TRAIAN LALESCU TYPE LIMITS WITH FIBONACCI AND LUCAS SEQUENCES AND GOLDEN RATIO

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Abstract. In this paper we present some certain Lalescu type limits with Fibonacci and Lucas numbers related to the golden ratio result.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$.

Theorem 1. Let $(a_n)_{n\geq 1}, a_1 = 1, a_{n+1} = (n+1)! \cdot a_n, \forall n \in \mathbb{N}^*$, then:

$$\lim_{n\to\infty} \frac{\sqrt[2n]{n! \cdot L_n^2}}{\sqrt[n^2]{a_n}} = \alpha e^{\frac{1}{4}}, \text{ where } \alpha = \frac{1+\sqrt{5}}{2}, \text{ i.e. is the golden ratio.}$$

Proof. First we prove the following:

Lemma. If $(x_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = x \in \mathbb{R}_+^*$, then $\lim_{n\to\infty} \sqrt[n^2]{x_n} = \sqrt{x}$.

Proof.

$$\ln\left(\lim_{n\to\infty} \sqrt[n^2]{x_n}\right) = \lim_{n\to\infty}(\ln \sqrt[n^2]{x_n}) = \lim_{n\to\infty}\left(\frac{1}{n^2}\cdot \ln x_n\right) \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1)^2 - n^2} =$$

$$= \lim_{n\to\infty} \frac{n}{2n+1}\cdot \frac{\ln x_{n+1} - \ln x_n}{n} = \frac{1}{2}\cdot \lim_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{n} \overset{\text{C-S}}{=} \frac{1}{2}\cdot \lim_{n\to\infty} \frac{\ln x_{n+2} - 2\ln x_{n+1} + \ln x_n}{(n+1) - n} =$$

$$= \frac{1}{2}\cdot \lim_{n\to\infty} \ln \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = \frac{1}{2}\cdot \ln \lim_{n\to\infty} \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = \frac{1}{2}\ln x = \ln\sqrt{x} \Rightarrow \lim_{n\to\infty} \sqrt[n^2]{x_n} = \sqrt{x}$$

□

(1)

$$\text{Now, } \frac{\sqrt[2n]{n! \cdot L_n^2}}{\sqrt[n^2]{a_n}} = \sqrt[2n^2]{\frac{(n!)^n \cdot L_n^{2n}}{a_n^2}} = \sqrt{\sqrt[n^2]{\frac{(n!)^n \cdot L_n^{2n}}{a_n^2}}} \Rightarrow \lim_{n\to\infty} \frac{\sqrt[2n]{n! \cdot L_n^2}}{\sqrt[n^2]{a_n}} = \sqrt{\lim_{n\to\infty} \sqrt[n^2]{\frac{(n!)^n \cdot L_n^{2n}}{a_n^2}}}$$

$$\text{But, } \lim_{n\to\infty} \frac{((n+2)!)^{n+2} \cdot L_{n+2}^{2(n+2)}}{a_{n+2}^2} \cdot \frac{(n!)^n \cdot L_n^{2n}}{a_n^2} \cdot \frac{a_{n+1}^4}{((n+1)!)^{2n+2} \cdot L_{n+1}^{4n+4}} =$$

$$(2) = \lim_{n\to\infty} \frac{((n+2)!)^{n+2}\cdot (n!)^n}{((n+1)!)^{2n+2}}\cdot\left(\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_n}{L_{n+1}}\right)^{2n} = \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^n\left(\alpha\cdot\frac{1}{\alpha}\right)^{2n} = e$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and we used the fact that $\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha$. From (1), (2) and Lemma yields that $\lim_{n\to\infty} \frac{\sqrt[2n]{n!\cdot L_n^2}}{\sqrt[n^2]{a_n}} = \alpha\sqrt{\sqrt{e}} = \alpha\sqrt[4]{e} = \alpha e^{\frac{1}{4}}$ and we are done! □

Key words and phrases. matrices.

Theorem 2. If $(a_n)_{n\geq 1}, a_1 = 1, a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^*$ then:

$$\lim_{n\to\infty} \frac{\sqrt[2n]{n! \cdot F_n^2}}{\sqrt[n^2]{a_n}} = \alpha e^{\frac{1}{4}}, \text{ where } \alpha = \frac{1+\sqrt{5}}{2}, \text{ i.e. is the golden ratio.}$$

Proof. First, we prove the following:

Lemma. If $(x_n)_{n\geq 1}, x_n \in \mathbb{R}_+^*$ and $\lim_{n\to\infty} \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = x \in \mathbb{R}_+^*$, then $\lim_{n\to\infty} \sqrt[n^2]{x_n} = \sqrt{x}$.

Proof.

$$\ln\left(\lim_{n\to\infty} \sqrt[n^2]{x_n}\right) = \lim_{n\to\infty} (\ln \sqrt[n^2]{x_n}) = \lim_{n\to\infty}\left(\lim_{n\to\infty} \frac{1}{n^2}\cdot \ln x_n\right) \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1)^2 - n^2} =$$

$$= \lim_{n\to\infty} \frac{n}{2n+1}\cdot \frac{\ln x_{n+1} - \ln x_n}{n} = \frac{1}{2}\cdot \lim_{n\to\infty} \frac{\ln x_{n+1} - \ln x_n}{n} \overset{\text{C-S}}{=} \frac{1}{2}\cdot \frac{\ln x_{n+2} - 2\ln x_{n+1} + \ln x_n}{(n+1)-n} =$$

$$= \frac{1}{2}\cdot \lim_{n\to\infty} \ln \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = \frac{1}{2}\cdot \ln \lim_{n\to\infty} \frac{x_{n+2}\cdot x_n}{x_{n+1}^2} = \frac{1}{2}\ln x = \ln\sqrt{x} \Rightarrow \lim_{n\to\infty} \sqrt[n^2]{x_n} = \sqrt{x}$$

□

(1)

Now, $$\frac{\sqrt[2n]{n! \cdot F_n^2}}{\sqrt[n^2]{a_n}} = \sqrt[2n^2]{\frac{(n!)^n \cdot F_n^{2n}}{a_n^2}} = \sqrt{\sqrt[n^2]{\frac{(n!)^n \cdot F_n^{2n}}{a_n^2}}} \Rightarrow \lim_{n\to\infty} \frac{\sqrt[2n]{n! \cdot F_n^2}}{\sqrt[n^2]{a_n}} = \sqrt{\lim_{n\to\infty} \sqrt[n^2]{\frac{(n!)^n \cdot F_n^{2n}}{a_n^2}}}$$

But, $$\frac{((n+2)!)^{n+2}\cdot F_{n+2}^{2(n+2)}}{a_{n+2}^2} \cdot \frac{(n!)^n \cdot F_n^{2n}}{a_n^2} \cdot \frac{a_{n+1}^4}{((n+1)!)^{2n+2}\cdot F_{n+1}^{4n+4}} =$$

(2)

$$= \lim_{n\to\infty} \frac{((n+2)!)^{n+2}\cdot (n!)^n}{((n+1)!)^{2n+2}} \left(\frac{F_{n+2}}{F_{n+1}}\cdot \frac{F_n}{F_{n+1}}\right)^{2n} = \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^n \left(\alpha \cdot \frac{1}{\alpha}\right)^{2n} = e; \text{ where } \alpha = \frac{1+\sqrt{5}}{2}$$

and we used the fact that $\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha$.

From (1), (2) and Lemma yields that $\lim_{n\to\infty} \frac{\sqrt[2n]{n! \cdot F_n^2}}{\sqrt[n^2]{a_n}} = \alpha\sqrt{\sqrt{e}} = \alpha\sqrt[4]{e} = \alpha e^{\frac{1}{4}}$, and we are done! □

Theorem 3. If $m, p \geq 0$, then:

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{((n+1)!)^{m+1} F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{(n!)^{m+1} F_n^{p(m+1)}}}{n^m} \right) = \left(\frac{\alpha^p}{e}\right)^{m+1},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

We have $$\lim_{n\to\infty} \frac{\sqrt[n]{n! F_n^p}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n! \cdot F_n^p}{n^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty} \frac{(n+1)! F_{n+1}^p}{(n+1)^{n+1}} \cdot \frac{n^n}{n! F_n^p} =$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(\frac{F_{n+1}}{F_n}\right)^p = \frac{1}{e}\cdot \left(\lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}\right)^p = \frac{\alpha^p}{e}$$

$$\text{where } \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

We denote $u_n = \frac{\sqrt[n+1]{((n+1)!)^{m+1}F_{n+1}^{p(m+1)}}}{\sqrt[n]{(n!)^{m+1}F_n^{p(m+1)}}}\cdot\frac{n^n}{(n+1)^m} = \left(\frac{\sqrt[n+1]{(n+1)!F_{n+1}^p}}{n+1}\right)^{m+1}\left(\frac{n}{\sqrt[n]{n!F_n^p}}\right)^{m+1}\cdot\frac{n+1}{n}$, so

$\lim_{n\to\infty} u_n = \left(\frac{\alpha^p}{e}\right)^{m+1}\cdot\left(\frac{e}{\alpha^p}\right)^{m+1}\cdot 1 = 1 \Rightarrow \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$. Also, we have that:

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\left(\frac{(n+1)!F_{n+1}^p}{n!F_n^p}\right)^{m+1}\left(\frac{n}{n+1}\right)^{n(m+1)}\left(\frac{1}{\sqrt[n+1]{(n+1)!F_{n+1}^p}}\right)^{m+1} =$$

$$= e\cdot\frac{1}{e^{m+1}}\cdot\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)^{p(m+1)}\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}^p}}\right)^{m+1} = \frac{1}{e^m}\cdot\alpha^{p(m+1)}\cdot\frac{e^{m+1}}{\alpha^{p(m+1)}} = e$$

Hence,

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((n+1)!)^{m+1}F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{(n!)^{m+1}F_n^{p(m+1)}}}{n^m}\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{(n!)^{m+1}F_n^{p(m+1)}}}{n^m}\cdot(u_n-1) = \lim_{n\to\infty}\frac{\sqrt[n]{(n!)^{m+1}F_n^{p(m+1)}}}{n^{m+1}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =$$

$$= \left(\frac{\alpha^p}{e}\right)^{m+1}\cdot 1\cdot\ln e = \left(\frac{\alpha^p}{e}\right)^{m+1}, \text{ and we are done!}$$

□

Theorem 4. If $m, p \geq 0$, then:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((n+1)!)^{m+1}L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{(n!)^{m+1}L_n^{p(m+1)}}}{n^m}\right) = \left(\frac{\alpha^p}{e}\right)^{m+1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!L_n^p}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!L_n^p}{n^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(n+1)!L_{n+1}^p}{(n+1)^{n+1}}\cdot\frac{n^n}{n!L_n^p} =$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\left(\frac{L_{n+1}}{L_n}\right)^p = \frac{1}{e}\cdot\left(\lim_{n\to\infty}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^n+\beta^n}\right)^p = \frac{\alpha^p}{e}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, L_n = \alpha^n + \beta^n$.

We denote $u_n = \frac{\sqrt[n+1]{((n+1)!)^{m+1}L_{n+1}^{p(m+1)}}}{\sqrt[n]{(n!)^{m+1}L_n^{p(m+1)}}}\cdot\frac{n^m}{(n+1)^m} =$

$= \left(\frac{\sqrt[n+1]{(n+1)!L_{n+1}^p}}{n+1}\right)^{m+1}\left(\frac{n}{\sqrt[n]{n!L_n^p}}\right)^{m+1}\cdot\frac{n+1}{n}$, so

$\lim_{n\to\infty} u_n = \left(\frac{\alpha^p}{e}\right)^{m+1}\cdot\left(\frac{e}{\alpha^p}\right)^{m+1}\cdot 1 = 1 \Rightarrow \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1$. Also, we have that:

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\left(\frac{(n+1)!L_{n+1}^p}{n!L_n^p}\right)^{m+1}\left(\frac{n}{n+1}\right)^{n(m+1)}\left(\frac{1}{\sqrt[n+1]{(n+1)!L_{n+1}^p}}\right)^{m+1} =$$

$$= e\cdot\frac{1}{e^{m+1}}\cdot\lim_{n\to\infty}\left(\frac{L_{n+1}}{L_n}\right)^{p(m+1)}\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!L_{n+1}^p}}\right)^{m+1} = \frac{1}{e^m}\cdot\alpha^{p(m+1)}\cdot\frac{e^{m+1}}{\alpha^{p(m+1)}} = e.$$

$$\text{Hence, }\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((n+1)!)^{m+1}L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{(n!)^{m+1}L_n^{p(m+1)}}}{n^m}\right)=$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{(n!)^{m+1}L_n^{p(m+1)}}}{n^m}\cdot(u_n-1) = \lim_{n\to\infty}\frac{\sqrt[n]{(n!)^{m+1}L_n^{p(m+1)}}}{n^{m+1}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =$$

$$=\left(\frac{\alpha^p}{e}\right)^{m+1}\cdot 1\cdot\ln e=\left(\frac{\alpha^p}{e}\right)^{m+1}\text{, and we are done!}$$

□

Theorem 5. If $m, p \geq 0$, then:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^{m+1}F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{((2n-1)!!)^{m+1}F_n^{p(m+1)}}}{n^m}\right)=\left(\frac{2\alpha^p}{e}\right)^{m+1},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!_n^p}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!\cdot F_n^p}{n^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}^p}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!F_n^p} =$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n\left(\frac{F_{n+1}}{F_n}\right)^p = \frac{2}{e}\cdot\left(\lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}\right)^p = \frac{2\alpha^p}{e}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$

$$\text{We denote } u_n = \frac{\sqrt[n+1]{((2n+1)!!)^{m+1}F_{n+1}^{p(m+1)}}}{\sqrt[n]{((2n-1)!!)^{m+1}F_n^{p(m+1)}}}\frac{n^m}{(n+1)^m} =$$

$$= \left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}^p}}{n+1}\right)^{m+1}\left(\frac{n}{\sqrt[n]{(2n-1)!!F_n^p}}\right)^{m+1}\cdot\frac{n+1}{n}\text{, so}$$

$$\lim_{n\to\infty}u_n=\left(\frac{2\alpha^p}{e}\right)^{m+1}\cdot\left(\frac{e}{2\alpha^p}\right)^{m+1}\cdot 1 = 1 \Rightarrow \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1.\text{ Also, we have that}$$

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{(2n+1)!!F_{n+1}^p}{(2n-1)!!F_n^p}\right)^{m+1}\left(\frac{n}{n+1}\right)^{nm}\left(\frac{1}{\sqrt[n+1]{(2n+1)!!F_{n+1}^p}}\right)^{m+1} =$$

$$= \frac{1}{e^m}\cdot\lim_{n\to\infty}\left(\frac{2n+1}{n+1}\right)^m\cdot\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)^{p(m+1)}\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!F_{n+1}^p}}\right)^{m+1} =$$

$$= \frac{2^{m+1}}{e^m}\cdot\alpha^{p(m+1)}\cdot\frac{e^{m+1}}{2^{m+1}\cdot\alpha^{p(m+1)}} = e$$

$$\text{Hence, }\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^{m+1}F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{((2n-1)!!)^{m+1}F_n^{p(m+1)}}}{n^m}\right)$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{((2n-1)!!)^{m+1}F_n^{p(m+1)}}}{n^m}\cdot(u_n-1) = \lim_{n\to\infty} \frac{\sqrt[n]{((2n-1)!!)^{m+1}F_n^{p(m+1)}}}{n^{m+1}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{((2n-1)!!)F_n^p}}{n}\right)^{m+1}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n = \left(\frac{2\alpha^p}{e}\right)^{m+1}\cdot 1\cdot \ln e = \left(\frac{2\alpha^p}{e}\right)^{m+1}, \text{ and we are done!}$$

□

Theorem 6. If $m, p \geq 0$, then:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^{m+1}L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{((2n-1)!!)^{m+1}L_n^{p(m+1)}}}{n^m}\right) = \left(\frac{2\alpha^p}{e}\right)^{m+1},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!L_n^p}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!\cdot L_n^p}{n^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(2n+1)!!L_{n+1}^p}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!L_n^p} =$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n\left(\frac{L_{n+1}}{L_n}\right)^p = \frac{2}{e}\cdot\left(\lim_{n\to\infty}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^n+\beta^n}\right)^p = \frac{2\alpha^p}{e}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, L_n = \alpha^n + \beta^n$ We denote:

$$u_n = \frac{\sqrt[n+1]{((2n+1)!!)^{m+1}L_{n+1}^{p(m+1)}}}{\sqrt[n]{((2n-1)!!)^{m+1}L_n^{p(m+1)}}}\cdot\frac{n^n}{(n+1)^m} = \left(\frac{\sqrt[n+1]{(2n+1)!!L_{n+1}^p}}{n+1}\right)^{m+1}\left(\frac{n}{\sqrt[n]{(2n-1)!!L_n^p}}\right)^{m+1}\frac{n+1}{n}, \text{ so}$$

$$\lim_{n\to\infty} u_n = \left(\frac{2\alpha^p}{e}\right)^{m+1}\cdot\left(\frac{e}{2\alpha^p}\right)^{m+1}\cdot 1 = 1 \Rightarrow \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1. \text{ Also, we have that:}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(2n+1)!!L_{n+1}^p}{(2n-1)!!L_n^p}\right)^{m+1}\left(\frac{n}{n+1}\right)^{nm}\left(\frac{1}{\sqrt[n+1]{(2n+1)!!L_{n+1}^p}}\right)^{m+1} =$$

$$= \frac{1}{e^m}\cdot\lim_{n\to\infty}\left(\frac{2n+1}{n+1}\right)^m\cdot\lim_{n\to\infty}\left(\frac{L_{n+1}}{L_n}\right)^{p(m+1)}\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!L_{n+1}^p}}\right)^{m+1} =$$

$$= \frac{2^{m+1}}{e^m}\cdot\alpha^{p(m+1)}\cdot\frac{e^{m+1}}{2^{m+1}\cdot\alpha^{p(m+1)}} = e$$

Hence, $\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{((2n+1)!!)^{m+1}L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{((2n-1)!!)^{m+1}L_n^{p(m+1)}}}{n^m}\right)$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{((2n-1)!!)^{m+1}L_n^{p(m+1)}}}{n^m}\cdot(u_n-1) = \lim_{n\to\infty} \frac{\sqrt[n]{((2n-1)!!)^{m+1}L_n^{p(m+1)}}}{n^{m+1}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{((2n-1)!!)L_n^p}}{n}\right)^{m+1}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n = \left(\frac{2\alpha^p}{e}\right)^{m+1}\cdot 1\cdot \ln e = \left(\frac{2\alpha^p}{e}\right)^{m+1}, \text{ and we are done!}$$

□

Theorem 7. If $(a_n)_{n\geq 0}, a_n \in \mathbb{R}_+^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a\in\mathbb{R}_+^*$ then:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left(\left(\sqrt[n+1]{a_{n+1}}\right)^{\frac{F_m}{F_{m+1}}}-\left(\sqrt[n]{a_n}\right)^{\frac{F_m}{F_{m+1}}}\right)n^{\frac{F_{m-1}}{F_{m+1}}}\right)\right)=\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_m=\frac{F_m}{F_{m+1}}$ we have $\lim_{m\to\infty}=\frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=\lim_{n\to\infty}\frac{a_{n+1}\cdot n^n}{a_n\cdot(n+1)^{n+1}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{a}{e}$$

Denoting $v_n=\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{u_m}$ we have $\lim_{n\to\infty}v_n=1$, so $\lim_{n\to\infty}\frac{v_n-1}{\ln v_n}=1$ and

$$\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n]{a_{n+1}}}\right)^{u_m}=$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\cdot n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\right)^{u_n}=\left(a\cdot\frac{e}{a}\cdot 1\right)^{u_m}=e^{u_m}$$

Denoting $B_n(m)=\left(\left(\sqrt[n+1]{a_{n+1}}\right)^{u_m}-\left(\sqrt[n]{a_n}\right)^{u_m}\right)n^{\frac{F_{m+1}-F_m}{F_{m+1}}}=\left(\sqrt[n]{a_n}\right)^{u_m}(v_n-1)n^{1-u_n}=$

$$=\left(\frac{\sqrt[n]{a_n}}{n}\right)^{u_m}\frac{v_n-1}{\ln v_n}\cdot n\cdot\ln v_n\cdot n\cdot\ln v_n=\left(\frac{\sqrt[n]{a_n}}{n}\right)^{u_m}\frac{v_n-1}{\ln v_n}\cdot\ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}B_n(m)\right)=\lim_{m\to\infty}\left(\left(\frac{a}{e}\right)^{u_m}\cdot 1\cdot\ln e^{u_m}\right)=\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha}$$

□

Theorem 8. If $(a_n)_{n\geq 0}, a_n\in\mathbb{R}_+^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a\in\mathbb{R}_+^*$, then:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left(\left(\sqrt[n+1]{a_{n+1}}\right)^{\frac{L_m}{L_{m+1}}}-\left(\sqrt[n]{a_n}\right)^{\frac{L_m}{L_{m+1}}}\right)n^{\frac{L_{m-1}}{L_{m+1}}}\right)\right)=\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_m=\frac{L_m}{L_{m+1}}$ we have $\lim_{m\to\infty}u_m=\frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=\lim_{n\to\infty}\frac{a_{n+1}\cdot n^n}{a_n\cdot(n+1)^{n+1}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{a}{e}.$$

Denoting $v_n=\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{u_m}$ we have $\lim_{n\to\infty}v_n=1$, so $\lim_{n\to\infty}\frac{v_n-1}{\ln v_n}=1$ and

$$\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\right)^{u_m}=$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\cdot n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\right)^{u_m}=\left(a\cdot\frac{e}{a}\cdot 1\right)^{u_m}=e^{u_m}$$

Denoting $B_n(m) = \left((\sqrt[n+1]{a_{n+1}})^{u_m} - (\sqrt[n]{a_n})^{u_m} \right) n^{\frac{L_{m+1}-L_m}{L_{m+1}}} = (\sqrt[n]{a_n})^{u_m} (v_n - 1) n^{1-u_m} =$

$$= \left(\frac{\sqrt[n]{a_n}}{n} \right)^{u_m} \frac{v_n - 1}{\ln v_n} \cdot n \cdot \ln v_n = \left(\frac{\sqrt[n]{a_n}}{n} \right)^{u_m} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} B_n(m) \right) = \lim_{m\to\infty} \left(\left(\frac{a}{e} \right)^{u_m} \cdot 1 \cdot \ln e^{u_m} \right) = \left(\frac{a}{e} \right)^{\frac{1}{\alpha}} \cdot \frac{1}{\alpha}$$

□

Theorem 9.

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \left(\left((\sqrt[n+1]{(n+1)!}^{\frac{F_m}{F_{m+1}}} - (\sqrt[n]{n!})^{\frac{F_m}{F_{m+1}}} \right) n^{\frac{F_{m-1}}{F_{m+1}}} = \frac{1}{\alpha \cdot e^{\frac{1}{\alpha}}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_m = \frac{F_m}{F_{m+1}}$ we have $\lim_{m\to\infty} u_m = \frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

Denoting $v_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^{u_m}$ we have $\lim_{n\to\infty} v_n = 1$, so $\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$ and

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} \right)^{u_m} = \lim_{n\to\infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} \right)^{u_m} = e^{u_m}.$$

Denoting $B_n(m) = \left((\sqrt[n+1]{(n+1)!})^{u_m} - (\sqrt[n]{n!})^{u_m} \right) n^{\frac{F_{m+1}-F_m}{F_{m+1}}} = (\sqrt[n]{n!})^{u_m} (v_n - 1) n^{1-u_m} =$

$$= \left(\frac{\sqrt[n]{n!}}{n} \right)^{u_m} \frac{v_n - 1}{\ln v_n} \cdot n \cdot \ln v_n = \left(\frac{\sqrt[n]{n!}}{n} \right)^{u_m} \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} B_n(m) \right) = \lim_{m\to\infty} \left(\left(\frac{1}{e} \right)^{u_m} \cdot 1 \cdot \ln e^{u_m} \right) = \left(\frac{1}{e} \right)^{\frac{1}{\alpha}} \cdot \frac{1}{\alpha} = \frac{1}{\alpha \cdot e^{\frac{1}{\alpha}}}.$$

□

Theorem 10.

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \left(\left((\sqrt[n+1]{(n+1)!})^{\frac{L_m}{L_{m+1}}} - (\sqrt[n]{n!})^{\frac{L_m}{L_{m+1}}} \right) n^{\frac{Lm-1}{L_{m+1}}} \right) \right) = \left(\frac{1}{e} \right)^{\frac{1}{\alpha}} \cdot \frac{1}{\alpha},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_m = \frac{L_m}{L_{m+1}}$ we have $\lim_{m\to\infty} u_m = \frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

Denoting $v_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{u_m}$ we have $\lim_{n\to\infty} v_n = 1$, so $\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$ and

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)^{u_m} = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{u_m} = e^{u_m}$$

Denoting $B_n(m) = \left((\sqrt[n+1]{(n+1)!})^{u_m} - (\sqrt[n]{n!})^{u_m}\right) n^{\frac{L_{m+1}-L_m}{L_{m+1}}} = (\sqrt[n]{n!})^{u_m}(v_n - 1)n^{1-u_m} =$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^{u_m}\frac{v_n - 1}{\ln v_n}\cdot n\cdot \ln v_n = \left(\frac{\sqrt[n]{n!}}{n}\right)^{u_m}\frac{v_n - 1}{\ln v_n}\cdot \ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty}(\lim_{n\to\infty} B_n(m)) = \lim_{m\to\infty}\left(\left(\frac{1}{e}\right)^{u_m}\cdot 1\cdot \ln e^{u_m}\right) = \left(\frac{1}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha} = \frac{1}{\alpha\cdot e^{\frac{1}{\alpha}}}.$$

□

Theorem 11.

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left((\sqrt[n+1]{(2n+1)!!})^{\frac{F_m}{F_{m+1}}} - (\sqrt[n]{(2n-1)!!})^{\frac{F_m}{F_{m+1}}}\right) n^{\frac{F_{m-1}}{F_{m+1}}}\right)\right)$$

$$= \left(\frac{2}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha}$$

Where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_n = \frac{F_m}{F_{m+1}}$ we have $\lim_{m\to\infty} u_m = \frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} = \lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}.$$

Denoting $v_n = \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^{u_m}$, we have $\lim_{n\to\infty} v_n = 1$, so $\lim_{n\to\infty}\frac{v_n-1}{\ln v_n} = 1$ and

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty}\left(\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)^{u_m} = \lim_{n\to\infty}\left(2\cdot\frac{e}{2}\right)^{u_m} = e^{u_m}$$

Denoting

$B_n(m) = \left((\sqrt[n+1]{(2n+1)!!})^{u_m} - (\sqrt[n]{(2n-1)!!})^{u_m}\right) n^{\frac{F_{m+1}-F_m}{F_{m+1}}} = (\sqrt[n]{(2n-1)!!})^{u_m}(v_n - 1)n^{1-u_m} =$

$$= \left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^{u_m}\cdot\frac{v_n - 1}{\ln v_n}\cdot n\cdot\ln v_n = \left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^{u_m}\frac{v_n - 1}{\ln v_n}\cdot\ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} B_n(m)\right) = \lim_{m\to\infty}\left(\left(\frac{2}{e}\right)^{u_m}\cdot 1\cdot\ln e^{u_m}\right) = \left(\frac{2}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha}.$$

□

Theorem 12.

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left((\sqrt[n+1]{(2n+1)!!})^{\frac{L_m}{L_{m+1}}}-(\sqrt[n]{(2n-1)!!})^{\frac{L_m}{L_{m+1}}}\right)n^{\frac{L_{m-1}}{L_{m+1}}}\right)\right)$$

$$=\left(\frac{2}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha}$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof. Denoting $u_m=\frac{L_m}{L_{m+1}}$ we have $\lim_{m\to\infty}u_m=\frac{1}{\alpha}$. Also, we have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2}{e}$$

Denoting $v_n=\left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right)^{u_m}$ we have $\lim_{n\to\infty}v_n=1$, so $\lim_{n\to\infty}\frac{v_n-1}{\ln v_n}=1$ and

$$\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\left(\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\right)^{u_m}=\lim_{n\to\infty}\left(2\cdot\frac{e}{2}\right)^{u_m}=e^{u_m}$$

Denoting

$$B_n(m)=\left((\sqrt[n+1]{(2n+1)!!})^{u_m}-(\sqrt[n]{(2n-1)!!})^{u_m}\right)n^{\frac{L_{m+1}-L_m}{L_{m+1}}}=(\sqrt[n]{(2n-1)!!})^{u_m}(v_n-1)n^{1-u_m}=$$

$$=\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^{u_m}\frac{v_n-1}{\ln v_n}\cdot n\cdot\ln v_n=\left(\frac{\sqrt[n]{(2n-1)!!}}{n}\right)^{u_m}\cdot\frac{v_n-1}{\ln v_n}\cdot\ln v_n^n.$$

Hence, the limit to compute is:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}B_n(m)\right)=\lim_{m\to\infty}\left(\left(\frac{2}{e}\right)^{u_m}\cdot 1\cdot\ln e^{u_m}\right)=\left(\frac{2}{e}\right)^{\frac{1}{\alpha}}\cdot\frac{1}{\alpha}$$

□

Theorem 13.

$$\lim_{m\to\infty}(\sqrt[3n+3]{(n+1)!L_{n+1}}-\sqrt[3n]{n!L_n})\sqrt[3]{n^2}=\frac{1}{3}\sqrt[3]{\frac{\alpha}{e}},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e., is the golden ratio.

Proof.

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!L_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!L_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!L_{n+1}\cdot n^n}{nL_n\cdot(n+1)^{n+1}}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}.$$

Denoting $u_n=\frac{\sqrt[3n+3]{(n+1)!L_{n+1}}}{\sqrt[3n]{n!L_n}}$, we have $\lim_{n\to\infty}u_n=1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}$ and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\sqrt[3]{\frac{(n+1)!L_{n+1}}{n!L_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!L_{n+1}}}}=$$

$$=\sqrt[3]{\lim_{n\to\infty}\left(\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!L_{n+1}}}\right)}=\sqrt[3]{\alpha\cdot\frac{e}{\alpha}}=\sqrt[3]{e}.$$

Denoting $B_n = \left(\sqrt[3n+3]{(n+1)!L_{n+1}} - \sqrt[3n]{n!L_n}\right)\sqrt[3]{n^2} = \sqrt[3n]{n!L_n} \cdot \sqrt[3]{n^2} \cdot (u_n - 1) =$

$$= \sqrt[3n]{n!L_n} \cdot \sqrt[3]{n^2} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \sqrt[3]{\frac{\sqrt[n]{n!L_n}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n.$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt[3]{\frac{\alpha}{e}} \cdot 1 \cdot \ln \sqrt[3]{e} = \frac{1}{3}\sqrt[3]{\frac{\alpha}{e}}.$$

□

Theorem 14.

$$\lim_{m\to\infty}\left(\sqrt[3n+3]{(n+1)!F_{n+1}} - \sqrt[3n]{n!F_n}\right)\sqrt[3]{n^2} = \frac{1}{3}\sqrt[3]{\frac{\alpha}{e}}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!F_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!F_n}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!F_{n+1} \cdot n^n}{nF_n \cdot (n+1)^{n+1}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}.$$

Denoting $u_n = \frac{\sqrt[3n+3]{(n+1)!F_{n+1}}}{\sqrt[3n]{n!F_n}}$ we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n-1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \sqrt[3]{\frac{(n+1)!F_{n+1}}{n!F_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!F_{n+1}}}} =$$

$$= \sqrt[3]{\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}}}\right)} = \sqrt[3]{\alpha \cdot \frac{e}{\alpha}} = \sqrt[3]{e}.$$

Denoting $B_n = \left(\sqrt[3n+3]{(n+1)!F_{n+1}} - \sqrt[3n]{n!F_n}\right)\sqrt[3]{n^2} = \sqrt[3n]{n!F_n} \cdot \sqrt[3]{n^2} \cdot (u_n - 1) =$

$$= \sqrt[3n]{n!F_n} \cdot \sqrt[3]{n^2} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \sqrt[3]{\frac{\sqrt[n]{n!F_n}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt[3]{\frac{\alpha}{e}} \cdot 1 \cdot \ln \sqrt[3]{e} = \frac{1}{3}\sqrt[3]{\frac{\alpha}{e}}$$

□

Theorem 15.

$$\lim_{m\to\infty}\left(\sqrt[3n+3]{(2n+1)!!L_{n+1}} - \sqrt[3n]{(2n-1)!!L_n}\right)\sqrt[3]{n^2} = \frac{1}{3}\sqrt[3]{\frac{2\alpha}{e}}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!L_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!L_n}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!L_{n+1} \cdot n^n}{(2n-1)!L_n \cdot (n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{L_{n+1}}{L_n} \cdot \frac{2n+1}{n+1} \Big(\frac{n}{n+1}\Big)^n = \frac{2\alpha}{e}$$

Denoting $u_n = \frac{\sqrt[n+1]{(2n+1)!!L_{n+1}}}{\sqrt[n]{(2n-1)!!L_n}}$ we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n-1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(2n+1)!!L_{n+1}}{(2n-1)!!L_n} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!L_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!L_{n+1}}} \cdot \frac{L_{n+1}}{L_n} = 2 \cdot \frac{e}{2\alpha} \cdot \alpha = e$$

Denoting

$$B_n = \Big(\sqrt[3n+3]{(2n+1)!!L_{n+1}} - \sqrt[3n]{(2n-1)!!L_n}\Big) \sqrt[3]{n^2} = \sqrt[3n]{(2n-1)!!L_n} \cdot \sqrt[3]{n^2} \cdot (\sqrt[3]{u_n} - 1) =$$

$$= \sqrt[3n]{(2n-1)!!L_n} \cdot \sqrt[3]{n^2} \cdot \frac{u_n-1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3n]{(2n-1)!!L_n} \cdot \sqrt[3]{n^2} \cdot \ln u_n \cdot \frac{u_n-1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3]{\frac{\sqrt[n]{(2n-1)!!L_n}}{n}} \cdot n \cdot \ln u_n \cdot \frac{u_n-1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3]{\frac{\sqrt[n]{(2n-1)!!L_n}}{n}} \cdot \ln u_n^n \cdot \frac{u_n-1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1}.$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt[3]{\frac{2\alpha}{e}} \cdot \ln e \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}\sqrt[3]{\frac{2\alpha}{e}}$$

Theorem 16.

$$\lim_{m\to\infty} \Big(\sqrt[3n+3]{(2n+1)!!F_{n+1}} - \sqrt[3n]{(2n-1)!!F_n} \sqrt[3]{n^2} = \frac{1}{3}\sqrt[3]{\frac{2\alpha}{e}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!F_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!F_n}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!F_{n+1} \cdot n^n}{(2n-1)!F_n \cdot (n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{2n+1}{n+1} \Big(\frac{n}{n+1}\Big)^n = \frac{2\alpha}{e}$$

Denoting $u_n = \frac{\sqrt[n+1]{(2n+1)!!F_{n+1}}}{\sqrt[n]{(2n-1)!!F_n}}$ we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n-1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(2n+1)!!F_{n+1}}{(2n-1)!!F_n} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!F_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!L_{n+1}}} \cdot \frac{F_{n+1}}{F_n} = 2 \cdot \frac{e}{2\alpha} \cdot \alpha = e$$

Denoting

$$B_n = (\sqrt[3n+3]{(2n+1)!!F_{n+1}} - \sqrt[3n]{(2n-1)!!F_n}) \sqrt[3]{n^2} = \sqrt[3n]{(2n-1)!!F_n} \cdot \sqrt[3]{n^2} \cdot (\sqrt[3]{u_n} - 1) =$$

$$= \sqrt[3n]{(2n-1)!!F_n} \cdot \sqrt[3]{n^2} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3n]{(2n-1)!!F_n} \cdot \sqrt[3]{n^2} \cdot \ln u_n \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3]{\frac{\sqrt[n]{(2n-1)!!F_n}}{n}} \cdot n \cdot \ln u_n \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

$$= \sqrt[3]{\frac{\sqrt[n]{(2n-1)!!F_n}}{n}} \cdot \ln u_n^n \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{1}{\sqrt[3]{u_n^2} + \sqrt[3]{u_n} + 1} =$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt[3]{\frac{2\alpha}{e}} \cdot \ln e \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}\sqrt[3]{\frac{2\alpha}{e}}$$

□

Theorem 17.

$$\lim_{n\to\infty} \left(\sqrt[2n+2]{(n+1)!F_{n+1}} - \sqrt[2n]{n!F_n} \right)\sqrt{n} = \frac{1}{2}\sqrt[\alpha]{e},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!F_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!F_n}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!F_{n+1} \cdot n^n}{nF_n \cdot (n+1)^{n+1}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}.$$

Denoting $u_n = \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{\sqrt[n]{n!F_n}}$ we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(n+1)!F_{n+1}}{n!F_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!F_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}}} = \alpha \cdot \frac{e}{\alpha} = e$$

Denoting $B_n = \left(\sqrt[2n+2]{(n+1)!F_{n+1}} - \sqrt[2n]{n!F_n} \right)\sqrt{n} = \sqrt[2n]{n!F_n} \cdot \sqrt{n} \cdot (\sqrt{u_n} - 1) =$

$$= \sqrt[2n]{n!F_n} \cdot \sqrt{n} \cdot \frac{u_n - 1}{\ln u} \cdot \frac{1}{\sqrt{u_n} + 1} \ln u_n =$$

$$= \frac{\sqrt[2n]{n!F_n}}{\sqrt{n}} \cdot n \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{1}{\sqrt{u_n} + 1} \ln u_n =$$

$$= \sqrt{\frac{\sqrt[n]{n!F_n}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{1}{\sqrt{u_n} + 1} \ln u_n^n$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt{\frac{\alpha}{e}} \cdot 1 \cdot \frac{1}{2} \cdot \ln e = \frac{1}{2}\sqrt{\frac{\alpha}{e}}$$

□

Theorem 18.

$$\lim_{n\to\infty}\left(\sqrt[2n+2]{(n+1)!L_{n+1}}-\sqrt[2n]{n!L_n}\right)\sqrt{n}=\frac{1}{2}\sqrt{\frac{\alpha}{e}},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!L_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!L_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!L_{n+1}\cdot n^n}{nL_n\cdot(n+1)^{n+1}}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}.$$

Denoting $u_n=\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{\sqrt[n]{n!L_n}}$ we have $\lim_{n\to\infty}u_n=1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!L_{n+1}}{n!L_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!L_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!L_{n+1}}}=\alpha\cdot\frac{e}{\alpha}=e$$

Denoting $B_n=\left(\sqrt[2n+2]{(n+1)!L_{n+1}}-\sqrt[2n]{n!L_n}\right)\sqrt{n}=\sqrt[2n]{n!L_n}\cdot\sqrt{n}\cdot(\sqrt{u_n}-1)=$

$$=\sqrt[2n]{n!L_n}\cdot\sqrt{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\frac{1}{\sqrt{u_n}+1}\ln u_n=$$

$$=\frac{\sqrt[2n]{n!L_n}}{\sqrt{n}}\cdot n\cdot\frac{u_n-1}{\ln u_n}\cdot\frac{1}{\sqrt{u_n}+1}\ln u_n=$$

$$=\sqrt{\frac{\sqrt[n]{n!L_n}}{n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\frac{1}{\sqrt{u_n}+1}\ln u_n^n.$$

Hence, the limit to compute is:

$$\lim_{n\to\infty}B_n=\sqrt{\frac{\alpha}{e}}\cdot 1\cdot\frac{1}{2}\cdot\ln e=\frac{1}{2}\sqrt{\frac{\alpha}{e}}.$$

□

Theorem 19.

$$\lim_{n\to\infty}\left(\sqrt[2n+2]{(2n+1)!!F_{n+1}}-\sqrt[2n]{(2n-1)!!F_n}\right)\sqrt{n}=\frac{1}{2}\sqrt{\frac{2\alpha}{e}},$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!F_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}\cdot n^n}{(2n-1)!F_n\cdot(n+1)^{n+1}}=$$

$$=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\cdot\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2\alpha}{e}$$

Denoting $u_n=\frac{\sqrt[2n+2]{(2n+1)!!F_{n+1}}}{\sqrt[2n]{(2n-1)!!F_n}}$ we have $\lim_{n\to\infty}u_n=1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ and

$$\lim_{n\to\infty}u_n^n=\left(\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{(2n-1)!!F_n}\cdot\frac{1}{\sqrt[n+1]{(2n+1)!!F_{n+1}}}\right)^{\frac{1}{2}}=$$

$$= \left(\lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!F_{n+1}}} \cdot \frac{F_{n+1}}{F_n} \right)^{\frac{1}{2}} = \left(2 \cdot \frac{e}{2\alpha} \cdot \alpha\right)^{\frac{1}{2}} = \sqrt{e}.$$

Denoting

$$B_n = (\sqrt[2n+2]{(2n+1)!!F_{n+1}} - \sqrt[2n]{(2n-1)!!F_n})\sqrt{n} = \sqrt[2n]{(2n-1)!!F_n} \cdot \sqrt{n} \cdot (u_n - 1) =$$

$$= \sqrt[2n]{(2n-1)!!F_n} \cdot \sqrt{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \sqrt{\frac{\sqrt[n]{(2n-1)!!}}{n}} \cdot \ln u_n^n \cdot \frac{u_n - 1}{\ln u_n}.$$

Hence, the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt{\frac{2\alpha}{e}} \cdot \ln\sqrt{e} \cdot 1 = \frac{1}{2}\sqrt{\frac{2\alpha}{e}}$$

□

Theorem 20.

$$\lim_{n\to\infty} (\sqrt[2n+2]{(2n+1)!!L_{n+1}} - \sqrt[2n]{(2n-1)!!L_n})\sqrt{n} = \frac{1}{2}\sqrt{\frac{2\alpha}{e}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, i.e. is the golden ratio.

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!L_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!L_n}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!L_{n+1} \cdot n^n}{(2n-1)!L_n \cdot (n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{L_{n+1}}{L_n} \cdot \frac{2n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{2\alpha}{e}$$

Denoting $u_n = \frac{\sqrt[2n+2]{(2n+1)!!L_{n+1}}}{\sqrt[2n]{(2n-1)!!L_n}}$ we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty} u_n^n = \left(\lim_{n\to\infty} \frac{(2n+1)!!L_{n+1}}{(2n-1)!!L_n} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!F_{n+1}}} \right)^{\frac{1}{2}} =$$

$$= \left(\lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!L_{n+1}}} \cdot \frac{L_{n+1}}{L_n} \right)^{\frac{1}{2}} = \sqrt{e}$$

Denoting

$$B_n = (\sqrt[2n+2]{(2n+1)!!L_{n+1}} - \sqrt[2n]{(2n-1)!!L_n})\sqrt{n} = \sqrt[2n]{(2n-1)!!L_n} \cdot \sqrt{n}(u_n - 1) =$$

$$= \sqrt[2n]{(2n-1)!!L_n} \cdot \sqrt{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \sqrt{\frac{\sqrt[n]{(2n-1)!!}}{n}} \cdot \ln u_n^n \cdot \frac{u_n - 1}{\ln u_n}.$$

Hence. the limit to compute is:

$$\lim_{n\to\infty} B_n = \sqrt{\frac{2\alpha}{e}} \cdot \ln\sqrt{e} \cdot 1 = \frac{1}{2}\sqrt{\frac{2\alpha}{e}}.$$

□

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