TRIGONOMETRIC AND GEOMETRIC INEQUALITIES WITH FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper we presents certain trigonometric and geometric inequalities with Fibonacci and Lucas numbers.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$.

Theorem 1.

$$\sin F_{2n+2} + \sin F_n^2 + \cos F_{n+2}^2 \leq \frac{3}{2}$$

Proof.

(1) We have $F_{n+2}^2 = F_{2n+2} + F_n^2$

If $a, b \in \mathbb{R}$, then $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$

$$= -2x^2 + 2xy + 1 = -2(x^2 - xy - \frac{1}{2}\Big) = -2\left(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\right) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.
If we take $x = F_{2n+2}, b = F_n^2$ then by (1) and (2) we obtain the desired inequality. □

Theorem 2.

$$\sin L_n^2 + \sin L_{n+1}^2 + \cos(L_{2n} + L_{2n+2}) \leq \frac{3}{2}$$

Proof.

(1) We have $L_n^2 + L_{n+1}^2 = L_{2n} + L_{2n+2}$

If $a, b \in \mathbb{R}$, then $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$

$$= -2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\left(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\right) =$$

(2) $$-2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.
If we take $a = L_n^2, b = L_{n+1}^2$ then, by (1) and (2) we obtain the desired inequality. □

Theorem 3.
$$\sin L_n^2 + \sin L_{n+1}^2 + \cos(5F_{2n+1}) \leq \frac{3}{2}$$

Proof.

(1) $$\text{We have } L_n^2 + L_{n+1}^2 = 5F_{2n+1}$$

If $a, b \in \mathbb{R}$ then, $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$
$$= -2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\left(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\right) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.
If we take $a = L_n^2, b = L_{n+1}^2$ then, by (1) and (2) we obtain the desired inequality.
□

Theorem 4.
$$\sin F_n^2 + \sin F_{n+1}^2 + \cos F_{2n+1} \leq \frac{3}{2}$$

Proof.

(1) $$\text{We have } F_n^2 + F_{n+1}^2 = F_{2n+1}$$

□

If $a, b \in \mathbb{R}$ then, $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$
$$= -2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\left(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\right) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$. If we take $a = F_n^2, b = F_{n+1}^2$ then, by (1) and (2) we obtain the desired inequality.

Theorem 5.
$$\sin F_{n-1}^2 + \sin F_{2n} + \cos F_{n+1}^2 \leq \frac{3}{2}$$

Proof.

(1) $$\text{We have } F_{n-1}^2 + F_{2n} = F_{n+1}^2$$

If $a, b \in \mathbb{R}$ then, $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$
$$= -2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\left(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\right) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.
If we take $a = F_{n-1}^2, b = F_{2n}$ then, by (1) and (2) we obtain the desired inequality.
□

Theorem 6.

$$\sin(F_m L_n) + \sin(F_n L_m) + \cos(2F_{m+n}) \leq \frac{3}{2}$$

Proof.

(1) We have $F_m L_n + F_n L_m = 2F_{m+n}$

If $a, b \in \mathbb{R}$, then $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$

$$= -2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\Bigg(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\Bigg) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.

If we take $a = F_m L_n, b = F_n L_m$, then by (1)and (2) we obtain the desired inequality. □

Theorem 7.

$$\sin F_{2n+3} + \sin(F_{n+1}F_n) + \cos(F_{n+3}F_{n+2}) \leq \frac{3}{2}$$

Proof.

(1)

We have $F_n F_{n+1} - F_{n-1}F_{n-2} = F_{2n-1}, \forall n \geq 2 \Leftrightarrow F_{2n+3} + F_{n+1}F_n = F_{n+3}F_{n+2}, \forall n$

If $a, b \in \mathbb{R}$, then $\sin a + \sin b + \cos(a+b) = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} + 1 - 2\sin^2\frac{a+b}{2} =$

$$-2x^2 + 2xy + 1 = -2\Big(x^2 - xy - \frac{1}{2}\Big) = -2\Bigg(\Big(x - \frac{y}{2}\Big)^2 - \frac{y^2}{4} - \frac{1}{2}\Bigg) =$$

(2) $$= -2\Big(x - \frac{y}{2}\Big)^2 + \frac{y^2}{2} + 1 \leq \frac{y^2}{2} + 1 \leq \frac{1}{2} + 1 = \frac{3}{2}$$

where $x = \sin\frac{a+b}{2}, y = \cos\frac{a-b}{2}$.

If we take $a = F_{2n+3}, b = F_n F_{n+1}$, then by (1) and (2) we obtain the desired inequality. □

Theorem 8. If ABC is a triangle with a, b, c the lengths of the sides, h_a, h_b, h_c the lengths of altitudes from A, B, C, r the length of the inradius and s the semiperimeter, then:

$$\frac{h_a}{F_n b + F_{n+1}c} + \frac{h_b}{F_n c + F_{n+1}a} + \frac{h_c}{F_n a + F_{n+1}b} \geq \frac{27}{2F_{n+2}} \cdot \frac{r}{s}, \forall n \in \mathbb{N}^*.$$

Proof. Let F be the area of ABC triangle. From Bergström's inequality we have:

$$\sum_{cyc} \frac{h_a}{F_n b + F_{n+1}c} = \sum_{cyc} \frac{a h_a}{F_n ab + F_{n+1}ac} = 2F \sum_{cyc} \frac{1}{F_n ab + F_{n+1}ac} \geq 18F \cdot \frac{1}{\sum_{cyc}(F_n ab + F_{n+1}ac)} =$$

(1) $$= 18F \cdot \frac{1}{F_n + F_{n+1}(ab + bc + ca)} = 18F \cdot \frac{1}{F_{n+2}(ab + bc + ca)}$$

(2) But $(a+b+c)^2 \geq 3(ab + bc + ca)$, for any positie real numbers a, b, c.

From (1), (2) and the formula $F = sr$, we obtain: □

$$\frac{h_a}{F_n b + F_{n+1}c} + \frac{h_b}{F_n c + F_{n+1}a} + \frac{h_c}{F_n a + F_{n+1}b} \geq \frac{18F}{F_{n+2}} \cdot \frac{3}{4s^2} = \frac{27sr}{2s^2 F_{n+2}} = \frac{27}{2F_{n+2}} \cdot \frac{r}{s}$$

The equality holds if and only if ABC triangle is equilateral.

Theorem 9. If ABC is a triangle with a, b, c the lengths of the sides, h_a, h_b, h_c the lengths of altitudes from A, B, C, r the length of the inradius and s the semiperimeter, then:

$$\frac{h_a}{L_n b + L_{n+1}c} + \frac{h_b}{L_n c + L_{n+1}a} + \frac{h_c}{L_n a + L_{n+1}b} \geq \frac{27}{2L_{n+2}} \cdot \frac{r}{s}, \forall n \in \mathbb{N}^*$$

Proof. Let F be the area of ABC triangle. From Bergström's inequality we have:

$$\sum_{cyc} \frac{h_a}{L_n b + L_{n+1}c} = \sum_{cyc} \frac{ah_a}{L_n ab + L_{n+1}ac} = 2F \sum_{cyc} \frac{1}{L_n ab + L_{n+1}ac} \geq 18F \cdot \frac{1}{\sum_{cyc}(L_n ab + L_{n+1}ac)} =$$

(1) $$= 18F \cdot \frac{1}{L_n + L_{n+1}(ab + bc + ca)} = 18F \cdot \frac{1}{L_{n+2}(ab + bc + ca)}$$

(2) But $(a + b + c)^2 \geq 3(ab + bc + ca)$, for any positive real numbers a, b, c.

From (1), (2) and the formula $F = sr$ we obtain:

$$\frac{h_a}{L_n b + L_{n+1}c} + \frac{h_b}{L_n c + L_{n+1}a} + \frac{h_c}{L_n a + L_{n+1}b} \geq \frac{18F}{L_{n+2}} \cdot \frac{3}{4s^2} = \frac{27sr}{2s^2 L_{n+2}} = \frac{27}{2L_{n+2}} \cdot \frac{r}{s}$$

The equality holds if and only if ABC triangle is equilateral.

□

Theorem 10. If ABC is a triangle with a, b, c the lengths of the sides, h_a, h_b, h_c the lengths of altitudes from A, B, C, r the length of the inradius and s the semiperimeter, then:

$$\frac{h_a}{F_n^2 b + F_{n+1}^2 c} + \frac{h_b}{F_n^2 c + F_{n+1}^2 a} + \frac{h_c}{F_n^2 a + F_{n+1}^2 b} \geq \frac{27}{2F_{2n+1}} \cdot \frac{r}{s}, \forall n \in \mathbb{N}^*$$

Proof. Let F be the area of ABC triangle. From Bergström's inequality we have:

$$\sum_{cyc} \frac{h_a}{F_n^2 b + F_{n+1}^2 c} = \sum_{cyc} \frac{ah_a}{F_n^2 ab + F_{n+1}^2 ac} = 2F \sum_{cyc} \frac{1}{F_n^2 ab + F_{n+1}^2 ac} \geq 18F \cdot \frac{1}{\sum_{cyc}(F_n^2 ab + F_{n+1}ac)} =$$

(1) $$= 18F \cdot \frac{1}{(F_n^2 + F_{n+1}^2)(ab + bc + ca)} = 18F \cdot \frac{1}{F_{2n+1}(ab + bc + ca)}$$

(2) But $(a + b + c)^2 \geq 3(ab + bc + ca)$, for any positive real numbers a, b, c.

From (1), (2) and the formula $F = sr$, we obtain:

$$\frac{h_a}{F_n^2 b + F_{n+1}^2 c} + \frac{h_b}{F_n^2 c + F_{n+1}^2 a} + \frac{h_c}{F_n^2 a + F_{n+1}^2 b} \geq \frac{18F}{F_{2n+1}} \cdot \frac{3}{4s^2} = \frac{27rs}{2F_{2n+1}s^2} = \frac{27}{2F_{2n+1}} \cdot \frac{r}{s}$$

The equality holds if and only if ABC triangle is equilateral.

□

References

[1] *Octogon Mathematical Magazine*, 2015-2019
[2] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro
[3] Mihály Bencze, Daniel Sitaru, Marian Ursărescu, *Olympic Mathematical Energy.* Studis Publishing House, Iaşi, 2018.
[4] Daniel Sitaru, George Apostolopoulos, *The Olympic Mathematical Marathon.* Cartea Romanească Publishing House, Piteşti, 2018.
[5] Mihály Bencze, Daniel Sitaru, *Quantum Mathematical Power.* Studis Publishing House, Iaşi, 2018.
[6] Daniel Sitaru, Marian Ursărescu, *Calculus Marathon.* Studis Publishing House, Iaşi, 2018.
[7] Daniel Sitaru, Mihály Bencze, *699 Olympic Mathematical Challenges.* Studis Publishing House, Iaşi, 2017.
[8] Daniel Sitaru, Marian Ursărescu, *Ice Math-Contests Problems.* Studis Publishing House, Iaşi, 2019.

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