PROPOSED PROBLEM

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Prove: If $a, b, c \geq 1$, then

$$e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}$$

Solution 1 by Tran Hong-Ben Tre University-Ben Tre-Vietnam.

$$\text{We have: } e^x \geq x + 1 (\forall x > 0)$$

$$\Rightarrow e^{ab} \geq ab + 1; e^{bc} \geq bc + 1; e^{ca} \geq ca + 1$$

$$\Rightarrow e^{ab} + e^{bc} + e^{ca} \geq 3 + ab + bc + ca$$

We must show that

$$3 + ab + bc + ca \geq 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}$$

$$\Leftrightarrow abc(ab + bc + ca) \geq ab^2 + bc^2 + ca^2$$

$$\Leftrightarrow c(ab)^2 + a(bc)^2 + b(ca)^2 \geq ab^2 + bc^2 + ca^2$$

$$\Leftrightarrow (ca^2 - a)b^2 + (ab^2 - b)c^2 + (bc^2 - c)a^2 \geq 0$$

$$\Leftrightarrow a(ca - 1)b^2 + b(ab - 1)c^2 + c(bc - 1)a^2 \geq 0$$

Which is true because:

$$a, b, c \geq 1 \Rightarrow ab \geq 1; bc \geq 1; ca \geq 1$$

□

Solution 2 by Sanong Huayrerai-Nathom Pathom College-Thailand.
Because $a, b, c \geq 1$, we have as follows

$$a^2bc \geq a^2c, ab^2c \geq b^2a, abc^2 \geq c^2b$$

$$\text{and } abc(a + b + c) \geq 3abc$$

$$\text{Hence } 2abc(a + b + c) \geq 3abc + a^2c + c^2b + b^2a$$

$$\text{Hence } 2(a + b + c) \geq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

$$\text{Hence } ab + 1 + bc + 1 + ca + 1 \geq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

$$\text{Hence } e^{ab} + e^{bc} + e^{ca} \geq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \text{ ok}$$

Therefore it is to be true.

□

Solution 3 by Ravi Prakash-New Delhi University-New Delhi-India.
For $x, y \geq 1, x^2 \geq 1 \Rightarrow x^2 y \geq y \Rightarrow xy \geq \frac{y}{x}$

$$\Rightarrow 1 + xy + \frac{1}{2!}x^2y^2 + \frac{1}{3!}x^3y^3 + \ldots > 1 + \frac{y}{x}$$

$$\Rightarrow e^{xy} > 1 + \frac{y}{x}$$

$$\therefore e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}$$

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