

$$\begin{cases}x\geq 0\Rightarrow x+2\geq 2\\ y\geq 0\Rightarrow y+3\geq 3\\ (x+2)(y+3)=8\end{cases}\Rightarrow\begin{cases}x<1\Rightarrow[x]=0\\ y\leq 1\Rightarrow[y]=0\ or\ [y]=1\end{cases}$$

$$i)\ [x]=0\ and\ [y]=0\overset{by\ (2)}{\Longrightarrow}\sqrt{xy}=\sqrt{xy}$$

$$ii) If\ x=0\ then\ y=1$$

$$iii) If\ x>0\ then\ y+3=\frac{8}{x+2}$$

$$y=\frac{2-3x}{x+2}>0\ then\ x<\frac{2}{3}\ldots(3)$$

$$\frac{2-3x}{x+2}<1\ \ then\ x>0\ldots(4)$$

From (3)+(4) we have:

$$x\in\left(0,\frac{2}{3}\right)\ and\ y=\frac{2-3x}{x+2}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Because:

$$x;y\geq 0 \text{ then } [x];[y]\geq 0 \text{ and } x\geq [x]; y\geq [y]$$

$$\{x\}:=x-[x];\{y\}:=y-[y]$$

Now, $\sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{[x]\cdot[y]}+\sqrt{\{x\}\cdot\{y\}}$

$$\overset{BCS}{\leq}\sqrt{\left(\sqrt{[x]}^2+\sqrt{\{x\}}^2\right)\cdot\left(\sqrt{[y]}^2+\sqrt{\{y\}}^2\right)}$$

$$\leq\sqrt{(\{x\}+[x])(\{y\}+[y])}=\sqrt{xy}$$

Equality for

$$x=y=\alpha>0 \text{ or } \begin{cases}x=0\\ y\geq 0\end{cases} \text{ or } \begin{cases}y=0\\ x\geq 0\end{cases} \text{ or } \begin{cases}x=[x]\\ y=[y]\end{cases}$$

If $x=0$ *then* $(0+2)(y+3)=0$ *and* $y=1\geq 0$

If $y=0$ *then* $(x+2)(0+3)=8$ *and* $x=\frac{2}{3}\geq 0$

$$(x+2)(y+3)=8\overset{x=y=\alpha}{\Longrightarrow}(\alpha+2)(\alpha+3)=8$$

$$\text{then } \alpha^2+5\alpha-2=0\Rightarrow\begin{cases}x=y=\dfrac{-5+\sqrt{33}}{2}\\ x=y=\dfrac{-5-\sqrt{33}}{2}\end{cases}$$

But: $x,y\geq 0\Rightarrow x=y=\frac{-5+\sqrt{33}}{2}$

$$x=[x]\in\mathbb{Z}^+;y=[y]\in\mathbb{Z}^+$$

$$(x+2)(y+3)=8\Rightarrow x=\frac{8}{y+3}-2\overset{y\in\mathbb{Z}^+}{\Longrightarrow}y+3\in\{4,8\}\overset{x,y\in\mathbb{Z}^+}{\Longrightarrow}y=1;x=0$$

So,

$$(x,y)\in\left\{(0;1);\left(\frac{2}{3};0\right);\left(\frac{-5+\sqrt{33}}{2};\frac{-5+\sqrt{33}}{2}\right)\right\}$$

91. Solve for real numbers:

$$\begin{cases}5\left(\sqrt[5]{x}+\sqrt[5]{y}\right)-2\left(\sqrt{x}+\sqrt{y}\right)=6\\ xz(z^4+10z^2+5)=y(5z^4+10z^2+1)\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Rahim Shahbazov-Baku-Azerbijan

It is clear that: $x>0,y>0$

Let: $x = a^{10}, y = b^{10} \to 5(a^2 + b^2) - 2(a^5 + b^5) = 6 \to$

$$2a^5 + 3 + 2b^5 + 3 = 5a^2 + 5b^2$$

$$\begin{cases} 2a^5 + 3 = a^5 + a^5 + 1 + 1 + 1 \geq 5\sqrt[5]{a^{10}} \\ 2b^5 + 3 \geq 5b^2 \end{cases} \to a = b = 1 \to x = y = 1$$

Then $2(z^4 + 10z^2 + 5) = 5z^4 + 10z^2 + 1 \leftrightarrow$

$$z^5 + 5z^4 + 10z^3 - 10z^2 + 5z - 1 = 0 \leftrightarrow (z-1)^5 = 0 \leftrightarrow z = 1.$$

So, $(x, y, z) = (1, 1, 1)$

92. $z_A, z_B, z_C \in \mathbb{C}^*$ **–different in pairs,** $|z_A| = |z_B| = |z_C| = 1$

$a = BC, b = CA, c = AB$. **Prove that:**

$$\left|\prod_{cyc} b(z_A - z_B) + c(z_A - z_C)\right| = (a + b + c)^3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vişescu-Romania

$$\left|\prod_{cyc} b(z_A - z_B) + c(z_A - z_C)\right| = (a + b + c)^3, |z_A| = |z_B| = |z_C| = 1 \Leftrightarrow$$

$$\prod_{cyc}\left|\frac{b(z_A - z_B) + c(z_A - z_C)}{a + b + c}\right| = 1 \Rightarrow \prod_{cyc}\left|\frac{(a + b + c)z_A}{a + b + c} - z_I\right| = 1 \Rightarrow$$

$$\prod_{cyc}|z_A - z_I| = 1 \Rightarrow |z_A - z_I| \cdot |z_B - z_I| \cdot |z_C - z_I| = 1 \Rightarrow AI \cdot BI \cdot CI = 1$$

$$\Rightarrow \frac{r^3}{sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}} = 1 \Rightarrow r^3 = sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2} \Rightarrow$$

$$r^3 = \frac{(s-a)(s-b)(s-c)}{abc} \Rightarrow r^3 = \frac{S^2}{4RsS} \Rightarrow r^3 = \frac{S}{4Rs} \Rightarrow r^3 = \frac{rs}{4Rs}$$

$$\Rightarrow r^2 = \frac{1}{4R} \overset{R=1}{\Longrightarrow} r = \frac{R}{2} \Rightarrow AB = BC = CA$$

93. Solve for real numbers:

$$x + 9^{\log_x 27} + x \cdot 9^{\log_x 27} = 279$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vişescu-Romania

$$x \in (0,1) \cup (1,\infty)$$

$$\textit{If } x \in (0,1) \Rightarrow \log_x 27 < 0 \Rightarrow 0 < \log_x 27 < 1; 0 < x < 1 \Rightarrow$$

$$x + 9^{\log_x 27} + x \cdot 9^{\log_x 27} \in (0,3) \Rightarrow \textit{not solution.}$$

$$\textit{If } x \in (1,\infty). \text{ Let } t = \log_x 27 > 0 \Rightarrow \frac{1}{\log_{27} x} = \frac{3}{\log_3 x} = t \Rightarrow$$

$$\log_3 x = \frac{3}{t} \Rightarrow x = 3^{\frac{3}{t}}.$$

$$\textit{The equation becomes: } 3^{\frac{3}{t}} + 3^{2t} + 3^{\frac{3}{t}+2t} = 279.$$

$$\textit{Let be the function } f:(0,\infty) \to \mathbb{R}, f(t) = 3^{\frac{3}{t}} + 3^{2t} + 3^{\frac{3}{t}+2t}$$

$$f'(t) = \left(-\frac{3}{t^2} \cdot 3^{\frac{3}{t}} + 2 \cdot 3^{2t} + \left(2 - \frac{3}{t^2}\right) 3^{\frac{3}{t}+2t}\right) log3$$

$$f''(t) = \left(\frac{9}{t^4} \cdot 3^{\frac{3}{t}} log3 + \frac{6}{t^3} \cdot 3^{\frac{3}{t}} + 4 \cdot 3^{2t} log3 + \left(2 - \frac{3}{t^2}\right)^2 3^{\frac{3}{t}+2t} log3 + \frac{6}{t^3} \cdot 3^{\frac{3}{t}+2t}\right) log3 > 0 \Rightarrow$$

$$f - \textit{convexe} \Rightarrow t \in \left\{1; \frac{3}{2}\right\} \Rightarrow x \in \{9; 27\}$$

94. Find all $x, y, z > 0$ such that:

$$4sinx \cdot siny \cdot sinz \cdot sin(x+y+z) = 1$$

Proposed by Daniel Sitaru-Romania

Solution by Marian Dincă-Romania

$$4sinx \cdot siny \cdot sinz \cdot sin(x+y+z) \le 1$$

$$2sinx \cdot siny = cos(x-y) - cos(x+y)$$

$$2sinz \cdot sin(x+y+z) = cos(x+y) - cos(2z+x+y)$$

$$[cos(x-y) - cos(x+y)][cos(x+y) - cos(2z+x+y)] =$$

$$= -cos^2(x+y) + cos(x+y)[cos(x+y) - cos(2z+x+y)]$$

$$- cos(x-y)cos(2z+x+y) - 1 \le 0$$

$$\cos^2(x+y) - \cos(x+y)[\cos(x+y) - \cos(2z+x+y)] + \cos(x-y)\cos(2z+x+y) + 1 \ge 0$$

Let: $\cos(x+y) = t$

$$t^2 - t[\cos(x+y) - \cos(2z+x+y)] + \cos(x-y)\cos(2z+x+y) + 1 \ge 0$$

$$\left[t - \frac{1}{2}(\cos(x-y) + \cos(2z+x+y))\right]^2 - $$
$$-\frac{1}{4}[\cos(x-y) + \cos(2z+x+y)]^2 + \cos(x-y)\cos(2z+x+y) + 1 \ge 0$$

$$\left[t - \frac{1}{2}(\cos(x-y) + \cos(2z+x+y))\right]^2 - $$
$$+\frac{4 - [\cos(x-y) - \cos(2z+x+y)]^2}{4} \ge 0$$

But: $\cos(x-y) - \cos(2z+x+y) = 2\sin(z+y)\sin(z+x)$. **So,**

$4 - [\cos(x-y) - \cos(2z+x+y)]^2 = 4 - 4\sin^2(z+y)\sin^2(z+x) \ge 0$ **because:**

$$0 \le \sin^2(z+y) \le 1; 0 \le \sin^2(z+x) \le 1$$

So, the inequality

$4\sin x \cdot \sin y \cdot \sin z \cdot \sin(x+y+z) \le 1$ **is true for any** $x, y, z \in \mathbb{R}$.

Equality holds when: $\sin(z+y) = \pm 1; \sin(z+x) = \pm 1$ **result:**

$\cos(x+z) = 0; \cos(y+z) = 0; t - \frac{1}{2}[\cos(x-y) + \cos(2z+x+y)] = 0$. **Or**

$$2\cos(x+y) = \cos(x-y) + \cos(2z+x+y) = 2\cos(x+z)\cos(y+z)$$

But: $\cos(x+z) = 0; \cos(y+z) = 0 \Rightarrow \cos(x+y) = 0$

So, $x + z = (2a+1)\frac{\pi}{2}; y + z = (2b+1)\frac{\pi}{2}; x + y = (2c+1)\frac{\pi}{2}$

$$x = (2a - 2b + 2c + 1)\frac{\pi}{4}; y = (-2a + 2b + 2c + 1)\frac{\pi}{4}$$

$$z = (2a + 2b - 2c + 1)\frac{\pi}{4}; a, b, c \in \mathbb{Z}$$

95. $$\begin{cases} f, g, h: \mathbb{R} \to \mathbb{R} \\ f(x) + g(x) + h(x) = 3x + 3, \forall x \in \mathbb{R} \\ f^2(x) + g^2(x) + h^2(x) = 3x^2 + 6x + 5, \forall x \in \mathbb{R} \\ f^3(x) + g^3(x) + h^3(x) = 3x^3 + 9x^2 + 15x + 9, \forall x \in \mathbb{R} \end{cases}$$

Solve for real numbers: $f(x) \cdot g(x) \cdot h(x) = 0$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

Denote: $$\begin{cases} f(x) + g(x) + h(x) = S_1 \\ f(x) \cdot g(x) + g(x) \cdot h(x) + h(x) \cdot f(x) = S_2 \\ f(x) \cdot g(x) \cdot h(x) = S_3 \end{cases}$$

$$S_1 = 3x + 3;\ S_1^2 - 2S_2 = 3x^2 + 6x + 5 \Rightarrow 9x^2 + 18x + 9 - 2S_2 = 3x^2 + 6x + 5 \Rightarrow 2S_2 = 6x^2 + 12x + 4 \Rightarrow S_2 = 3x^2 + 6x + 2$$

$$S_1^3 - 3S_1S_2 + 3S_3 = 3x^3 + 9x^2 + 15x + 9$$

$$27(x^3 + 3x^2 + 3x + 1) - 9(x + 1)(3x^2 + 6x + 2) + 3S_3 = 3x^3 + 9x^2 + 15x + 9$$

$$3S_3 = 3x^3 + 9x^2 + 6x \Rightarrow S_3 = x^3 + 3x^2 + 2x$$

$$f(x)\cdot g(x)\cdot h(x) = x^3 + 3x^2 + 2x = 0 \Leftrightarrow$$

$$x(x^2 + 3x + 2) = 0 \Rightarrow x \in \{-2; -1; 0\}$$

96. Find $x, y, z > 0$ such that:

$$\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} + \frac{(1+y^2)(1+z^2)}{(1+y)(1+z)} + \frac{(1+z^2)(1+x^2)}{(1+z)(1+x)} + 24\sqrt{2} = 36$$

Proposed by Daniel Sitaru-Romania

Solution by Remus Florin Stanca-Romania

Let's prove that: $\sum \frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} = 36 - 24\sqrt{2}$, *let*

$$x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}, A, B, C \in \left[0, \frac{\pi}{2}\right],$$

$$\sin A = \frac{2x}{x^2+1}, \sin B = \frac{2y}{y^2+1}, \sin C = \frac{2z}{z^2+1}$$

$$\sin A + \frac{2}{x^2+1} = 2\cdot\frac{x+1}{x^2+1} \Rightarrow \sin A + 2\cos^2\frac{A}{2} - 1 + 1 = 2\cdot\frac{x+1}{x^2+1} \Rightarrow$$

$$\sin A + \cos A + 1 = 2\cdot\frac{x+1}{x^2+1} \Rightarrow \frac{x^2+1}{x+1} = \frac{2}{\sin A + \cos A + 1} \Rightarrow$$

$$\sum_{cyc}\frac{(1+x^2)(1+y^2)}{(1+x)(1+y)} = \sum_{cyc}\frac{4}{(\sin A + \cos A + 1)(\sin B + \cos B + 1)} \geq 36 - 24\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}\frac{1}{(\sin A + \cos A + 1)(\sin B + \cos B + 1)} \geq 9 - 6\sqrt{2};\ (1)$$

$$\sum_{cyc}\frac{1}{(\sin A + \cos A + 1)(\sin B + \cos B + 1)} \overset{Bergstrom}{\geq} \frac{9}{\sum_{cyc}(\sin A + \cos A + 1)(\sin B + \cos B + 1)};\ (2)$$

Let's prove that:

$$\frac{9}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)} \geq 9-6\sqrt{2}$$

We need to prove that:

$$\frac{3}{\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1)} \geq 3-2\sqrt{2} = \frac{1}{(\sqrt{2}+1)^2} \Leftrightarrow$$

$$\sum_{cyc}(sinA+cosA+1)(sinB+cosB+1) \leq 9+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}(sinAsinB+sinAcosB+sinA+sinAsinB+cosAcosB+cosA+sinB+cosB+1) \leq 9+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}\big(cos(A-B)+sin(A+B)\big)+2(sinA+sinB+sinC)+2(cosA+cosB+cosC) \leq 6+6\sqrt{2} \Leftrightarrow$$

$$\sum_{cyc}\left(cos(A-B)+cos\left(\frac{\pi}{2}-A-B\right)\right)+2(sinA+sinB+sinC)+$$

$$+2(cosA+cosB+cosC) \leq 6+6\sqrt{2} \Leftrightarrow$$

$$2\sum_{cyc}(cos\left(\frac{\pi}{4}-B\right)cos\left(\frac{\pi}{4}-A\right)+2\sqrt{2}\sum_{cyc}cos\left(\frac{\pi}{4}-A\right) \leq 6+6\sqrt{2}$$

Let: $cos\left(\frac{\pi}{4}-A\right)=x_1;\ cos\left(\frac{\pi}{4}-B\right)=x_2;\ cos\left(\frac{\pi}{4}-C\right)=x_3 \Leftrightarrow$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3) \leq 3+3\sqrt{2};\ (3)$$

$$\frac{(x_1+x_2+x_3)^2}{3} \geq x_1x_2+x_2x_3+x_3x_1 \Rightarrow$$

$$(x_1x_2+x_2x_3+x_3x_1)+\sqrt{2}(x_1+x_2+x_3) \leq \frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3) \overset{(3)}{\Rightarrow}$$

We need to prove that:

$\frac{(x_1+x_2+x_3)^2}{3}+\sqrt{2}(x_1+x_2+x_3) \leq 3+3\sqrt{2}$ ***let*** $x_1+x_2+x_3=s$ ***and we know that***

$0 \leq s \leq 3 \Rightarrow s^2+3s\sqrt{2}-9-9\sqrt{2} \leq 0 \overset{(1),(2)}{\Longrightarrow}$ ***we prove that (1),(2) are true.***

But we have in the equality case $s^2+3s\sqrt{2}-9-9\sqrt{2}=0;\ \Delta=54+36\sqrt{2} \Rightarrow s=3,$

because $s \geq 0 \Rightarrow$

$$\sum_{cyc}cos\left(\frac{\pi}{4}-A\right)=3 \Rightarrow 2\sum_{cyc}sin^2\left(\frac{\frac{\pi}{4}-A}{2}\right)=0 \Rightarrow A=B=C=\frac{\pi}{4} \Rightarrow$$

$$x = y = z = tan\frac{\pi}{8}$$

$$tan\frac{\pi}{8} = \frac{sin\frac{\pi}{8}}{cos\frac{\pi}{8}} = \frac{\sqrt{\frac{1 - cos\frac{\pi}{4}}{2}}}{\sqrt{\frac{1 + cos\frac{\pi}{4}}{2}}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}} = \sqrt{2} - 1 \Rightarrow x = y = z = \sqrt{2} - 1.$$

97. $x^9 + \frac{15}{8}x^6 + \frac{75}{64}x^3 - x + \frac{445}{512} = 0$

If S −sum of all real roots then find $[S]$, $[*]$ −GIF

Proposed by Rajeev Rastogi-India

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$x^9 + \frac{15}{8}x^6 + \frac{75}{64}x^3 - x + \frac{445}{512} = 0; (1)$$

Put: $t = x^3$; (2), we have the given equation equivalent to:

$$t^3 + \frac{15}{8}t^2 + \frac{75}{64}t - x + \frac{445}{512} = 0 \Rightarrow x = t^3 + \frac{15}{8}t^2 + \frac{75}{64}t + \frac{445}{512} = \left(t + \frac{5}{8}\right)^3 + \frac{5}{8}; (3)$$

From (2),(3), we have $x^3 + x = t + \left(t + \frac{5}{8}\right)^3 + \frac{5}{8}; (4)$

Put $f(t) = t^3 + t$, we have $f'(t) = 3t^2 + 1 > 0$ so $f(t)$ −is a increasing function.

So, from (4) we get $x = t + \frac{5}{8} \Rightarrow x = x^3 + \frac{5}{8} \Rightarrow x^3 - x + \frac{5}{8} = 0 \Rightarrow 8x^3 - 8x + 5 = 0.$

Put $g(x) = 8x^3 - 8x + 5$, $g'(x) = 24x^2 - 8.$

So, $g'(x) = 0 \Leftrightarrow x = \pm\frac{\sqrt{3}}{3}$. *We have* $g\left(-\frac{\sqrt{3}}{3}\right) = 5 + \frac{16\sqrt{3}}{9}$ *and* $\left(\frac{\sqrt{3}}{3}\right) = 5 - \frac{16\sqrt{3}}{9}$.

So, $8x^3 - 8x + 5 = 0$ *has only one root in* $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$.

On the other hand, we have $f(-2)f(-1) < 0$ then the equation (1) has only one root in $(-2, -1)$.

So, $[S] = -2$.

98. Find the function $f: \mathbb{R} \to \mathbb{R}$ continuous such that:

$$f(x) + f(x^2) = 2, \forall x \in \mathbb{R}.$$

Proposed by Sridhar Rao-India

Solution 1 by Ravi Prakash-New Delhi-India

$$f(x) + f(x^2) = 2, \forall x \in \mathbb{R}; (1)$$

Replace x by $-x$ we get: $f(-x) + f((-x)^2) = 2 \Rightarrow$

$$f(-x)+f(x^2)=2;\forall x\in\mathbb{R};(2)$$

From (1),(2) we get: $f(-x)=f(x),\forall x\in\mathbb{R}$

Also, from (1), $2f(0)=2\Rightarrow f(0)=1$

$$2f(1)=2\Rightarrow f(1)=1$$

Let $0<x<1$. ***From (1) we have:***

$$f(x)=2-f(x^2)=2-[2-f((x^2)^2)]=f(x^4);(3)\Rightarrow$$

$$f(x)=f(x^4)=f(x^{16})=\dots=f\left(x^{4^n}\right),\forall n\in\mathbb{N}$$

As $0<x<1, x^{4^n}\underset{n\to\infty}{\longrightarrow}0$

Since f ***−is continuous on*** $\mathbb{R}\Rightarrow\lim_{n\to\infty}f\left(x^{4^n}\right)=f(0)=1\Rightarrow$

$$f(x)=1,\forall x\in(0,1)$$

Let $x>1$***, then from (3)***

$$f(x)=f\left(x^{\frac{1}{4}}\right)=f\left(x^{\frac{1}{16}}\right)=\dots=f\left(x^{\frac{1}{4^n}}\right),\forall n\in\mathbb{N}$$

As $x>1, x^{\frac{1}{4^n}}\underset{n\to\infty}{\longrightarrow}1, f$ ***−continuous on*** $\mathbb{R}\Rightarrow\lim_{n\to\infty}f\left(x^{\frac{1}{4^n}}\right)=f(1)=1\Rightarrow f(x)=1,\forall x>1$

Thus, $f(x)=1,\forall x\in\mathbb{R}$.

Solution 2 by Kamel Benaicha-Algiers-Algerie

$$\begin{cases} f:\mathbb{R}\to\mathbb{R} \\ f(x)+f(x^2)=2,\forall x\in\mathbb{R}\end{cases}\Rightarrow f(x)-1=1-f(x^2)$$

$$f(-x)=f((-x)^2)=f(-x)+f(x^2)=2$$

So, $f(x)=f(-x),\forall x\in\mathbb{R}, f$ ***−continuous, then***

$$f(0)=f(1)=1;(x^2=x\Rightarrow x\in\{0,1\})$$

Put: $g(x)=f(x)-1\Rightarrow g(x)=-g(x^2), g$ ***−continuous, then***

$$g(-1)=g(0)=g(1)=0$$

$$g(-x)=f(-x)-1-f(x)-1=g(x)$$

We have: $g(x)=g(\sqrt{x})=g\left(x^{\frac{1}{2^2}}\right)=\dots=(-1)^m g\left(x^{\frac{1}{2^m}}\right)$ ***then, for*** $m=2n$:

$$g(x)=g\left(x^{\frac{1}{2^n}}\right)$$

If $x\neq 0, \lim_{n\to\infty}g(x)=\lim_{n\to\infty}g\left(x^{\frac{1}{2^n}}\right)=g\left(\lim_{n\to\infty}x^{\frac{1}{2^n}}\right)=g(1)=0$

So, $\forall x\in\mathbb{R}, g(x)=0\Rightarrow f(x)=1$

99. Solve for real numbers:

$$\begin{cases} xy(4xy-1)^2+16xy=16z^2 \\ yz(4yz-1)^2+16yz=16x^2 \\ zx(4zx-1)^2+16zx=16y^2 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Bedri Hajrizi-Mitrovica-Kosovo

It's clear that trivial solution is $(0,0,0)$. *Suppose that* $(x,y,z)\neq(0,0,0)$

Produce all we get:

$$\prod((4xy-1)^2+16)=16^3$$

$$16^3=\prod((4xy-1)^2+16)\geq 16^3\Rightarrow\begin{cases} xy=\frac{1}{4} \\ yz=\frac{1}{4} \\ zx=\frac{1}{4} \end{cases}$$

Produce all $x^2y^2z^2=\left(\frac{1}{4}\right)^3$ *and* $xy=\frac{1}{4}\Rightarrow x^2y^2z^2=\frac{1}{16}z^2\Rightarrow$

$$\frac{1}{16}z^2=\frac{1}{64}\Rightarrow z=\pm\frac{1}{2}$$

Similarly: $x=y=\pm\frac{1}{2}$.

Finally solutions are:

$$(0,0,0),\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

100. Suppose:

$$\begin{cases} tan^{-1}\left(\frac{x+y}{1-xy}\right)=\sqrt[3]{tan^{-1}z} \\ tan^{-1}\left(\frac{y+z}{1-yz}\right)=\sqrt[3]{tan^{-1}x} \\ tan^{-1}\left(\frac{z+x}{1-xz}\right)=\sqrt[3]{tan^{-1}y} \end{cases}$$

Find: $\Omega=x+y+z;x,y,z\in\mathbb{R}$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

For $x,y,z\in\mathbb{R}\Rightarrow\exists a,b,c\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *such that:*

$$x=\tan a, y=\tan b, z=\tan c$$

$$\tan^{-1}\left(\frac{\tan a+\tan b}{1-\tan a\cdot\tan b}\right)=\sqrt[3]{\tan^{-1}(\tan c)}\Rightarrow\begin{cases}a+b=\sqrt[3]{c};(1)\\b+c=\sqrt[3]{a};(2)\\c+a=\sqrt[3]{b};(3)\end{cases}$$

$$(1)-(2)\Rightarrow a-c=\sqrt[3]{c}-\sqrt[3]{a}$$

$$\text{If } a>c\Rightarrow\begin{cases}a-c>0\\\sqrt[3]{c}-\sqrt[3]{a}<0\end{cases}\Rightarrow a-c\neq\sqrt[3]{c}-\sqrt[3]{a}$$

$$\text{If } a<c\Rightarrow\begin{cases}a-c<0\\\sqrt[3]{c}-\sqrt[3]{a}>0\end{cases}\Rightarrow a-c\neq\sqrt[3]{c}-\sqrt[3]{a}\Rightarrow a=c.$$

Similarly we get: $b=c; a=b\Rightarrow a=b=c\Rightarrow 2a=\sqrt[3]{a}\Rightarrow$

$$8a^3=a\Rightarrow a(8a^2-1)=0\Rightarrow a\in\left\{-\frac{1}{2\sqrt{2}},0,\frac{1}{2\sqrt{2}}\right\}$$

$$(i)a=b=c=0\Rightarrow x=y=z=0\Rightarrow x+y+z=0$$

$$(ii)a=b=c=\frac{1}{2\sqrt{2}}\Rightarrow x=y=z=\tan\left(\frac{1}{2\sqrt{2}}\right)\Rightarrow x+y+z=3\tan\left(\frac{1}{2\sqrt{2}}\right)$$

$$(iii)a=b=c=-\frac{1}{2\sqrt{2}}\Rightarrow x=y=z=-\tan\left(\frac{1}{2\sqrt{2}}\right)\Rightarrow x+y+z=-3\tan\left(\frac{1}{2\sqrt{2}}\right)$$

Solution 2 by Ravi Prakash-New Delhi-India

Let: $a,b,c\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *be such that* $=\tan a, y=\tan b, z=\tan c$

The first equation can be written as:

$$\frac{\tan a+\tan b}{1-\tan a\cdot\tan b}=\tan\left(c^{\frac{1}{3}}\right)\Rightarrow\tan(a+b)=\tan\left(c^{\frac{1}{3}}\right)\Rightarrow a+b=c^{\frac{1}{3}}\Rightarrow(a+b)^3=c$$

Analogous,

$(b+c)^3=a, (a+c)^3=b$. *Let:* $d=a+b+c$.

Then a,b,c *satisfy the equation:* $(d-t)^3=t, t\in\mathbb{R}$

$$t^3-3dt^2+(3d^2+1)t-d^3=0;(1)$$

$$\Rightarrow a+b+c=3d\Rightarrow d=3d\Rightarrow d=0\Rightarrow a+b+c=0$$

$\Rightarrow(1)$ *becomes:* $t^3+t=0\Rightarrow t=0$ *thus* $a=b=c=0\Rightarrow x+y+z=0$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru