# RMM - Calculus Marathon 901 - 1000

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**901.** **Let:**

$$\emptyset_n = \int_0^{+\infty} \frac{1}{1 + x + x^2 + \cdots + x^n} dx$$

**Prove that:** $\emptyset_{13} + \emptyset_6 = \frac{2\pi}{7}\left(\sin\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{14}\right)\right)$

*Proposed by Mohammed Bouras-Marocco*

*Solution by Samir HajAli-Damascus-Syria*

$$\emptyset_n = \int_0^{+\infty} \frac{1}{1 + x + x^2 + \cdots + x^n} dx = \int_0^{+\infty} \frac{1 - x}{1 - x^{n+1}} dx$$

$$= \int_0^1 \frac{1 - x}{1 - x^{n+1}} dx + \int_1^{+\infty} \frac{1 - x}{1 - x^{n+1}} dx = I_1 + I_2$$

$$I_2 = \int_1^{+\infty} \frac{1 - x}{1 - x^{n+1}} dx \overset{x=\frac{1}{t}}{=} \int_0^1 \frac{t^{n-2} - t^{n-1}}{1 - t^{n+1}} dt$$

$$= \int_0^1 \frac{1 - t^{n-1}}{1 - t^{n+1}} dt - \int_0^1 \frac{1 - t^{n-2}}{1 - t^{n+1}} dt$$

*Now, let put:*

$$I(p,q) = \int_0^1 \frac{1 - x^p}{1 - x^q} dx = \sum_{k=0}^{\infty} \int_0^1 (x^{kq} - x^{p+kq}) dx = \sum_{k=0}^{\infty} \left(\frac{1}{kq + 1} + \frac{1}{p + kq + 1}\right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{kq + 1} - \frac{1}{kq}\right) + \sum_{k=1}^{\infty} \left(\frac{1}{kq} - \frac{1}{p + kq + 1}\right) + \frac{p}{p + 1}$$

$$= \frac{p}{p + 1} - \frac{1}{q^2} \sum_{k=1}^{\infty} \frac{1}{k\left(k + \frac{1}{q}\right)} + \frac{p + 1}{q^2} \sum_{k=1}^{\infty} \frac{1}{k\left(k + \frac{p + 1}{q}\right)}$$

$$= \frac{p}{p + 1} - \frac{1}{q}\left[\psi\left(\frac{1}{q} + 1\right) - \psi\left(\frac{p + 1}{q} + 1\right)\right]$$

*For* $p = 1, q = n + 1$:

$$I_1=\frac{1}{2}-\frac{1}{n+1}\left[\psi\left(\frac{1}{n+1}+1\right)-\psi\left(\frac{2}{n+1}+1\right)\right]$$

$$=\frac{1}{n+1}\left[\psi\left(\frac{2}{n+1}\right)-\psi\left(\frac{1}{n+1}\right)\right]$$

$$for\ p=n-1,q=n+1\ and\ p=n-2,q=n+1$$

$$I_2=\frac{1}{n+1}\left[\psi\left(\frac{n}{n+1}\right)-\psi\left(\frac{1}{n+1}\right)\right]-\frac{1}{n+1}\left[\psi\left(\frac{n-1}{n+1}\right)-\psi\left(\frac{1}{n+1}\right)\right]$$

$$=\frac{1}{n+1}\left[\psi\left(\frac{n}{n+1}\right)-\psi\left(\frac{n-1}{n+1}\right)\right]$$

***Hence:***

$$\emptyset_n=\frac{1}{n+1}\left[\psi\left(\frac{2}{n+1}\right)-\psi\left(\frac{1}{n+1}\right)+\psi\left(\frac{n}{n+1}\right)-\psi\left(\frac{n-1}{n+1}\right)\right]$$

$$=\frac{1}{n+1}\left[\psi\left(\frac{2}{n+1}\right)-\psi\left(1-\frac{2}{n+1}\right)+\psi\left(1-\frac{1}{n+1}\right)-\psi\left(\frac{1}{n+1}\right)\right]$$

$$=\frac{1}{n+1}\left[-\pi cot\left(\frac{2\pi}{n+1}\right)+\pi cot\left(\frac{\pi}{n+1}\right)\right]=\frac{\pi}{n+1}csc\left(\frac{2\pi}{n+1}\right)$$

***Therefore:***

$$\emptyset_{13}+\emptyset_6=\frac{\pi}{14}csc\left(\frac{2\pi}{14}\right)+\frac{\pi}{7}csc\left(\frac{2\pi}{7}\right)=\frac{\pi}{7}\left(\frac{1}{2}csc\left(\frac{\pi}{7}\right)+csc\left(\frac{2\pi}{7}\right)\right)\sim 1,0912$$

**902.** $a=\frac{1}{\sqrt{2}}csc\left(\frac{n-1}{2n^2}\right)\cdot\left(sin\left(\frac{n+1}{2n^2}\right)+cos\left(\frac{n+1}{2n^2}\right)\right); b=\frac{tan\left(\frac{1}{2n}\right)-1}{\sqrt{2}};$

$c=\frac{tan\left(\frac{1}{2n^2}\right)-1}{\sqrt{2}}$**. Prove that:** $\frac{a+1}{a-1}=\frac{b+1}{b-1}\cdot\frac{c-1}{c+1}$

***Proposed by Mohammed Bouras-Morocco***

***Solution by Izumi Ainsworth-Lima-Peru***

$$a=\frac{1}{\sqrt{2}}csc\left(\frac{n-1}{2n^2}\right)\cdot\left(sin\left(\frac{n+1}{2n^2}\right)+cos\left(\frac{n+1}{2n^2}\right)\right)$$

$$=\frac{1}{\sqrt{2}sin\left(\frac{n-1}{2n^2}\right)}\cdot\sqrt{2}sin\left(\frac{\pi}{4}+\frac{n+1}{2n^2}\right)=\frac{sin\left(\frac{\pi}{4}+\frac{n+1}{2n^2}\right)}{sin\left(\frac{n-1}{2n^2}\right)}$$

***Now,*** $\frac{a+1}{a-1}=\frac{\sin\left(\frac{\pi}{4}+\frac{n+1}{2n^2}\right)+\sin\left(\frac{n-1}{2n^2}\right)}{\sin\left(\frac{\pi}{4}+\frac{n+1}{2n^2}\right)-\sin\left(\frac{n-1}{2n^2}\right)}=\frac{2\sin\left(\frac{\pi}{8}+\frac{1}{2n}\right)\cos\left(\frac{\pi}{8}+\frac{1}{2n^2}\right)}{2\cos\left(\frac{\pi}{8}+\frac{1}{2n}\right)\sin\left(\frac{\pi}{8}+\frac{1}{2n^2}\right)}$

$$=\tan\left(\frac{\pi}{8}+\frac{1}{2n}\right)\cot\left(\frac{\pi}{8}+\frac{1}{2n^2}\right)=\left(\frac{\tan\frac{\pi}{8}+\tan\frac{1}{2n}}{1-\tan\frac{\pi}{8}\tan\frac{1}{2n}}\right)\left(\frac{1-\tan\frac{\pi}{8}\tan\frac{1}{2n^2}}{\tan\frac{\pi}{8}+\tan\frac{1}{2n^2}}\right)$$

$$=\left(\frac{\sqrt{2}-1+\tan\frac{1}{2n}}{1-(\sqrt{2}-1)\tan\frac{1}{2n}}\right)\left(\frac{1-(\sqrt{2}-1)\tan\frac{1}{2n^2}}{\sqrt{2}-1+\tan\frac{1}{2n^2}}\right)$$

***Divide the numerator of the first member and the denominator of the second member by*** $\sqrt{2}$

***Divide the denominator of the first member and the numerator of the second member by*** $\sqrt{2}-1$

$$\frac{a+1}{a-1}=\left(\frac{1+\frac{\tan\frac{1}{2n}-1}{\sqrt{2}}}{\tan\frac{1}{2n}-\frac{1}{\sqrt{2}-1}}\right)\left(\frac{\tan\frac{1}{2n^2}-\frac{1}{\sqrt{2}-1}}{1+\frac{\tan\frac{1}{2n^2}-1}{\sqrt{2}}}\right)$$

$$=\left(\frac{1+\frac{\tan\frac{1}{2n}-1}{\sqrt{2}}}{\tan\frac{1}{2n}-(\sqrt{2}+1)}\right)\left(\frac{\tan\frac{1}{2n^2}-(\sqrt{2}+1)}{1+\frac{\tan\frac{1}{2n^2}-1}{\sqrt{2}}}\right)$$

***Divide the denominator of the first member and the numerator of the second member by*** $\sqrt{2}$

$$\frac{a+1}{a-1}=\left(\frac{\frac{\tan\frac{1}{2n}-1}{\sqrt{2}}+1}{\frac{\tan\frac{1}{2n}-1}{\sqrt{2}}-1}\right)\left(\frac{\frac{\tan\frac{1}{2n^2}-1}{\sqrt{2}}-1}{\frac{\tan\frac{1}{2n^2}-1}{\sqrt{2}}+1}\right)$$

***But***

$$b=\frac{\tan\left(\frac{1}{2n}\right)-1}{\sqrt{2}};\ c=\frac{\tan\left(\frac{1}{2n^2}\right)-1}{\sqrt{2}}$$

$\frac{a+1}{a-1} = \frac{b+1}{b-1} \cdot \frac{c-1}{c+1}$. ***Proved.***

**903.** **If $f \in C([0,1]), f(0) = f(1), n \in \mathbb{N}^*$ then: $\exists c_1, c_2, \dots, c_n$ −different in pairs such that:**

$$2f'(c_1) + 6f'(c_2) + \dots + n(n+1)f'(c_n) = 0$$

***Proposed by Marian Ursărescu-Romania***

***Solution by Remus Florin Stanca-Romania***

***We apply Lagrange's theorem and we have that:***

$$\frac{f\left(\frac{(k+1)(k+2)(k+3)}{n(n+1)(n+2)}\right) - f\left(\frac{k(k+1)(k+2)}{n(n+1)(n+2)}\right)}{\frac{(k+1)(k+2)\cdot 3}{n(n+1)(n+2)}} = f'(c_k) \quad (1)$$

$$c_k \in \left[\frac{k(k+1)(k+2)}{n(n+1)(n+2)}, \frac{(k+1)(k+2)(k+3)}{n(n+1)(n+2)}\right] \text{ and } \frac{k(k+1)(k+2)}{n(n+1)(n+2)} \in [0,1],$$

$$k = \overline{0, n-1} \overset{(1)}{\Rightarrow} \sum_{k=0}^{n-1} 3f'(c_{k+1}) \cdot \frac{k(k+1)(k+2)}{n(n+1)(n+2)}$$

$$= \sum_{k=0}^{n-1}\left(f\left(\frac{(k+1)(k+2)(k+3)}{n(n+1)(n+2)}\right) - f\left(\frac{k(k+1)(k+2)}{n(n+1)(n+2)}\right)\right)$$

$$= f(1) - f(0) = 0 \Rightarrow 2f'(c_1) + 6f'(c_2) + \dots + n(n+1)f'(c_n) = 0$$ ***. Proved.***

**904.** **Prove that:**

$$\int_0^\infty \frac{Li_{-3}(-x)}{\sqrt{x}} \cdot \frac{log(2+x)}{(2+x)} dx =$$

$$\pi\left(\frac{111}{4} - \frac{95}{2\sqrt{2}} + \frac{3}{4}\left(52\sqrt{2}log(2) - 49log(1+\sqrt{2}\right)\right)$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$\Omega = \int_0^\infty \frac{Li_{-3}(-x)}{\sqrt{x}} \cdot \frac{\log(2+x)}{(2+x)} dx$$

$$Li_{-3}(x) = \sum_{n=1}^{\infty} n^3 x^n = \frac{x^3 + 4x^2 + x}{(1-x)^4}$$

$$= -\frac{1}{1-x} + \frac{7}{(1-x)^2} - \frac{12}{(1-x)^3} + \frac{6}{(1-x)^4}$$

$$\Omega \overset{t=\sqrt{x}}{\cong} -2\int_0^{+\infty} \left(\frac{1}{1+t^2} - \frac{7}{(1+t^2)^2} + \frac{12}{(1+t^2)^3} - \frac{6}{(1+t^2)^4}\right) \cdot \frac{\log(2+x^2)}{(2+x^2)} dx$$

$$= -2(\Omega_1 - 7\Omega_2 + 12\Omega_3 - 6\Omega_4)$$

$$\Omega(a) = \int_0^{+\infty} \frac{\log(2+x^2)}{(2+x^2)(a+x^2)} dx ; 1 > a > 0$$

$$= \frac{1}{a-2}\left(\int_0^{+\infty} \frac{\log(2+x^2)}{2+x^2} dx - \int_0^{+\infty} \frac{\log(2+x^2)}{a+x^2} dx\right)$$

$$= \frac{\pi}{2(a-2)}\left(\frac{3\log(2)}{\sqrt{2}} - \frac{1}{\sqrt{a}}\left(\log\left(\frac{\sqrt{2}+\sqrt{a}}{\sqrt{2}-\sqrt{a}}\right) + \log(2-a)\right)\right)$$

$$= \frac{\pi}{2(a-2)}\left(\frac{3\log(2)}{\sqrt{2}} - \frac{2\log(\sqrt{2}+\sqrt{a})}{\sqrt{a}}\right)$$

$$\Omega_1 = \pi\log(1+\sqrt{2}) - \frac{3\pi\log(2)}{2\sqrt{2}}$$

$$\Omega_2 = -\frac{d}{da}\left(\frac{\pi}{2(a-2)}\left(\frac{3\log(2)}{\sqrt{2}} - \frac{2\log(\sqrt{2}+\sqrt{a})}{\sqrt{a}}\right)\right) \underbrace{=}_{a=1}$$

$$= \frac{3\pi\log(2)}{2\sqrt{2}} - \pi\log(1+\sqrt{2}) - \frac{\pi}{2}\left(\frac{1}{1+\sqrt{2}} - \log(1+\sqrt{2})\right)$$

$$= \frac{3\pi\log(2)}{2\sqrt{2}} - \frac{\pi}{2}\log(1+\sqrt{2}) + \frac{1-\sqrt{2}}{2}\pi$$

$$\Omega_3 = \frac{d^2}{2da^2}\left(\frac{\pi}{2(a-2)}\left(\frac{3log(2)}{\sqrt{2}}-\frac{2log(\sqrt{2}+\sqrt{a})}{\sqrt{a}}\right)\right)\underbrace{=}_{a=1}$$

$$= -\frac{3\pi log(2)}{2\sqrt{2}}-\frac{\pi}{8}\left(4-3\sqrt{2}-7log(1+\sqrt{2})\right)$$

$$\Omega_4 = -\frac{d^3}{6da^3}\left(\frac{\pi}{2(a-2)}\left(\frac{3log(2)}{\sqrt{2}}-\frac{2log(\sqrt{2}+\sqrt{a})}{\sqrt{a}}\right)\right)\underbrace{=}_{a=1}$$

$$= \frac{3\pi log(2)}{2\sqrt{2}}+\frac{\pi}{48}\left(35-31\sqrt{2}-27log(1+\sqrt{2})\right)$$

$$\Omega = \frac{78\pi log(2)}{\sqrt{2}}+\frac{111}{4}\pi-\frac{95\sqrt{2}}{4}\pi-\frac{147}{4}\pi log(1+\sqrt{2})$$

$$= \pi\left(\frac{111}{4}-\frac{95}{2\sqrt{2}}-\frac{3}{4}\left(52\sqrt{2}log(2)-49log(1+\sqrt{2})\right)\right)$$

***Finally:***

$$\int_0^\infty \frac{Li_{-3}(-x)}{\sqrt{x}}\cdot\frac{log(2+x)}{(2+x)}dx = \pi\left(\frac{111}{4}-\frac{95}{2\sqrt{2}}-\frac{3}{4}\left(52\sqrt{2}log(2)-49log(1+\sqrt{2})\right)\right)$$

***Note:***

$$I(a,b) = \int_0^{+\infty}\frac{log(b+x^2)}{a+x^2}dx\,;0<a<b.$$

$$\frac{\partial I(a,b)}{\partial b} = \int_0^{+\infty}\frac{dx}{(a+x^2)(b+x^2)} = \frac{1}{b-a}\left(\int_0^{+\infty}\frac{dx}{a+x^2}-\int_0^{+\infty}\frac{dx}{b+x^2}\right)$$

$$= \frac{1}{b-a}\left(\frac{\pi}{2\sqrt{a}}-\frac{\pi}{2\sqrt{b}}\right) = \frac{\pi}{2\sqrt{ab}}\cdot\frac{1}{\sqrt{a}+\sqrt{b}}$$

$$I(a,2) = \frac{\pi}{\sqrt{a}}log(\sqrt{a}+\sqrt{2})+I(a,0)-\frac{\pi}{2\sqrt{a}}log(a)$$

$$I(a,0) = 2\int_0^{+\infty}\frac{log(x)}{a+x^2}dx \overset{x=t\sqrt{a}}{\cong} \frac{2}{\sqrt{a}}\int_0^{+\infty}\frac{log(t)+log(\sqrt{a})}{1+t^2}dt = \frac{\pi log(a)}{2\sqrt{a}}$$

$$\int_0^{+\infty} \frac{log(2+x^2)}{a+x^2}dx = \frac{\pi log(\sqrt{2}+\sqrt{a})}{\sqrt{a}}; 0 < a < 1$$

**Let:** $a = b = 2$, **then:** $\int_0^{+\infty} \frac{log(2+x^2)}{2+x^2}dx = \frac{3\pi log(2)}{2\sqrt{2}}$

**905. For any complex numbers $a, b, c, n$ with $Re[a, b, c, n] > 0$; $Re[a + b + c] > 0$, define**

$$f_n(a, b, c) = \int_0^{\infty} \frac{(1 - e^{-ax})(1 - e^{-bx})e^{-nx}}{1 - e^{-cx}}dx$$

**then show that:**

$$e^{\int f_n(a,b,c)dn} = \frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution 1 by Kamel Benaicha-Algiers-Algerie***

$$f_n(a, b, c) = \int_0^{\infty} \frac{(1 - e^{-ax})(1 - e^{-bx})e^{-nx}}{1 - e^{-cx}}dx \overset{t=e^{-cx}}{=} \frac{1}{c}\int_0^1 \frac{\left(1 - t^{\frac{a}{c}}\right)\left(1 - t^{\frac{b}{c}}\right)t^{\frac{n}{c}-1}}{1 - t}dt$$

$$= \frac{1}{c}\int_0^1 \frac{t^{\frac{n}{c}-1} - t^{\frac{a+n}{c}-1} + t^{\frac{a+b+n}{c}-1} - t^{\frac{b+n}{c}-1}}{1 - t}dt$$

$$= \frac{1}{c}\int_0^1 \frac{\left(t^{\frac{n}{c}-1} - 1\right) + \left(1 - t^{\frac{a+n}{c}-1}\right) + \left(t^{\frac{a+b+n}{c}-1} - 1\right) + \left(1 - t^{\frac{b+n}{c}-1}\right)}{1 - t}dt$$

**We know that:** $\int_0^1 \frac{t^{\alpha-1}-1}{t-1}dt = H_{\alpha-1} = \psi(\alpha) + \gamma$

$$f_n(a, b, c) = \frac{1}{c}\left(\psi\left(\frac{a+n}{c}\right) + \psi\left(\frac{b+n}{c}\right) - \psi\left(\frac{a+b+n}{c}\right) - \psi\left(\frac{n}{c}\right)\right)$$

$$\int f_n(a,b,c)\,dn = \frac{1}{c}\int\left(\psi\left(\frac{a+n}{c}\right)+\psi\left(\frac{b+n}{c}\right)-\psi\left(\frac{a+b+n}{c}\right)-\psi\left(\frac{n}{c}\right)\right)dn$$

$$= log\Gamma\left(\frac{a+n}{c}\right)+log\Gamma\left(\frac{b+n}{c}\right)-log\Gamma\left(\frac{n}{c}\right)-log\Gamma\left(\frac{a+b+n}{c}\right)+K$$

$$= log\left(\frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}\right)+K, K\in\mathbb{C}$$

***So,*** $exp(\int f_n(a,b,c)\,dn) = exp\left(log\left(\frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}\right)+K\right)$

***Then:*** $e^{\int f_n(a,b,c)dn} = \frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}$

***Solution 2 by Tobi Josua-Nigeria***

$$f_n(a,b,c) = \int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-ax}}{1-e^{-cx}}dx = \int_0^\infty \frac{(1-e^{-a(x)})(e^{-ax}-e^{-(b+a)x})}{1-e^{-cx}}dx$$

$$= \int_0^\infty\int_a^0 e^{-yx}dy\int_{b+a}^a e^{-xt}dt\frac{x^2}{(1-e^{-cx})}dx = \sum_{k=0}^\infty\int_a^0 dy\int_{b+a}^a dt\int_0^\infty x^2e^{-x(y+t+ck)}dx$$

$$= \sum_{k=0}^\infty\int_a^0 dy\int_{b+a}^a dt\frac{\Gamma(3)}{(y+t+ck)^3} = 2\int_a^0 dy\int_{b+a}^a dt\sum_{k=0}^\infty\frac{1}{(y+t+ck)^3}$$

$$= -\frac{2}{c^3}\int_a^0 dy\int_{b+a}^a\frac{1}{2}\psi'\left(\frac{y+t}{c}\right)dt = -\frac{c}{c^3}\int_a^0 dy\int_{b+a}^a\frac{\partial}{\partial t}\psi'\left(\frac{y+t}{c}\right)$$

$$= -\frac{1}{c^2}\int_0^a dy\,\psi'\left(\frac{y+t}{c}\right)\Big|_{b+a}^a = -\frac{1}{c^2}\int_a^0\left[\psi'\left(\frac{y+n}{c}\right)-\psi'\left(\frac{y+n+b}{c}\right)\right]dy$$

$$= -\frac{c}{c^2}\left[\psi'\left(\frac{y+n}{c}\right)-\psi'\left(\frac{y+n+b}{c}\right)\right]\Big|_a^0$$

$$= \frac{1}{c}\left[-\psi\left(\frac{n}{c}\right)+\psi\left(\frac{n+b}{c}\right)+\psi\left(\frac{n+a}{c}\right)-\psi\left(\frac{n+a+b}{c}\right)\right]\ldots\ldots(\Omega)$$

***Hence,*** $\int f_n(a,b,c)\,dn = \frac{1}{c}\int\left(\psi\left(\frac{a+n}{c}\right)+\psi\left(\frac{b+n}{c}\right)-\psi\left(\frac{a+b+n}{c}\right)-\psi\left(\frac{n}{c}\right)\right)dn$

$$= log\Gamma\left(\frac{a+n}{c}\right) + log\Gamma\left(\frac{b+n}{c}\right) - log\Gamma\left(\frac{n}{c}\right) - log\Gamma\left(\frac{a+b+n}{c}\right)$$

$$= log\left(\frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}\right)$$

So, $exp(\int f_n(a,b,c)\,dn) = exp\left(log\left(\frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}\right)\right)$

Then: $e^{\int f_n(a,b,c)dn} = \frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}$

**906.** Let for $Re(n) > 0$

$$f(n) = \int_0^\infty \left(\frac{e^{-nx}}{1+x} - \frac{e^{-nx^2}}{1+x^2} - \frac{e^{-nx^3}}{1+x^3}\right)\frac{dx}{\sqrt{x}}$$

**Show that:**

$$\sum_{n=1}^\infty \left(\frac{\partial^2}{\partial n^2} f(n) - \frac{\partial}{\partial n} f(n)\right)$$

$$= \frac{1}{6}\left(6\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) - 3\zeta\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right) - 2\zeta\left(\frac{7}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)$$

*Proposed by Srinivasa Raghava-AIRMC-India*

*Solution 1 by Kamel Benaicha-Algiers-Algerie*

$$f(n) = \int_0^\infty \left(\frac{e^{-nx}}{1+x} - \frac{e^{-nx^2}}{1+x^2} - \frac{e^{-nx^3}}{1+x^3}\right)\frac{dx}{\sqrt{x}}$$

$$\frac{\partial f(n)}{\partial n} = -\int_0^\infty \left(\frac{xe^{-nx}}{1+x} - \frac{x^2e^{-nx^2}}{1+x^2} - \frac{x^3e^{-nx^3}}{1+x^3}\right)\frac{dx}{\sqrt{x}}$$

$$= -\int_0^\infty \frac{e^{-nx} - e^{-nx^2} - e^{-nx^3}}{\sqrt{x}}dx + f(n)$$

$$\frac{\partial^2 f(n)}{\partial n^2}=\int_0^\infty\left(\sqrt{x}e^{-nx}-x\sqrt{x}e^{-nx^2}-x^2\sqrt{x}e^{-nx^3}\right)dx+\frac{\partial f(n)}{\partial n}$$

$$\Omega(n)=\frac{\partial^2 f(n)}{\partial n^2}-\frac{\partial f(n)}{\partial n}=\int_0^\infty\left(\sqrt{x}e^{-nx}-x\sqrt{x}e^{-nx^2}-x^2\sqrt{x}e^{-nx^3}\right)dx$$

$$=\int_0^\infty\left(t^{\frac{1}{2}}-\frac{1}{2}t^{\frac{1}{4}}-\frac{1}{3}t^{\frac{1}{6}}\right)e^{-nt}dt\overset{z=nt}{\cong}\int_0^\infty\left(\frac{1}{n^{\frac{3}{2}}}z^{\frac{3}{2}-1}-\frac{1}{2n^{\frac{5}{4}}}z^{\frac{5}{4}-1}-\frac{1}{3n^{\frac{7}{6}}}z^{\frac{7}{6}-1}\right)e^{-z}$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{n^{\frac{3}{2}}}-\frac{\Gamma\left(\frac{5}{4}\right)}{2n^{\frac{5}{4}}}-\frac{\Gamma\left(\frac{7}{6}\right)}{3n^{\frac{7}{6}}}$$

$$\Omega=\sum_{n=1}^\infty\Omega(n)=\sum_{n=1}^\infty\frac{\Gamma\left(\frac{3}{2}\right)}{n^{\frac{3}{2}}}-\frac{\Gamma\left(\frac{5}{4}\right)}{2n^{\frac{5}{4}}}-\frac{\Gamma\left(\frac{7}{6}\right)}{3n^{\frac{7}{6}}}$$

$$=\frac{1}{6}\left(6\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)-3\zeta\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)-2\zeta\left(\frac{7}{6}\right)\Gamma\left(\frac{7}{6}\right)\right)$$

***Solution 2 by Ekpo Samuel-Nigeria***

***If*** $Re(n)>0$, ***consider:*** $f(n)=\int_0^\infty\left(\frac{e^{-nx}}{1+x}-\frac{e^{-nx^2}}{1+x^2}-\frac{e^{-nx^3}}{1+x^3}\right)\frac{dx}{\sqrt{x}}$

$$=\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{\sqrt{x}}-\underbrace{\int_0^\infty\left(\frac{e^{-nx^2}}{1+x^2}\right)\frac{dx}{\sqrt{x}}}_{let\ x^2\to x}-\underbrace{\int_0^\infty\left(\frac{e^{-nx^3}}{1+x^3}\right)\frac{dx}{\sqrt{x}}}_{let\ x^3\to x}$$

$$f(n)=\underbrace{\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{\sqrt{x}}}_{f_1(n)}-\frac{1}{2}\underbrace{\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{x^{3/4}}}_{f_2(n)}-\frac{1}{3}\underbrace{\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{x^{5/6}}}_{f_3(n)}$$

$f(n)=f_1(n)-\frac{1}{2}f_2(n)-\frac{1}{3}f_3(n)$***. Considering :***

$$f_1(n)=\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{\sqrt{x}}=\int_0^\infty e^{-nx}\int_0^\infty e^{-xt}e^{-t}dt\frac{dx}{\sqrt{x}}=\int_0^\infty e^{-t}\int_0^\infty e^{-x(n+t)}x^{-1/2}dxdt\overset{y=x(n+t)}{=}$$

$$=\int_0^\infty\frac{e^{-t}}{\sqrt{n+t}}\int_0^\infty e^{-y}y^{-\frac{1}{2}}dydt=\sqrt{\pi}\int_0^\infty\frac{e^{-t}}{\sqrt{n+t}}dt\overset{a=n+t}{=}\sqrt{\pi}\int_0^\infty\frac{e^{-(a-n)}}{\sqrt{a}}da$$

$$=\sqrt{\pi}e^{n}\int_{n}^{\infty}\frac{e^{-(\sqrt{a})^{2}}}{\sqrt{a}}da\overset{x=\sqrt{a}}{=}2\sqrt{\pi}e^{n}\int_{\sqrt{n}}^{\infty}e^{-x^{2}}dx=\pi e^{n}erfc(\sqrt{n})$$

$f_1(n)=\pi e^{n}erfc(\sqrt{n})$. ***Considering:***

$$f_2(n)=\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{x^{\frac{3}{4}}}=\int_0^\infty e^{-nx}\int_0^\infty e^{-xt}e^{-t}dt\frac{dx}{x^{\frac{3}{4}}}=\int_0^\infty e^{-t}\int_0^\infty e^{-x(n+t)}x^{-\frac{3}{4}}dxdt=$$

$$\overset{y=x(n+t)}{=}\int_0^\infty\frac{e^{-t}}{\sqrt[4]{n+t}}\int_0^\infty e^{-y}y^{-\frac{3}{4}}dydt=\Gamma\left(\frac{1}{4}\right)\int_0^\infty\frac{e^{-t}}{\sqrt[4]{n+t}}dt\overset{a=n+t}{=}\Gamma\left(\frac{1}{4}\right)\int_n^\infty\frac{e^{-(a-n)}}{\sqrt[4]{a}}da$$

$$=e^{n}\Gamma\left(\frac{1}{4}\right)\int_n^\infty e^{-a}a^{-1/4}da;\ recall\ \Gamma(a,z)=\int_z^\infty e^{-t}t^{a-1}dt\ we\ get:$$

$f_2(n)=e^{n}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4},n\right)$. ***Considering:***

$$f_3(n)=\int_0^\infty\left(\frac{e^{-nx}}{1+x}\right)\frac{dx}{x^{\frac{5}{6}}}=\int_0^\infty e^{-nx}\int_0^\infty e^{-xt}e^{-t}dt\frac{dx}{x^{\frac{5}{6}}}=\int_0^\infty e^{-t}\int_0^\infty e^{-x(n+t)}x^{-\frac{5}{6}}dxdt=$$

$$\overset{y=x(n+t)}{=}\int_0^\infty\frac{e^{-t}}{\sqrt[6]{n+t}}\int_0^\infty e^{-y}y^{-\frac{5}{6}}dydt=\Gamma\left(\frac{1}{6}\right)\int_0^\infty\frac{e^{-t}}{\sqrt[6]{n+t}}dt\overset{a=n+t}{=}\Gamma\left(\frac{1}{6}\right)\int_n^\infty\frac{e^{-(a-n)}}{\sqrt[6]{a}}da$$

$$=e^{n}\Gamma\left(\frac{1}{6}\right)\int_n^\infty e^{-a}a^{-\frac{1}{6}}da;\ recall\ \Gamma(a,z)=\int_z^\infty e^{-t}t^{a-1}dt\ we\ get:$$

$f_3(n)=e^{n}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6},n\right)$. ***Recall:*** $f(n)=f_1(n)-\frac{1}{2}f_2(n)-\frac{1}{3}f_3(n)$

$$f(n)=\pi e^{n}erfc(\sqrt{n})-\frac{1}{2}e^{n}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4},n\right)-\frac{1}{3}e^{n}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6},n\right)$$

***Note that:*** $\frac{d}{dn}\Gamma(a,n)=-e^{-n}n^{a-1}$ $and$ $\frac{d}{dn}erfc(n)=-\frac{2e^{-n^{2}}}{\sqrt{\pi}}$

$$\frac{d}{dn}f(n)=\pi e^{n}erfc(\sqrt{n})-\sqrt{\frac{\pi}{n}}-\frac{1}{2}e^{n}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4},n\right)-\frac{1}{3}e^{n}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6},n\right)+\frac{\Gamma\left(\frac{1}{6}\right)}{3\sqrt[6]{n}}$$

$$\frac{d^{2}}{dn^{2}}f(n)=\pi e^{n}erfc(\sqrt{n})-\sqrt{\frac{\pi}{n}}+\frac{\sqrt{\pi}}{2n^{\frac{3}{2}}}-\frac{1}{2}e^{n}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4},n\right)+\frac{\Gamma\left(\frac{1}{4}\right)}{4\sqrt[4]{n}}-\frac{\Gamma\left(\frac{1}{4}\right)}{8n^{\frac{5}{4}}}-$$

$$-\frac{1}{3}e^n\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6},n\right)+\frac{\Gamma\left(\frac{1}{6}\right)}{3\sqrt[6]{n}}-\frac{\Gamma\left(\frac{1}{6}\right)}{18n^{\frac{7}{6}}}$$

$$\frac{d^2}{dn^2}f(n)-\frac{d}{dn}f(n)=\frac{\sqrt{\pi}}{2n^{\frac{3}{2}}}-\frac{\Gamma\left(\frac{1}{4}\right)}{8n^{\frac{5}{4}}}-\frac{\Gamma\left(\frac{1}{6}\right)}{18n^{\frac{7}{6}}}$$

$$\sum_{n=1}^{\infty}\left(\frac{d^2}{dn^2}f(n)-\frac{d}{dn}f(n)\right)=\sum_{n=1}^{\infty}\frac{\sqrt{\pi}}{2n^{\frac{3}{2}}}-\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{1}{4}\right)}{8n^{\frac{5}{4}}}-\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{1}{6}\right)}{18n^{\frac{7}{6}}}=$$

$$=\frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right)-\frac{\Gamma\left(\frac{1}{4}\right)}{8}\zeta\left(\frac{5}{4}\right)-\frac{\Gamma\left(\frac{1}{6}\right)}{18}\zeta\left(\frac{7}{6}\right)\overset{\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}}{=}$$

$$=\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)-\frac{\Gamma\left(\frac{1}{4}\right)}{8}\zeta\left(\frac{5}{4}\right)-\frac{\Gamma\left(\frac{1}{6}\right)}{18}\zeta\left(\frac{7}{6}\right)\dots\dots(1)$$

***Using that:*** $\Gamma(n)=\frac{\Gamma(n+1)}{n}$

$$\Gamma\left(\frac{1}{4}\right)=4\Gamma\left(\frac{5}{4}\right)\ and\ \Gamma\left(\frac{1}{6}\right)=6\Gamma\left(\frac{7}{6}\right)$$

$$=\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)-\frac{4\Gamma\left(\frac{5}{4}\right)}{8}\zeta\left(\frac{5}{4}\right)-\frac{6\Gamma\left(\frac{7}{6}\right)}{18}\zeta\left(\frac{7}{6}\right)=\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)-\frac{\Gamma\left(\frac{5}{4}\right)}{2}\zeta\left(\frac{5}{4}\right)-\frac{\Gamma\left(\frac{7}{6}\right)}{3}\zeta\left(\frac{7}{6}\right)$$

$$=\frac{1}{6}\left(6\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)-3\Gamma\left(\frac{5}{4}\right)\zeta\left(\frac{5}{4}\right)-2\Gamma\left(\frac{7}{6}\right)\zeta\left(\frac{7}{6}\right)\right)$$

**907. Let for any positive integer $n\geq 1$**

$$F(n)=\int_{-\pi}^{\pi}\frac{cos^n x}{1+e^{x^3}}dx$$

**Prove the following sums:**

$$\sum_{n=1}^{\infty}\frac{F(n)}{n}=log(2);\ \sum_{n=1}^{\infty}\frac{F(n)}{n^2}=\frac{\pi^2}{24}-\frac{log^2(2)}{2}$$

***Proposed by Srinivasa Raghava-AIRMC-India***

*Solution by Kamel Benaicha-Algiers-Algerie*

$$F(n)=\int_{-\pi}^{\pi}\frac{cos^n x}{1+e^{x^3}}dx=\int_{-\pi}^{0}\frac{cos^n x}{1+e^{x^3}}dx+\int_{0}^{\pi}\frac{cos^n x}{1+e^{x^3}}dx=\int_{0}^{\pi}cos^n x\,dx$$

$$S_1=\sum_{n=1}^{\infty}\frac{F(n)}{n}=\int_0^{\pi}\sum_{n=1}^{\infty}\frac{cos^n x}{n}dx=-\int_0^{\pi}log(1-cosx)dx$$

$$=-\pi log(2)-2\int_0^{\pi}log\left(sin\frac{x}{2}\right)dx\overset{t=\frac{x}{2}}{=}=-\pi log(2)-4\int_0^{\frac{\pi}{2}}log(sin\,t)dx$$

$$=-\pi log(2)-4\int_0^{\frac{\pi}{2}}log(cos\,t)dx=\pi log(2)$$

$$S_2=\sum_{n=1}^{\infty}\frac{F(n)}{n^2}=\int_0^{\pi}\sum_{n=1}^{\infty}\frac{cos^n x}{n^2}dx=\int_0^{\pi}Li_2(cosx)dx=\int_0^{\pi}Li_2(-cosx)dx$$

$$=\frac{1}{2}\int_0^{\pi}Li_2(cos^2x)dx-\int_0^{\pi}Li_2(cosx)dx=\frac{1}{4}\int_0^{\frac{\pi}{2}}Li_2(cos^2x)dx+\frac{1}{4}\int_0^{\frac{\pi}{2}}Li_2(sin^2x)dx$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\left(Li_2(cos^2x)+Li_2(1-cos^2x)\right)dx=\frac{1}{4}\int_0^{\frac{\pi}{2}}\left(\frac{\pi^2}{6}-4log(cosx)\cdot log(sinx)\right)dx$$

$$=\frac{\pi^3}{48}-\int_0^{\frac{\pi}{2}}\left(log(cosx)\cdot log(sinx)\right)dx$$

$$\therefore\left(log(sinx)+log(cosx)\right)^2=log^2(sinx)+log^2(cosx)+2log(sinx)\cdot log(cosx)$$

$$\therefore\left(log(sin2x)+log(2)\right)^2=log^2(sinx)+log^2(cosx)+2log(sinx)\cdot log(cosx)$$

$$S_2=\frac{\pi^3}{48}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\left(log(sin2x)+log(2)\right)^2-(log^2(cosx)+log^2(sinx)\right)dx$$

$$=\frac{\pi^3}{48}-\frac{1}{2}\left(\int_0^{\pi} log^2(sint)dt-log(2)\int_0^{\pi} log(sint)dt\right)-\frac{\pi}{4}log^2(2)+\int_0^{\frac{\pi}{2}} log^2(sinx)dx$$

$$=\frac{\pi^3}{48}+\frac{1}{2}\int_0^{\frac{\pi}{2}} log^2(sinx)dx-\frac{3\pi}{4}log^2(2)$$

$$=\frac{\pi^3}{48}+\frac{1}{2}\left[\frac{d^2}{ds^2}\int_0^{\frac{\pi}{2}} sin^s x\,dx\right]_{s=0}-\frac{3\pi}{4}log^2(2)=\frac{\pi^3}{24}-\frac{\pi}{2}log^2(2)$$

***Notes:*** $Li_2(-x)=\sum\frac{x^{2n}}{4n^2}-\sum\frac{x^{2n+1}}{(2n+1)^2}=\frac{1}{2}Li_2(x^2)-Li_2(x)$

$$Li_2(x)+Li_2(1-x)=\frac{\pi^2}{6}-log(x)log(1-x)$$

$$\int_0^{\pi} log^n(sinx)dx=\int_0^{\pi/2} log^n(sinx)dx+\int_{\pi/2}^{\pi} log^n(sinx)dx\overset{x\to\pi-x}{\cong}2\int_0^{\pi/2} log^n(sinx)dx$$

$$\int_0^{\pi/2} log(sinx)dx=-\frac{\pi}{2}log(2)$$

**908. If $j\geq 0, q\geq 2$ is an even integer and positive integer respectively.**

**Prove:**

$$\int_0^1\sum_{i=1}^{\infty}\sum_{k=2}^{i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(k^n m+k^m n)(i-k+2)^q}\cdot\frac{x^j+x^{j-1}+\cdots+x+1}{x^{j+2}+x^{j+1}+\cdots+x^2+x+1}dx$$

$$=\frac{\pi}{2(j+3)}tan\left(\frac{\pi(j+1)}{2(j+3)}\right)\big(\zeta(2)+2\zeta(3)+\zeta(4)\big)\zeta(q)$$

***Proposed by Naren Bhandari-Bajura-Nepal***

***Solution by Sergio Esteban-Argentina***

***Interchanging the sum and integral justified by Fubini's theorem***

$$\Omega=\sum_{i=1}^{\infty}\sum_{k=2}^{i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(k^nm+k^mn)(i-k+2)^q}\cdot\int_0^1\frac{x^j+x^{j-1}+\dots+x+1}{x^{j+2}+x^{j+1}+\dots+x^2+x+1}dx$$

$$=\sum_{k=2}^{i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(k^nm+k^mn)}\sum_{i=1}^{\infty}\frac{1}{i^q}\cdot\int_0^1\frac{1-x^{j+1}}{1-x^{j+3}}dx$$

***i)*** $\sum_{k=2}^{i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(k^nm+k^mn)}=\frac{\zeta(2)+2\zeta(3)+\zeta(4)}{2}$ ***,this is Limit-285 that we can find on the official RMM website.***

***ii)***

***iii)*** $\sum_{i=1}^{\infty}\frac{1}{i^q}=\zeta(q)$ ***by the Riemman zeta function definition.***

***iv) by notable rations or ''cocientes notables''***

$I_n=\int_0^1\frac{x^j+x^{j-1}+\dots+x+1}{x^{j+2}+x^{j+1}+\dots+x^2+x+1}dx=\int_0^1\frac{1-x^{j+1}}{1-x^{j+3}}dx$***, by a power series expansion of the integrand and then integrate term-by-term to arrive at an infinite sum:***

***The integrand is*** $\frac{1-x^{j+1}}{1-x^{j+3}}=(1-x^{j+1})=\left(1+x^{j+3}+x^{2(j+3)}+x^{3(j+3)}+\cdots\right)$, ***then***

$$I_n=\int_0^1(1-x^{j+1})\sum_{k=0}^{\infty}x^{k(j+3)}dx=\sum_{k=0}^{\infty}\int_0^1\left(x^{k(j+3)}-x^{k(j+3)+(j+1)}\right)dx$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{k(j+3)+1}-\frac{1}{k(j+3)+(j+1)+1}\right)=\frac{1}{j+3}\sum_{k=1}^{\infty}\left(\frac{1}{k-\frac{j+2}{j+3}}-\frac{1}{k-\frac{1}{j+3}}\right)$$

$$=\frac{1}{j+3}\left\{\psi\left(1-\frac{1}{j+3}\right)-\psi\left(1-\frac{j+2}{j+3}\right)\right\}=\frac{1}{j+3}\left\{\psi\left(1-\frac{1}{j+3}\right)-\psi\left(\frac{1}{j+3}\right)\right\}$$

***By Euler's reflection formula*** $=\frac{\pi}{j+3}cot\left(\frac{\pi}{j+3}\right)=\frac{\pi}{j+3}tan\left(\frac{\pi(j+1)}{j+3}\right)$

***By (i),(ii),(iii) we have:***

$$\Omega=\frac{\pi}{2(j+3)}tan\left(\frac{\pi(j+1)}{2(j+3)}\right)(\zeta(2)+2\zeta(3)+\zeta(4))\zeta(q)$$

**909.** If $x, y, z \geq 0$ then:

$$\sum_{cyc} \frac{(x+1)(y+1)}{(x+2)(y+2)} = \frac{3}{4} \Rightarrow \sum_{cyc} \sqrt{(x+1)(y+1)} = 3$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Florentin Vişescu-Romania***

$$x \geq 0 \Rightarrow x+2 \geq 2 \Rightarrow \frac{1}{x+2} \leq \frac{1}{2} \Rightarrow -\frac{1}{x+2} \geq -\frac{1}{2} \Rightarrow 1-\frac{1}{x+2} \geq 1-\frac{1}{2} \Rightarrow$$

$$1-\frac{1}{x+2} \geq \frac{1}{2} \text{ and } 1-\frac{1}{y+2} \geq \frac{1}{2} \Rightarrow \left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{y+2}\right) \geq \frac{1}{4}$$

$$\sum_{cyc}\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{y+2}\right) \geq \frac{3}{4}, \text{ but } \sum_{cyc}\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{y+2}\right) = \frac{3}{4} \Rightarrow$$

$$1-\frac{1}{x+2} = 1-\frac{1}{y+2} = 1-\frac{1}{z+2} \Leftrightarrow x=y=z=0 \Rightarrow \sum_{cyc}\sqrt{(x+1)(y+1)} = 3$$

***Solution 2 by Ravi Prakash-New Delhi-India***

$$\sum_{cyc} \frac{(x+1)(y+1)}{(x+2)(y+2)} = \frac{3}{4}; \quad (1)$$

*Rewrite (1) as* $\sum_{cyc}\left(\frac{(x+1)(y+1)}{(x+2)(y+2)} - \frac{1}{4}\right) = 0; \quad (2)$

*For* $x, y \geq 0$ *we have:*

$4(x+1)(y+1)-(x+2)(y+2) = 3xy+2x+2y \geq 0$, *equality for* $x=y=0$

$\frac{(x+1)(y+1)}{(x+2)(y+2)} - \frac{1}{4} \geq 0$ *equality for* $x=y=0$

*As each terms of* $LHS_{(2)} \geq 0$

*(2) can hold iff* $x=y=z=0 \Rightarrow \sum_{cyc}\sqrt{(x+1)(y+1)} = 3$

**910.** Prove the following general result:

$$\lim_{n\to\infty} \frac{n^k}{\sqrt[n]{(n^k+1^k)(n^k+2^k)(n^k+3^k)\dots(2n^k)}} = \frac{e^k}{2exp\left(\Phi\left(-1,1,\frac{1}{k}\right)\right)}$$

and hence for $k=6$

$$\lim_{n\to\infty}\frac{n^6}{\sqrt[n]{(n^6+1^6)(n^6+2^6)(n^6+3^6)\dots(2n^6)}}=\frac{e^6}{2\left(2+\sqrt{3}\right)^{\sqrt{3}}e^{\pi}}$$

**where $k\in\mathbb{N}$ and $\Phi(z,a,b)$ is Lerch transcendent.**

***Proposed by Narendra Bhandari-Bajura-Nepal***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$\Omega(k)=\lim_{n\to\infty}\frac{n^k}{\sqrt[n]{(n^k+1^k)(n^k+2^k)(n^k+3^k)\dots(2n^k)}}$$

$$=\lim_{n\to\infty}\frac{n^k}{e^{\log\left(\sqrt[n]{(n^k+1^k)(n^k+2^k)(n^k+3^k)\dots(2n^k)}\right)}}=\lim_{n\to\infty}\frac{n^k}{e^{\frac{1}{n}\sum_{p=1}^{n}\log\left(n^k\left(1+\left(\frac{p}{n}\right)^k\right)\right)}}$$

$$=\lim_{n\to\infty}\frac{n^k}{e^{\frac{1}{n}\sum_{k=1}^{n}\log(n^k)+\frac{1}{n}\sum_{p=1}^{n}\log\left(1+\left(\frac{p}{n}\right)^k\right)}}=\lim_{n\to\infty}\frac{n^k}{e^{\log(n^k)}e^{\frac{1}{n}\sum_{p=1}^{n}\log\left(1+\left(\frac{p}{n}\right)^k\right)}}$$

$$=\lim_{n\to\infty}\frac{1}{e^{\frac{1}{n}\sum_{p=1}^{n}\log\left(1+\left(\frac{p}{n}\right)^k\right)}}=\frac{1}{e^{\lim_{n\to\infty}\frac{1}{n}\sum_{p=1}^{n}\log\left(1+\left(\frac{p}{n}\right)^k\right)}}=\frac{1}{e^{\int_0^1\log(1+x^k)dx}}$$

$$=\frac{1}{e^{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{nk}dx}}=\frac{1}{e^{\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n(nk+1)}}}$$

$$\frac{1}{n(nk+1)}=\frac{1}{n}-\frac{k}{nk+1}=\frac{1}{n}-\frac{1}{n+\frac{1}{k}}$$

$$\Omega(k)=\frac{1}{e^{\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{k}{nk+1}\right)}}=\frac{1}{e^{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\frac{1}{k}}}}$$

$$=\frac{1}{e^{\log 2-k+\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\frac{1}{k}}}}=\frac{1}{2e^{-k}e^{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{k}}}}=\frac{e^k}{2\exp\left(\Phi\left(-1,1,\frac{1}{k}\right)\right)}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\frac{1}{k}}=-k+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{k}}$$

$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$ ***denote Lerch transcendent function.***

***Put $k=6$, we get:***

$$\Phi\left(-1,1,\frac{1}{6}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{6}}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{12}}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}+\frac{1}{12}}\right)$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{12}\right)\left(n+\frac{1}{2}+\frac{1}{12}\right)}$$

***Using Gauss theorem or residue theorem, we get:***

$$\Phi\left(-1,1,\frac{1}{6}\right)=\frac{1}{2}\left(\psi\left(\frac{1}{2}+\frac{1}{12}\right)-\psi\left(\frac{1}{12}\right)\right)=\pi+\sqrt{3}log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$=\pi-\sqrt{3}log\left(\frac{\left(\sqrt{3}+1\right)^2}{2}\right)=\pi+\sqrt{3}log\left(\frac{4+2\sqrt{3}}{2}\right)=\pi+log\left(\left(2+\sqrt{3}\right)^{\sqrt{3}}\right)$$

$$\Omega(6)=\frac{e^6}{2\left(2+\sqrt{3}\right)^{\sqrt{3}}e^{\pi}}$$

**911. Find:**

$$\int_0^{\frac{\pi}{4}} x^4\log(sin(2x))\,dx$$

***Proposed by Mokhtar Khassani-Mostaganem-Algerie***

***Solution by Avishek Mitra-West Bengal-India***

$$I=\int_0^{\frac{\pi}{4}} x^4\log(sin(2x))\,dx=\int_0^{\frac{\pi}{4}} x^4\left[-\log(2)-\sum_{n=1}^{\infty}\frac{\cos(4nx)}{n}\right]dx$$

$$=-\log(2)\int_0^{\frac{\pi}{4}} x^4dx-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}} x^4\cos(4nx)\,dx$$

$$I_1=\int_0^{\frac{\pi}{4}} x^4\cos(4nx)\,dx=\left[x^4\frac{\sin(4nx)}{4n}\right]\Bigg|_0^{\frac{\pi}{4}}-4\int_0^{\frac{\pi}{4}} x^3\frac{\sin(4nx)}{4n}dx$$

$$= -\frac{1}{n}\int_0^{\frac{\pi}{4}} x^3 \sin(4nx)\,dx$$

$$= -\frac{1}{n}\left[\left(-x^3\frac{\cos(4nx)}{4n}\right)\Bigg|_0^{\frac{\pi}{4}} + 3\int_0^{\frac{\pi}{4}} x^2\frac{\cos(4nx)}{4n}dx\right]$$

$$= -\frac{1}{n}\left[-\frac{\pi^3}{64}\cdot\frac{\cos(4nx)}{4n} + \frac{3}{4n}\int_0^{\frac{\pi}{4}} x^2\cos(4nx)\,dx\right]$$

$$= \frac{(-1)^n\pi^3}{256n^2} - \frac{3}{4n^2}\left[\left(x^2\frac{\sin(4nx)}{4n}\right)\Bigg|_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} x\frac{\sin(4nx)}{4n}dx\right]$$

$$= \frac{(-1)^n\pi^3}{256n^2} - \frac{3}{4n^2}\left[-\frac{1}{2n}\int_0^{\frac{\pi}{4}} x\sin(4nx)dx\right]$$

$$= \frac{(-1)^n\pi^3}{256n^2} - \frac{3}{8n^3}\left[\left(-x\cdot\frac{\cos(4nx)}{4n}\right)\Bigg|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\cdot\frac{\cos(4nx)}{4n}dx\right]$$

$$= \frac{(-1)^n\pi^3}{256n^2} - \frac{3}{8n^3}\left[-\frac{\pi}{4}\cdot\frac{(-1)^n}{4n} + \frac{1}{16n^2}(\sin(4nx))\Bigg|_0^{\frac{\pi}{4}}\right] = \frac{(-1)^n\pi^3}{256n^2} - \frac{3\pi}{128}\cdot\frac{(-1)^{n-1}}{n^4}$$

$$I = -\log(2)\frac{x^5}{5}\Bigg|_0^{\frac{\pi}{4}} + \frac{\pi^3}{256}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \frac{3\pi}{128}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5}$$

$$= -\log(2)\frac{\pi^5}{5\cdot 4^5} + \frac{\pi^3}{256}\eta(3) - \frac{3\pi}{128}\eta(5)$$

$$= -\log(2)\cdot\frac{\pi^5}{5120} + \frac{\pi^3}{256}\cdot\frac{3}{4}\zeta(3) - \frac{3\pi}{128}\cdot\frac{15}{16}\zeta(5)$$

$$= -\log(2)\cdot\frac{\pi^5}{5120} + \frac{3\pi^3}{1024}\zeta(3) - \frac{15\pi}{2048}\zeta(5)$$

**912. Find without softs:**

$$\Omega=\int_0^{\infty}\frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx$$

*Proposed by Abdul Mukhtar-Nigeria*

*Solution by Kamel Benaicha-Algiers-Algerie*

$$\Omega=\int_0^{+\infty}\frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx=\int_0^{+\infty}\frac{\log\left(1+x^4+x^6(1+x^4)\right)dx}{(1+x)(1-x+x^2)+3x(1+x)}$$

$$=\int_0^{+\infty}\frac{\log\left((1+x^4)(1+x^6)\right)dx}{(1+x)^3}=\int_0^{+\infty}\frac{\log(1+x^4)dx}{(1+x)^3}+\int_0^{\infty}\frac{\log(1+x^6)dx}{(1+x)^3}$$

$$\stackrel{I.B.P.}{=}2\int_0^{+\infty}\frac{x^3dx}{(1+x)^2(1+x^4)}+3\int_0^{+\infty}\frac{x^5dx}{(1+x)^2(1+x^6)}$$

$$\frac{x^3}{(1+x)^2(1+x^4)}=\frac{a_1x^3+a_2x^2+a_3x+a_4}{1+x^4}+\frac{a_5}{1+x}+\frac{a_6}{(1+x)^2}$$

***By comparison, we get:***

$$\frac{x^3}{(1+x)^2(1+x^4)}=\frac{1}{2}\left(\frac{-x^3+2x^2-x}{1+x^4}+\frac{1}{1+x}-\frac{1}{(1+x)^2}\right)$$

***And:***

$$\frac{x^5}{(1+x)^2(1+x^6)}=\frac{b_1x^5+b_2x^4+b_3x^3+b_4x^2+b_5x+b_6}{1+x^6}+\frac{b_7}{1+x}+\frac{b_8}{2(1+x)^2}$$

***By comparison, we get:***

$$\frac{x^5}{(1+x)^2(1+x^6)}=\frac{1}{6}\cdot\frac{-6x^5+9x^4-6x^3+3x^2-3}{1+x^6}+\frac{1}{1+x}-\frac{1}{2(1+x)^2}$$

***So:***

$$\Omega=\int_0^{+\infty}\left(\frac{1}{1+x}-\frac{x^3}{1+x^4}\right)dx+3\int_0^{+\infty}\left(\frac{1}{1+x}-\frac{x^5}{1+x^6}\right)dx$$

$$+\int_0^{+\infty}\frac{2x^2-x}{1+x^4}dx-\frac{5}{2}\int_0^{+\infty}\frac{dx}{(1+x)^2}+\frac{1}{2}\int_0^{+\infty}\frac{9x^4-6x^3+3x^2-3}{1+x^6}dx$$

$$= log\left(\frac{1+x}{\sqrt[4]{1+x^4}}\right)\Bigg|_0^{+\infty} + 3log\left(\frac{1+x}{\sqrt[6]{1+x^6}}\right)\Bigg|_0^{+\infty} + \int_0^{+\infty}\frac{2x^2-x}{1+x^4}dx + \frac{5}{2}\cdot\frac{1}{1+x}\Bigg|_0^{+\infty} + \frac{1}{2}\int_0^{+\infty}\frac{9x^4-6x^3+3x^2-3}{1+x^6}dx$$

$$= \frac{1}{4}\int_0^{+\infty}\frac{2t^{-\frac{1}{4}}-t^{-\frac{1}{2}}}{1+t}dt + \frac{1}{12}\int_0^{+\infty}\frac{9t^{-\frac{1}{6}}-6t^{-\frac{2}{3}}+3t^{-\frac{1}{2}}-3t^{-\frac{5}{6}}}{1+t}ddt - \frac{5}{2}$$

$$= \frac{\pi}{4}\left(\frac{2}{sin\frac{\pi}{4}} - \frac{1}{sin\frac{\pi}{2}}\right) + \frac{\pi}{4}\left(\frac{3}{sin\frac{\pi}{6}} - \frac{2}{sin\frac{2\pi}{3}} + \frac{1}{sin\frac{\pi}{2}} - \frac{1}{sin\frac{5\pi}{6}}\right) - \frac{5}{2}$$

$$= \frac{\pi}{4}\left(2\sqrt{2} - 1 + 6 - \frac{4\sqrt{3}}{3} + 1 - 2\right) - \frac{5}{2} = \frac{\pi}{12}\left(12 + 6\sqrt{2} - 4\sqrt{3}\right) - \frac{5}{2}$$

*So,*

$$\Omega = \int_0^{\infty}\frac{log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx = \frac{\pi}{12}\left(12+6\sqrt{2}-4\sqrt{3}\right) - \frac{5}{2}$$

**913. Find:**

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\left(\frac{x\cdot\log x}{1-x^2+x^4}\right)dx\right)$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution 1 by Kamel Benaicha-Algiers-Algerie***

$$I = \int_0^1\frac{x\ln(x)}{1-x^2+x^4}dx = \frac{1}{4}\int_0^1\frac{(1+t)}{1+t^3}\ln(t)\,dt$$

$$= \frac{1}{4}\sum_{n=0}^{+\infty}(1)^n\int_0^1\left(t^{3n+1}+t^{3n}\right)\ln(t)\,dt$$

***Let*** $z = -\ln(t) \Rightarrow t = e^{-z} \Rightarrow dt = -e^{-z}dz$

$$= -\frac{1}{4}\sum_{n=0}^{+\infty}(1)^n\int_0^{+\infty}z\left(e^{-(3n+2)z}+e^{-3(n+1)z}\right)dz = -\frac{1}{4}\sum_{n}^{+\infty}(1)^n\left(\frac{1}{(3n+2)^2}+\frac{1}{(3n+1)^2}\right)$$

$$= -\frac{1}{4}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{(3n+2)^2} + \sum_{n=1}^{+\infty}\frac{(-1)^n}{(3n+1)^2}\right) = -\frac{1}{36}\left(\sum_{n=0}^{+\infty}\frac{(-1)^n}{\left(n+\frac{2}{3}\right)^2} + \sum_{n=1}^{+\infty}\frac{(-1)^n}{\left(n+\frac{1}{3}\right)^2}\right)$$

$$= -\frac{1}{144}\left(\sum_{n=0}^{+\infty}\left(\frac{1}{\left(n+\frac{1}{3}\right)^2}+\frac{1}{\left(n+\frac{1}{6}\right)^2}-\frac{1}{\left(n+\frac{5}{5}\right)^2}-\frac{1}{\left(n+\frac{2}{3}\right)}\right)\right)$$

$$= -\frac{1}{144}\left(\Psi^{(1)}\left(\frac{1}{3}\right)+\Psi^{(1)}\left(\frac{1}{6}\right)-\Psi^{(1)}\left(\frac{5}{6}\right)-\Psi^{(1)}\left(\frac{2}{3}\right)\right)$$

## Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\left(\frac{x\cdot\log x}{1-x^2+x^4}\right)dx\right) = \int_0^1 \frac{x(1+x^2)\log x}{(1+x^2)(1-x^2+x^4)}dx$$

$$= \int_0^1 \frac{(x+x^3)\log x}{1+x^6}dx = \int_0^1 (x+x^3)\log x\sum_{k=1}^{\infty}(-1)^{k-1}x^{6(k-1)}dx$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{6(k-1)}(x+x^6)[D_a(x^a)]_{a=0}\,dx$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}D_a\left[\int_0^1 x^a x^{6(k-1)}(x+x^3)dx\right]\Bigg|_{a=0}$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}D_a\left[\int_0^1 (x^{a+6k-5}+x^{a+6k-3})dx\right]\Bigg|_{a=0}$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}D_a\left[\frac{x^{a+6k-4}}{a+6k-4}+\frac{x^{a+6k-2}}{a+6k-2}\right]\Bigg|_0^1\Bigg|_{a=0}$$

$$= -\sum_{k=1}^{\infty}(-1)^{k-1}D_a\left(\frac{1}{a+6k-4}+\frac{1}{a+6k-2}\right)\Bigg|_{a=0}$$

$$= -\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{1}{(6k-4)^2}+\frac{1}{(6k-2)^2}\right] = -\frac{1}{4}\sum_{k=1}^{\infty}(-1)^{k-1}D_a\left[\frac{1}{(3k-2)^2}+\frac{1}{(3k-1)^2}\right]$$

$$= -\frac{1}{4}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(3k-2)^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(3k-1)^2}\right] = -\frac{1}{144}\left[\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{2}{3}\right)+\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{5}{6}\right)\right]$$

$$= -\frac{1}{144}\left[\psi_1\left(\frac{1}{6}\right)+\psi_1\left(1-\frac{1}{6}\right)+\psi_1\left(\frac{1}{3}\right)+\psi_1\left(1-\frac{1}{3}\right)-2\psi_1\left(\frac{2}{3}\right)-2\psi_1\left(\frac{5}{6}\right)\right]$$

$$= -\frac{1}{144}\left\{D_a^2\{log[\Gamma(x)\Gamma(1-x)]\}|_{x=\frac{1}{6}} + D_a^2\{log[\Gamma(y)\Gamma(1-y)]\}|_{x=\frac{1}{3}} - 2\left[\psi_1\left(\frac{5}{6}\right) + \psi_1\left(\frac{2}{3}\right)\right]\right\}$$

$$= -\frac{1}{144}\left\{D_x^2\left\{log\left[\frac{\pi}{sin(\pi x)}\right]\right\}\Big|_{x=\frac{1}{6}} + D_y^2\left\{log\left[\frac{\pi}{sin(\pi y)}\right]\right\}\Big|_{y=\frac{1}{3}} - 2\left[\psi_1\left(\frac{5}{6}\right) + \psi_1\left(\frac{2}{3}\right)\right]\right\}$$

$$= -\frac{1}{144}\left\{D_x\left[-\pi\frac{cos(\pi x)}{sin(\pi x)}\right]\Big|_{x=\frac{1}{6}} + D_y\left[-\pi\frac{cos(\pi y)}{sin(\pi y)}\right]\Big|_{x=\frac{1}{6}} - 2\left[\psi_1\left(\frac{5}{6}\right) + \psi_1\left(\frac{2}{3}\right)\right]\right\}$$

$$= -\frac{1}{72}\left[\frac{8}{3}\pi^2 - \psi_1\left(\frac{5}{6}\right) - \psi_1\left(\frac{2}{3}\right)\right]$$

$$\psi_\pi(x) = \{log[\Gamma(x)]\}^{(\pi+1)}$$

*Solution 3 by Tobi Joshua-Nigeria*

$$\int_0^1\left(\frac{x\cdot log\,x}{1-x^2+x^4}\right)dx \overset{x=x^2}{\equiv} \frac{1}{4}\int_0^1\frac{logx}{1-x+x^2}dx$$

$$= \frac{1}{4}\int_0^1\frac{(x+1)logx}{(1-x+x^2)(x+1)}dx = \frac{1}{4}\int_0^1\frac{(x+1)logx}{x^3+1}dx$$

$$= \frac{1}{4}\frac{\partial}{\partial a}\Big|_{a=0}\int_0^1\frac{x^{a+1}+x^a}{x^3+1}dx = \frac{1}{4}\frac{\partial}{\partial a}\Big|_{a=0}\int_0^1\left(x^{a+3k}+x^{a+3k+1}\right)dx\sum_{k=0}^\infty(-1)^k$$

$$= \frac{1}{4}\frac{\partial}{\partial a}\Big|_{a=0}\sum_{k=0}^\infty\left(\frac{(-1)^k}{3k+a+1} + \frac{(-1)^k}{3k+a+2}\right)$$

$$= -\frac{1}{36}\left[-\sum_{k=0}^\infty\frac{(-1)^k}{\left(k+\frac{1}{3}\right)^2} - \sum_{k=0}^\infty\frac{(-1)^k}{\left(k+\frac{2}{3}\right)^2}\right] = -\frac{1}{144}\left[-\psi'\left(\frac{1}{3}\right) + \psi'\left(\frac{2}{3}\right) - \psi'\left(\frac{1}{6}\right) + \psi'\left(\frac{5}{6}\right)\right]$$

**914. Show that:**

$$\int_0^1\left\{\frac{x^2+1}{x+2}\right\}\left\{\frac{x+\sqrt{x}}{x+1}\right\}dx = 10\sqrt{2}tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + log\left(\frac{59049}{4096}\right) - \pi - \frac{47}{6}$$

**where $\{x\}$ =part fractional of x.**

***Proposed by Mokhtar Kassani-Mostaganem-Algerie***

***Solution by Dawid Bialek-Poland***

***Using the definition of the fractional part function*** $\{x\} = x - [x]$***, we see that:***

$$\left\{\frac{x^2+1}{x+2}\right\} = \frac{x^2+1}{x+2} \text{ and } \left\{\frac{x+\sqrt{x}}{x+1}\right\} = \frac{x+\sqrt{x}}{x+1}, \forall 0 \le x \le 1;$$

***So:***

$$I = \int_0^1 \left\{\frac{x^2+1}{x+2}\right\}\left\{\frac{x+\sqrt{x}}{x+1}\right\}dx = \int_0^1 \frac{x^2+1}{x+2}\cdot\frac{x+\sqrt{x}}{x+1}dx = \int_0^1 \frac{x^3+\left(\sqrt{x}\right)^5+x+\sqrt{x}}{(x+1)(x+2)}dx$$

$$= \underbrace{\int_0^1 \frac{x^3+\left(\sqrt{x}\right)^5+x+\sqrt{x}}{(x+1)}dx}_{I_1} + \underbrace{\int_0^1 \frac{x^3+\left(\sqrt{x}\right)^5+x+\sqrt{x}}{(x+2)}dx}_{I_2} \quad (1)$$

$$I_1 = \int_0^1 \frac{x^3+\left(\sqrt{x}\right)^5+x+\sqrt{x}}{(x+1)}dx = \underbrace{\int_0^1 \frac{x^3}{x+1}dx}_{I_{11}} + \underbrace{\int_0^1 \frac{\left(\sqrt{x}\right)^5}{x+1}dx}_{I_{12}} + \underbrace{\int_0^1 \frac{x}{x+1}dx}_{I_{13}} + \underbrace{\int_0^1 \frac{\sqrt{x}}{x+1}dx}_{I_{14}}$$

$$I_{11} = \int_0^1 \frac{x^3}{x+1}dx \underbrace{=}_{t=x+1} \int_1^2 \frac{(t-1)^3}{t}dt = \int_1^2 \frac{t^3-3t^2+3t-1}{t}dt$$

$$= \left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t - log(t)\right)\Big|_1^2 = \frac{5}{6} - log2$$

$$I_{12} = \int_0^1 \frac{\left(\sqrt{x}\right)^5}{x+1}dx \underbrace{=}_{t=\left(\sqrt{x}\right)^5} \frac{2}{5}\int_0^1 \frac{\left(\sqrt[5]{t}\right)^2}{1+\left(\sqrt[5]{t}\right)^2}dt = \frac{2}{5}\int_0^1 dt + \frac{2}{5}\int_0^1 \frac{1}{1+\left(\sqrt[5]{t}\right)^2}dt$$

$$\underbrace{=}_{u=\sqrt[5]{t}} \frac{2}{5} - 2\int_0^1 \frac{u^4}{1+u^2}du = \frac{2}{5} - 2\int_0^1 \frac{(u^2-1)(u^2+1)+1}{1+u^2}du$$

$$= \frac{2}{5} - 2\int_0^1 (u^2-1)du - 2\int_0^1 \frac{1}{1+u^2}du = \frac{2}{5} - 2\left(\frac{u^3}{3} - u + tan^{-1}u\right)\Big|_0^1 = \frac{26}{15} - \frac{\pi}{2}$$

$$I_{13} = \int_0^1 \frac{x}{x+1}dx = \int_0^1 \frac{x+1-1}{x+1}dx = \left(x - log(x+1)\right)\Big|_0^1 = 1 - log2$$

$$I_{14}=\int_0^1\frac{\sqrt{x}}{x+1}dx\underbrace{=}_{t=\sqrt{x}}2\int_0^2\frac{t^2}{1+t^2}dt=2\int_0^2\frac{1+t^2-1}{1+t^2}dt=2(t-tan^{-1}t)\Big|_0^1=2-\frac{\pi}{2}$$

$$I_1=I_{11}+I_{12}+I_{13}+I_{14}=\frac{167}{30}-2log2-\pi \quad (2)$$

$$I_2=\int_0^1\frac{x^3+(\sqrt{x})^5+x+\sqrt{x}}{(x+2)}dx=\underbrace{\int_0^1\frac{x^3}{x+2}dx}_{I_{21}}+\underbrace{\int_0^1\frac{(\sqrt{x})^5}{x+2}dx}_{I_{22}}+\underbrace{\int_0^1\frac{x}{x+2}dx}_{I_{23}}+\underbrace{\int_0^1\frac{\sqrt{x}}{x+2}dx}_{I_{24}}$$

$$I_{21}=\int_0^1\frac{x^3}{x+2}dx\underbrace{=}_{t=x+2}\int_2^3\frac{t^3-6t^2+12t-8}{t}dt=\left(\frac{t^3}{3}-3t^2+12t-8log(t)\right)\Big|_2^3=\frac{10}{3}+log\left(\frac{256}{6561}\right)$$

$$I_{22}=\int_0^1\frac{(\sqrt{x})^5}{x+2}dx\underbrace{=}_{t=(\sqrt{x})^5}\frac{2}{5}\int_0^1\frac{(\sqrt[5]{t})^2}{2+(\sqrt[5]{t})^2}dt=\frac{2}{5}\int_0^1dt-\frac{4}{5}\int_0^1\frac{1}{2+(\sqrt[5]{t})^2}dt$$

$$=\frac{2}{5}\int_0^1dt-\frac{2}{5}\int_0^1\frac{1}{1+\left(\frac{\sqrt[5]{t}}{\sqrt{2}}\right)^2}dt\underbrace{=}_{u=\frac{\sqrt[5]{t}}{\sqrt{2}}}\frac{2}{5}-8\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{u^4}{1+u^2}du$$

$$=\frac{2}{5}-8\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{(u^2-1)(u^2+1)+1}{1+u^2}du$$

$$=\frac{2}{5}-8\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}(u^2-1)du-8\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{1+u^2}du=\frac{2}{5}-8\sqrt{2}\left(\frac{u^3}{3}-u-tan^{-1}u\right)\Big|_0^{\frac{1}{\sqrt{2}}}=\frac{106}{15}-8\sqrt{2}tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$I_{23}=\int_0^1\frac{x}{x+2}dx=\int_0^1\frac{x+2-2}{x+2}dx=\left(x-2log(x+2)\right)\Big|_0^1=1+log\left(\frac{4}{9}\right)$$

$$I_{24}=\int_0^1\frac{\sqrt{x}}{x+2}dx\underbrace{=}_{t=\sqrt{x}}2\int_0^1\frac{t^2}{2+t^2}dt=2\int_0^1\frac{2+t^2-2}{2+t^2}dt=2-2\int_0^1\frac{1}{1+\left(\frac{t}{\sqrt{2}}\right)^1}dt$$

$$=2-2\sqrt{2}(tan^{-1}u)\Big|_0^{\frac{1}{\sqrt{2}}}=2-2\sqrt{2}tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$I_2=I_{21}+I_{22}+I_{23}+I_{24}=\frac{67}{5}-log\left(\frac{59049}{1024}\right)-10\sqrt{2}tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad (3)$$

***Rewriting (1) with (2) and (3), we get:***

$$I = I_1 - I_2 = 10\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \log\left(\frac{59049}{1024}\right) - \frac{47}{6} - \pi$$

**915. For any $n > 0$, prove the relation:**

$$\int_0^\infty \log(1+x^2)\cot^{-1}\left(\frac{x}{n}\right)\frac{dx}{x} = -\pi Li_2(-n)$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$\Omega(n) = \int_0^\infty \log(1+x^2)\cot^{-1}\left(\frac{x}{n}\right)\frac{dx}{x}$$

$$A(n) = \int_0^\infty \frac{\log(1+x^2)}{x}\cot^{-1}(nx)dx$$

$$A'(n) = -\int_0^\infty \frac{\log(1+x^2)}{1+n^2x^2}dx$$

$$= -\frac{1}{n}\int_0^\infty \frac{\log(1+x^2)}{1+n^2x^2}dx + 2\frac{\log(n)}{n}\int_0^\infty \frac{1}{1+x^2}dx$$

$$I(n) = \int_0^\infty \frac{\log(n^2+x^2)}{1+x^2}dx$$

$$I'(n) = 2n\int_0^\infty \frac{dx}{(n^2+x^2)(1+x^2)} = \frac{2n}{1-n^2}\left(\int_0^\infty \frac{dx}{n^2+x^2} - \int_0^\infty \frac{dx}{1+x^2}\right)$$

$$= \frac{n\pi}{1-n^2}\left(\frac{1}{n}-1\right) = \frac{\pi}{n+1} \Rightarrow I(n) = \pi\log(1+n)$$

$$A'(n) = -\pi\frac{\log(1+n)}{n} + \pi\frac{\log(n)}{n} = -\pi\frac{\log\left(1+\frac{1}{n}\right)}{n} = -\pi\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\cdot\frac{1}{n^2}$$

$$A(n)=\pi\int_0^{\frac{1}{n}}\frac{\log(1+t)}{t}dt=-\pi Li_2\left(-\frac{1}{n}\right)$$

$$\Omega(n)=A\left(\frac{1}{n}\right)=-\pi Li_2(-n)$$

$$\int_0^{\infty}\log(1+x^2)\cot^{-1}\left(\frac{x}{n}\right)\frac{dx}{x}=-\pi Li_2(-n)$$

**916.** Let $S(x)=\int_0^x\frac{(x^2+y+1)^2}{(x^2-y+1)^3}dy$ then evaluate the integral in a closed-form

$$\int_{-\infty}^{+\infty}\frac{S(x)}{x}dx$$

*Proposed by Srinivasa Raghava-AIRMC-India*

*Solution by Kamel Benaicha-Algiers-Algerie*

$$S(x)=\int_0^x\frac{(x^2+y+1)^2}{(x^2-y+1)^3}dy\overset{IBP}{\cong}\frac{1}{2}\left(\left(\frac{x^2+x+1}{x^2-x+1}\right)^2-1\right)-\int_0^x\frac{(x^2+y+1)^2}{(x^2-y+1)^3}dx$$

$$\overset{IBP}{\cong}\frac{1}{2}\left(\left(\frac{x^2+x+1}{x^2-x+1}\right)^2+1\right)-\left(\frac{x^2+x+1}{x^2-x+1}\right)+\int_0^x\frac{1}{x^2-y+1}dx$$

$$\overset{IBP}{\cong}\frac{1}{2}\left(\left(\frac{x^2+x+1}{x^2-x+1}\right)^2+1\right)-\left(\frac{x^2+x+1}{x^2-x+1}\right)-\log(x^2-x+1)+\log(x^2+1)$$

$$=\frac{1}{2}\left(1+\frac{2x}{x^2-x+1}\right)^2-\left(1+\frac{2x}{x^2-x+1}\right)-\log(x^2-x+1)+\log(x^2+1)+\frac{1}{2}$$

$$=\frac{2x^2}{(x^2-x+1)^2}-\log(x^2-x+1)+\log(x^2+1)$$

*So:*

$$\int_{-\infty}^{+\infty}\frac{S(x)}{x}dx=2\int_{-\infty}^{+\infty}\frac{xdx}{(x^2-x+1)^2}+\int_{-\infty}^{+\infty}\frac{\log(x^2+1)-\log(x^2-x+1)}{x}dx$$

$$= 2\int_{-\infty}^{+\infty}\frac{xdx}{(x^2-x+1)^2}+\int_{-\infty}^{+\infty}\frac{log(x^2+1)-log(x^2-x+1)}{x}dx$$

$$+\int_{-\infty}^{+\infty}\frac{log(x^2+x+1)-log(1+x^2)}{x}dx$$

$$=\int_{-\infty}^{+\infty}\frac{(2x-1)dx}{(x^2-x+1)^2}+\int_{-\infty}^{+\infty}\frac{dx}{\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}$$

$$+2\int_{-\infty}^{+\infty}\frac{log(x^2+x-1)-log(x-x+1^2)}{x}dx$$

$$\overset{z=\frac{2x-1}{\sqrt{3}},t=\frac{1}{x}}{\cong}\frac{16}{3\sqrt{3}}\int_0^{+\infty}\frac{dt}{(t^2+1)^2}+2\int_0^1\frac{log\left(\frac{1-x^3}{1-x}\right)-log\left(\frac{x^3+1}{x+1}\right)}{x}dx$$

$$=\frac{4\pi}{3\sqrt{3}}-2\int_0^1\frac{log(1-x)-log(1+x)}{x}dx+2\int_0^1\frac{log(1-x^3)-log(1+x^3)}{x^3}\cdot x^2dx$$

$$=\frac{4\pi}{3\sqrt{3}}+2\big(Li_2(1)-Li_2(-1)\big)+\frac{2}{3}\int_0^1\frac{log(1-t)-log(1+t)}{t}dt$$

$$=\frac{4\pi}{3\sqrt{3}}+\frac{4}{3}\big(Li_2(1)-Li_2(-1)\big)$$

$$\int_{-\infty}^{+\infty}\frac{S(x)}{x}dx=\frac{4\pi}{3\sqrt{3}}+\frac{\pi^2}{3}$$

***Note:***

$$\int_0^{+\infty}\frac{dt}{1+t^2}=\frac{\pi}{2}\overset{IBP}{\cong}2\int_0^{+\infty}\frac{t^2dt}{(1+t^2)^2}\Rightarrow\int_0^{+\infty}\frac{t^2dt}{(1+t^2)^2}=\frac{\pi}{4}$$

**917. Calculate:**

$$\int_0^1 \frac{\tan^{-1}(x)}{x\sqrt{1+x^2}}dx - \frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)}{x\sqrt{1+x}}dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}\tanh^{-1}\left(x\sqrt{\frac{1-x^2}{1+x^2}}\right)dx$$

*Proposed by Cornel Ioan Vălean-Romania*

*Solution by proposer*

$$\int_0^1 \frac{\tan^{-1}(x)-\tan^{-1}(x^2)}{x\sqrt{1+x^2}}dx = \int_0^1\left(\int_x^1 \frac{1}{(1+x^2y^2)\sqrt{1+x^2}}dy\right)dx$$

$$= \int_0^1\left(\int_0^y \frac{1}{(1+x^2y^2)\sqrt{1+x^2}}dx\right)dy \overset{x\to\frac{x}{\sqrt{1-x^2}}}{\cong} \int_0^1 \frac{1}{1-y^2}\left(\int_0^{\frac{y}{\sqrt{1+y^2}}} \frac{1}{\left(\frac{1}{\sqrt{1-y^2}}\right)^2 - x^2}dx\right)dy$$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\tanh^{-1}\left(x\sqrt{\frac{1-x^2}{1+x^2}}\right)dx$. *We conclude that:*

$$\int_0^1 \frac{\tan^{-1}(x)}{x\sqrt{1+x^2}}dx - \frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)}{x\sqrt{1+x}}dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}\tanh^{-1}\left(x\sqrt{\frac{1-x^2}{1+x^2}}\right)dx = 0$$

*For a slightly different approach, we can start by differentiating*

$$f(y) = \int_0^1 \frac{1}{\sqrt{1-x^2}}\tanh^{-1}\left(y\sqrt{\frac{1-y^2}{1+y^2}}\right)dx$$

*and integrating back from* $y = 0$ *to* $y = x$.

**918. Prove that:**

$$\int_1^\infty \frac{\log(x)}{x^2}\cot^{-1}\left(x+\frac{2}{x}\right)dx$$

$$= \frac{1}{8}\left(8\tan^{-1}\left(\frac{1}{3}\right)\right) - 2\left(\log^2 2 + \log\left(\frac{5}{4}\right)\right) - Li_2\left(-\frac{1}{4}\right)$$

*Proposed by Srinivasa Ragahava-AIRMC-India*

**Solution 1 by Kamel Benaicha-Algiers-Algerie**

$$\Omega=\int_1^\infty \frac{\log(x)}{x^2}\cot^{-1}\left(x+\frac{2}{x}\right)dx \overset{t=\frac{1}{x}}{\cong} -\int_0^1 \log(t)\cot^{-1}\left(2t+\frac{1}{t}\right)dt$$

$$\overset{IBP}{\cong} \cot^{-1}(3)-\int_0^1 \frac{(t\log(t)-t)\left(2-\frac{1}{t^2}\right)}{1+\left(2t+\frac{1}{t}\right)^2}dt$$

$$=\tan^{-1}\left(\frac{1}{3}\right)-\int_0^1 \frac{2t^3\log(t)-t\log(t)-2t^3+t}{1+5t^2+4t^4}dt$$

$$\overset{t\to t^2}{\cong} \tan^{-1}\left(\frac{1}{3}\right)-\frac{1}{16}\int_0^1 \frac{(2t-1)\log(t)-(2t-1)}{(t+1)\left(t+\frac{1}{4}\right)}dt$$

$$\frac{2t-1}{(t+1)\left(t+\frac{1}{4}\right)}=\frac{a}{t+1}+\frac{b}{t+\frac{1}{4}}\ldots(E)$$

$$(E)\cdot(t+1),\ t=-1\to a=4;\ t=0\to b=-2$$

$$\Omega=\tan^{-1}\left(\frac{1}{3}\right)-\left(\frac{1}{4}\int_0^1 \frac{\log(t)}{t+1}dt-\frac{1}{8}\int_0^1 \frac{\log(t)}{t+\frac{1}{4}}dt-\frac{1}{2}\int_0^1 \frac{dt}{1+t}+\frac{1}{4}\int_0^1 \frac{dt}{t+\frac{1}{4}}\right)$$

$$=\tan^{-1}\left(\frac{1}{3}\right)+\frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\,dt+\frac{1}{2}\int_0^1 \frac{\log(t)}{4t+1}dt+\frac{1}{2}\log 2-\frac{1}{4}\left(\log\left(\frac{5}{4}\right)-\log\left(\frac{1}{4}\right)\right)$$

$$=\tan^{-1}\left(\frac{1}{3}\right)+\frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\,dt+\frac{1}{8}\int_0^4 \frac{\log\left(\frac{z}{4}\right)}{z+1}dz-\frac{1}{4}\log\left(\frac{5}{4}\right)$$

$$=\tan^{-1}\left(\frac{1}{3}\right)+\frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\,dt+\frac{1}{8}\int_0^4 \frac{\log(z)}{z+1}dz-\frac{1}{4}\log\left(\frac{5}{4}\right)-\frac{1}{4}\log(2)\log(5)$$

$$=\tan^{-1}\left(\frac{1}{3}\right)+\frac{1}{4}\int_0^1 \frac{\log(1+t)}{t}\,dt-\frac{1}{8}\int_0^4 \frac{\log(z)}{z}dz-\frac{1}{4}\log\left(\frac{5}{4}\right)$$

$$=\tan^{-1}\left(\frac{1}{3}\right)+\frac{1}{8}\int_0^1 \frac{\log(1+t)}{t}\,dt-\frac{1}{8}\int_{\frac{1}{4}}^1 \frac{\log\left(1+\frac{1}{z}\right)}{z}dz-\frac{1}{4}\log\left(\frac{5}{4}\right)$$

$$= \tan^{-1}\left(\frac{1}{3}\right) + \frac{1}{8}\int_0^1 \frac{\log(1+t)}{t}\,dt - \frac{1}{8}\int_0^1 \frac{\log(1+z)}{z}\,dz - \frac{1}{8}\int_{\frac{1}{4}}^0 \frac{\log(1+z)}{z}dz - \frac{1}{8}\int_{\frac{1}{4}}^1 \frac{\log(z)}{z}\,dz - -\frac{1}{4}\log\left(\frac{5}{4}\right) =$$

$$= \tan^{-1}\left(\frac{1}{3}\right) + \frac{1}{8}\int_0^{\frac{1}{4}} \frac{\log(1+z)}{z}\,dz - \frac{1}{16}\log^2 4 - \frac{1}{4}\log\left(\frac{5}{4}\right)$$

$$= \tan^{-1}\left(\frac{1}{3}\right) - \frac{1}{8}Li_2\left(-\frac{1}{4}\right) + \frac{1}{4}\log^2(2)$$

$$= \frac{1}{8}\left(\tan^{-1}\left(\frac{1}{3}\right) - 2\left(\log^2(2) + \log\left(\frac{5}{4}\right)\right) - Li_2\left(-\frac{1}{4}\right)\right)$$

***Solution 2 by Dawid Bialek-Poland***

$$\int_1^\infty \frac{\log(x)}{x^2}\cot^{-1}\left(x + \frac{2}{x}\right)dx \overset{\substack{\cot^{-1}(x)=\tan^{-1}\left(\frac{1}{x}\right) \\ x>0}}{\cong} \int_1^\infty \frac{\log(x)}{x^2}\tan^{-1}\left(\frac{x}{x^2+2}\right)dx$$

$$\overset{x=\frac{1}{t}}{\cong} -\int_0^1 \log(t)\tan^{-1}\left(\frac{t}{1+2t^2}\right)dt = -\int_0^1 \log(t)\tan^{-1}\left(\frac{2t-t}{1+(2t)t}\right)dt$$

$$\overset{(*)}{\cong} \int_0^1 \log(t)\tan^{-1}(t)\,dt - \int_0^1 \log(t)\tan^{-1}(2t)\,dt = I_1 - I_2 \ldots\ldots (1)$$

$$I(a) = \int_0^1 \log(t)\tan^{-1}(a\cdot t)\,dt \overset{IBP}{\cong} \left[\log(t)\cdot\left(t\cdot\tan^{-1}(a\cdot t) - \frac{\log(1+a^2\cdot t^2)}{2a}\right)\right]_0^1$$

$$-\int_0^1 \tan^{-1}(a\cdot t)\,dt + \frac{1}{2a}\int_0^1 \frac{\log(1+a^2\cdot t^2)}{t}dt =$$

$$= \left[t\cdot\tan^{-1}(a\cdot t)\cdot\frac{\log(1+a^2\cdot t^2)}{2a}\right]_0^1 + \frac{1}{2a}\int_0^1 \frac{\log(1+a^2\cdot t^2)}{t}dt$$

$$= \frac{\log(1+a^2)}{2a} - \tan^{-1}(a) + \frac{1}{2a}\int_0^1 \frac{\log(1+a^2\cdot t^2)}{t}dt$$

$$\overset{a^2\cdot t^2=-u}{\cong}\frac{\log(1+a^2)}{2a}-\tan^{-1}(a)-\frac{1}{4a}\left(\underbrace{-\int_0^{-a^2}\frac{\log(1-u)}{u}du}_{Li_2(u)}\right)$$

$$=\frac{\log(1+a^2)}{2a}-\tan^{-1}(a)-\frac{1}{4a}[Li_2(u)]_0^{-a^2}$$

$$=\frac{\log(1+a^2)}{2a}-\tan^{-1}(a)-\frac{Li_2(-a^2)}{4a}\dots.(**)$$

$$I_1=I(a=1)\overset{(**)}{\cong}\frac{\log(2)}{2}-\tan^{-1}(1)-\frac{Li_2(-1)}{4}$$

$$I_2(a=2)\overset{(**)}{\cong}\frac{\log(5)}{4}-\tan^{-1}(2)-\frac{Li_2(-4)}{8}\dots\dots(2)$$

$$\int_1^{\infty}\frac{\log(x)}{x^2}\cot^{-1}\left(x+\frac{2}{x}\right)dx=I_1-I_2$$

$$=\frac{Li_2(-4)}{8}-\frac{Li_2(-1)}{4}+\tan^{-1}(2)-\tan^{-1}(1)+\frac{\log(2)}{2}-\frac{\log(5)}{4}$$

$$\overset{(*)}{\cong}\frac{Li_2(-4)}{8}-\frac{Li_2(-1)}{4}+\tan^{-1}\left(\frac{1}{3}\right)-\frac{1}{4}\log\left(\frac{5}{4}\right)$$

$$\overset{(**)}{\cong}\frac{1}{8}\left(8\tan^{-1}\left(\frac{1}{3}\right)\right)-2\left(\log^2 2+\log\left(\frac{5}{4}\right)\right)-Li_2\left(-\frac{1}{4}\right)$$

**919. Evaluate the integral in a closed form:**

$$\int_0^{\frac{\pi}{2}}\left(\log(\tan x)+\log^2(\tan x)+\log^3(\tan x)\right)\sin^3(2x)dx$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution 1 by Tobi Josua-Nigeria***

$$I=\int_0^{\frac{\pi}{2}}\left(\log(\tan x)+\log^2(\tan x)+\log^3(\tan x)\right)\sin^3(2x)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sum_{k=1}^{3} \frac{\partial^k}{\partial s^k}\right)_{s=0} (tanx)^s sin^3(2x)dx = 8\int_0^{\frac{\pi}{2}} \left(\sum_{k=1}^{3} \frac{\partial^k}{\partial s^k}\right)_{s=0} \left(\frac{sinx}{cosx}\right)^s sin^3xcos^3xdx$$

$$= 8\int_0^{\frac{\pi}{2}} \left(\sum_{k=1}^{3} \frac{\partial^k}{\partial s^k}\right)_{s=0} sin^{3-s}xcos^{3+s}xdx = 8\left(\sum_{k=1}^{3} \frac{\partial^k}{\partial s^k}\right)_{s=0} \int_0^{\frac{\pi}{2}} sin^{3-s}xcos^{3+s}xdx$$

***Recall that:*** $\int_0^{\frac{\pi}{2}} cos^a(x)sin^a(x)dx = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{2\Gamma\left(\frac{a+b+2}{2}\right)}, a, b \geq 0$

$$I = 8\left(\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2} + \frac{\partial^3}{\partial s^3}\right)_{s=0} \frac{\Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right)}{2\Gamma\left(\frac{8}{2}\right)}$$

$$= \frac{4}{6}\left(\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2} + \frac{\partial^3}{\partial s^3}\right)_{s=0} \Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right) =$$

$$= \frac{4}{6}\left[\left(\frac{\partial}{\partial s}\right)_{s=0} \Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right) + \left(\frac{\partial^2}{\partial s^2}\right)_{s=0} \Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right) + \left(\frac{\partial^3}{\partial s^3}\right)_{s=0} \Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right)\right]$$

***All odd derivate is zero for symmetric case***

$$I = \frac{4}{6}\left[0 + \left(\frac{\partial^2}{\partial s^2}\right)_{s=0} \Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right) + 0\right]$$

$$= \frac{4}{6}\left[\frac{1}{4}\Gamma\left(\frac{s+4}{2}\right)\Gamma\left(\frac{4-s}{2}\right)\left\{\left(H_{\frac{2-s}{2}} - H_{\frac{s+2}{2}}\right)^2 + \psi'\left(\frac{4-s}{2}\right) + \psi'\left(\frac{4+s}{2}\right)\right\}\right]_{s=0}$$

$$= \frac{1}{6}[\Gamma(2)\Gamma(2)\{2\psi'(2)\}] = \left[\left\{\left(\frac{\pi^2-6}{18}\right)\right\}\right]$$

***Solution 2 by Ekpo Samuel-Nigeria***

$$sin(2x) = \frac{2tanx}{1+tan^2x}$$

***Consider:***

$$I(n) = \int_0^{\frac{\pi}{2}} \left(log^n\left(tan(x)\right)\right) sin^3(2x)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\log^n(\tan(x))\right)\left(\frac{2\tan(x)}{1+\tan^2(x)}\right)^3 dx \overset{u=\tan x}{\cong} 8\int_0^{\infty} \left(\log^n(u)\right)\left(\frac{2u}{1+u^2}\right)^3 \frac{du}{1+u^2}$$

$$= 8\int_0^{\infty} \left(\log^n(u)\right)\frac{u^3}{(1+u^2)^4}du \overset{u=x}{\cong} 8\frac{\partial^n}{\partial a^n}\int_0^{\infty} \frac{x^{3+a}}{(1+x^2)^4}dx\Big|_{a=0}$$

$$\therefore \int_0^{\infty} x^{m-1}\left(1+x^k\right)^{-n}dx = \frac{1}{k}\beta\left(\frac{m}{k}, n-\frac{m}{k}\right)$$

***Hence*** $I(n) = 8\frac{\partial^n}{\partial a^n}\left\{\frac{1}{2}\beta\left(\frac{4+a}{2}, 4-\frac{4+a}{2}\right)\right\} = 4\frac{\partial^n}{\partial a^n}\left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}$

$$I(1) = \int_0^{\frac{\pi}{2}} \left(\log(\tan(x))\right)\sin^3(2x)dx = 4\frac{\partial^n}{\partial a^n}\left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\Big|_{a=0}$$

$$= 4\frac{\partial^n}{\partial a^n}\left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\left(\psi\left(\frac{4+a}{2}\right)-\psi\left(\frac{4-a}{2}\right)\right)\Big|_{a=0}$$

$$= \left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\left(\left(\psi\left(\frac{4+a}{2}\right)-\psi\left(\frac{4-a}{2}\right)\right)^2 + \left(\psi'\left(\frac{4+a}{2}\right)+\psi'\left(\frac{4-a}{2}\right)\right)\right)\Big|_{a=0}$$

***When*** $a = 0; I(2) = \{\beta(2,2)\}\left(\left(\psi(2)-\psi(2)\right)^2 + \left(\psi'(2)+\psi'(2)\right)\right) = \frac{1}{3}\left(\frac{\pi^2}{6}-1\right)$

$$\int_0^{\frac{\pi}{2}} \left(\log^2(\tan(x))\right)\sin^3(2x)dx = \frac{1}{3}\left(\frac{\pi^2}{6}-1\right)$$

$$\int_0^{\frac{\pi}{2}} \left(\log^3(\tan(x))\right)\sin^3(2x)dx = 4\frac{\partial^3}{\partial a^3}\left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\Big|_{a=0}$$

$$= 4\frac{\partial}{\partial a}\left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\left(\left(\psi\left(\frac{4+a}{2}\right)-\psi\left(\frac{4-a}{2}\right)\right)^2 + \left(\psi'\left(\frac{4+a}{2}\right)+\psi'\left(\frac{4-a}{2}\right)\right)\right)\Big|_{a=0}$$

$$= \left\{\beta\left(\frac{4+a}{2}, \frac{4-a}{2}\right)\right\}\left(\frac{1}{2}\left(2\left(\psi\left(\frac{4+a}{2}\right)-\psi\left(\frac{4-a}{2}\right)\right)\left(\psi'\left(\frac{4+a}{2}\right)-\psi'\left(\frac{4-a}{2}\right)\right)\right)\right) +$$

$$\left(\left(\psi\left(\frac{4+a}{2}\right)-\psi\left(\frac{4-a}{2}\right)\right)^2+\left(\psi'\left(\frac{4+a}{2}\right)+\psi'\left(\frac{4-a}{2}\right)\right)\right)\frac{1}{2}\left\{\beta\left(\frac{4+a}{2},\frac{4-a}{2}\right)\right\}\left(\psi\left(\frac{4+a}{2}\right)-\right.$$

***ψ4−a2a=0***

$$a=0; I(3)=0; \int_0^{\frac{\pi}{2}}\left(\log^3(\tan(x))\right)\sin^3(2x)dx=0$$

**Hence:** $I(1)+I(2)+I(3)=\frac{1}{3}\left(\frac{\pi^2}{6}-1\right)$

$$I=\int_0^{\frac{\pi}{2}}\left(\log(\tan x)+\log^2(\tan x)+\log^3(\tan x)\right)\sin^3(2x)dx=\frac{1}{3}\left(\frac{\pi^2}{6}-1\right)$$

**Solution 3 by Kamel Benaicha-Algiers-Algerie**

$$\Omega=\int_0^{\frac{\pi}{2}}\left(\log(\tan x)+\log^2(\tan x)+\log^3(\tan x)\right)\sin^3(2x)dx$$

**Put:** $t=\tan x, \sin(2x)=2\sin x\cos x=\frac{2}{1+t^2}\left(\cos x=\frac{1}{\sqrt{1+\tan^2 x}}\right)$

$$\Omega=2^3\int_0^{+\infty}\frac{\log(t)+\log^2(t)+\log^3(t)}{(1+t^2)^4}t^3dt\overset{t=\frac{1}{z}}{\cong}2^3\int_0^{+\infty}\frac{-\log(z)+\log^2(z)-\log^3(z)}{(1+z^2)^4}z^3dz$$

$$\Omega=2^3\int_0^{+\infty}\frac{\log^2(t)}{(1+t^2)^4}t^3dt\overset{z=t^2}{\cong}\int_0^{+\infty}\frac{z\log^2(z)}{(1+z)^4}dz=\int_0^{+\infty}\frac{t\log^2 t}{(1+t)^4}dt$$

$$\overset{IBP}{\cong}\frac{1}{3}\int_0^{+\infty}\frac{2\log(t)+\log^2(t)}{(1+t)^3}dt\overset{t=\frac{1}{x}}{\cong}\frac{1}{3}\int_0^{+\infty}\frac{x\left(\log^2(x)-2\log(x)\right)}{(1+x)^3}dx$$

$$\overset{IBP}{\cong}\frac{1}{6}\int_0^{+\infty}\frac{\log^2(x)-2}{(1+x)^2}dx=\frac{1}{6}\int_0^{+\infty}\frac{\log^2(x)}{(1+x)^2}dx+\frac{1}{3}\cdot\frac{1}{1+x}\Big|_0^{+\infty}=\frac{1}{6}\int_0^{+\infty}\frac{\log^2(x)}{(1+x)^2}dx-\frac{1}{3}$$

**Or:**

$$I=\int_0^{+\infty}\frac{log^2(x)}{(1+x)^2}dx=\int_0^1\frac{log^2(x)}{(1+x)^2}dx+\int_1^{+\infty}\frac{log^2(x)}{(1+x)^2}dx$$

$$=2\int_0^1\frac{log^2(x)}{(1+x)^2}dx\overset{IBP}{=}2\left(\lim_{\varepsilon\to0_+}\left(\frac{log^2(\varepsilon)}{\varepsilon}+2\int_\varepsilon^1\frac{log(t)}{t(1+t)}dt\right)\right)$$

$$=2\left(\lim_{\varepsilon\to0_+}\left(\frac{log^2(\varepsilon)}{\varepsilon}+2\int_\varepsilon^1\frac{log(t)}{t}dt-2\int_\varepsilon^1\frac{log(t)}{1+t}dt\right)\right)$$

$$=2\left(\lim_{\varepsilon\to0_+}\left(-\frac{\varepsilon log^2\varepsilon}{1+\varepsilon}\right)-2\int_\varepsilon^1\frac{log(t)}{1+t}dt\right)=-4Li_2(-1)=\frac{\pi^2}{3}$$

$$\Omega=\frac{1}{6}I-\frac{1}{3}=\frac{\pi^2}{18}-\frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}}\left(log(tanx)+log^2(tanx)+log^3(tanx)\right)sin^3(2x)dx=\frac{\pi^2}{18}-\frac{1}{3}$$

***Solution 4 by Kamel Benaicha-Algiers-Algerie***

$$\Omega=\int_0^{\frac{\pi}{2}}\left(log(tanx)+log^2(tanx)+log^3(tanx)\right)sin^3(2x)dx$$

$$I(\alpha)=\int_0^{\frac{\pi}{2}}tan^\alpha(x)sin^3(2x)dx=2^3\int_0^{+\infty}sin^{3+\alpha}(x)cos^{3-\alpha}(x)dx$$

$$=2^3\int_0^{+\infty}sin^{2\left(2+\frac{\alpha}{2}\right)-1}(x)cos^{2\left(2-\frac{\alpha}{2}\right)-1}(x)dx=4\frac{\Gamma\left(2+\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)}{\Gamma(4)}$$

$$=\frac{2}{3}\Gamma\left(2+\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)$$

$$\Omega=\frac{dI(\alpha)}{d\alpha}+\frac{d^2I(\alpha)}{d\alpha^2}+\frac{d^3I(\alpha)}{d\alpha^3}\bigg|_{\alpha=0}=\frac{2}{3}\left(\frac{1}{4}2\Gamma^2(2)\psi'(2)\right)=\frac{1}{3}\psi'(2)$$

$$=\frac{1}{3}(\psi(1+x))'\Big|_{x=1}=\frac{\pi^2}{18}-\frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} \left(log(tanx) + log^2(tanx) + log^3(tanx)\right) sin^3(2x)dx = \frac{\pi^2}{18} - \frac{1}{3}$$

***Note:***

$$\frac{dI(\alpha)}{d\alpha} = \frac{1}{2}\left(\psi\left(2+\frac{\alpha}{2}\right) - \psi\left(2+\frac{\alpha}{2}\right)\right)\Gamma\left(2+\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)$$

$$\frac{d^2I(\alpha)}{d\alpha^2} = \Gamma\left(2+\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)\left(\left(\psi\left(2+\frac{\alpha}{2}\right) - \psi\left(2+\frac{\alpha}{2}\right)\right)^2 + \left(\psi'\left(2+\frac{\alpha}{2}\right) + \psi'\left(2+\frac{\alpha}{2}\right)\right)\right)$$

$$\frac{d^3I(\alpha)}{d\alpha^3} = \frac{1}{8}\Gamma\left(2+\frac{\alpha}{2}\right)\Gamma\left(2-\frac{\alpha}{2}\right)\cdot$$

$$\cdot\left(\left(\psi\left(2+\frac{\alpha}{2}\right) - \psi\left(2+\frac{\alpha}{2}\right)\right)^3 + 2\left(\psi'\left(2+\frac{\alpha}{2}\right) - \psi'\left(2+\frac{\alpha}{2}\right)\right)\right)\cdot\cdot\left(\psi'\left(2+\frac{\alpha}{2}\right) - \psi'\left(2+\frac{\alpha}{2}\right)\right)$$

$$+\left(\psi'\left(2+\frac{\alpha}{2}\right) - \psi'\left(2+\frac{\alpha}{2}\right)\right)\left(\psi\left(2+\frac{\alpha}{2}\right) - \psi\left(2+\frac{\alpha}{2}\right)\right) + \left(\psi''\left(2+\frac{\alpha}{2}\right) - \psi''\left(2+\frac{\alpha}{2}\right)\right)$$

$$\Omega = \left.\frac{dI(\alpha)}{d\alpha}\right|_{\alpha=0} = 0,$$

$$\left.\frac{d^2I(\alpha)}{d\alpha^2}\right|_{\alpha=0} = \frac{1}{4}(2\Gamma^2(2)\psi''(2)) = \frac{1}{2}(\zeta(2)-1)$$

$$\left.\frac{d^3I(\alpha)}{d\alpha^3}\right|_{\alpha=0} = 0$$

**920. Find:**

$$\Omega(a) = \int_0^\infty \frac{x}{(1+x^4)(1+ax)}dx, a > 0$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution by Sergio Esteban-Argentina***

$$\Omega(a) = \int_0^\infty \frac{x}{(1+x^4)(1+ax)}dx;\ \Omega = \int \frac{x}{(1+x^4)(1+ax)}dx$$

***By partial factors***

$$\Omega=\frac{1}{a^4+1}\underbrace{\int\frac{a^2x^3-ax^2+x+a^3}{x^4+1}dx}_{\Omega_1}-\frac{a^3}{a^4+1}\underbrace{\int\frac{1}{ax+1}dx}_{\Omega_2}$$

$$\Omega_1=\int\frac{a^2x^3-ax^2+x+a^3}{x^4+1}dx=\int\frac{a^2x^3-ax^2+x+a^3}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}dx=\frac{1}{2\sqrt{2}}(I_1-I_2)$$

$$I_1=\int\frac{a(a^2+\sqrt{2}a+1)x+\sqrt{2}a^3+a^2-1}{x^2+\sqrt{2}x+1}dx$$

$$I_2=\int\frac{a(a^2-\sqrt{2}a+1)x-\sqrt{2}a^3+a^2-1}{x^2-\sqrt{2}x+1}dx$$

***We calculate*** $I_1$ ***but we note that:*** $a(a^2+\sqrt{2}a+1)x+\sqrt{2}a^3+a^2-1$ ***can be written as*** $\frac{a(a^2+\sqrt{2}a+1)}{2}(2x+\sqrt{2})+\sqrt{2}a^3-\frac{a(a^2+\sqrt{2}a+1)}{\sqrt{2}}+a^2-1$

$$I_1=\frac{a(a^2+\sqrt{2}a+1)}{\sqrt{2}}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx+\left(\sqrt{2}a^3-\frac{a(a^2+\sqrt{2}a+1)}{\sqrt{2}}+a^2-1\right)\int\frac{1}{x^2+\sqrt{2}x+1}dx$$
$$=\frac{a(a^2+\sqrt{2}a+1)}{\sqrt{2}}log(x^2+\sqrt{2}x+1)+\left(\sqrt{2}a^3-\frac{a(a^2+\sqrt{2}a+1)}{\sqrt{2}}+a^2-1\right)\sqrt{2}tan^{-1}(\sqrt{2}x+1)$$

***We calculate*** $I_2$

***Analogously we can notice that*** $a(a^2-\sqrt{2}a+1)x-\sqrt{2}a^3+a^2-1$ ***can be written as***

$$\frac{a(a^2-\sqrt{2}a+1)}{2}(2x-\sqrt{2})-\sqrt{2}a^3+\frac{a(a^2-\sqrt{2}a+1)}{\sqrt{2}}+a^2-1$$

***We separate the denominator as above and we have***

$$I_2=\frac{a(a^2-\sqrt{2}a+1)}{2}log(x^2-\sqrt{2}x+1)$$
$$+\left(-\sqrt{2}a^3+\frac{a(a^2-\sqrt{2}a+1)}{\sqrt{2}}+a^2-1\right)\sqrt{2}tan^{-1}(\sqrt{2}x-1)$$

***We calculate***

$\Omega_2=\int\frac{1}{ax+1}dx$ ***by substitution*** $u=ax+1;\ \frac{du}{a}=dx$

$\Omega_2=\frac{log(ax+1)}{a}$ ***then, ordering and simplifying***

$$\Omega=\frac{1}{a^4+1}(I_1-I_2)\cdot\frac{1}{2\sqrt{2}}-\frac{a^3}{a^4+1}\cdot\frac{log(ax+1)}{a}+C$$
$$=[-4\sqrt{2}a^2(log(ax+1))+a(a^2+\sqrt{2}a+1)log(x^2+\sqrt{2}x+1)$$

$$-a(a^2-\sqrt{2}a+1)log(x^2-\sqrt{2}x+1)+2(a^3-a-\sqrt{2})tan^{-1}(\sqrt{2}x+1)$$

$$+2(a^3-a+\sqrt{2})tan^{-1}(\sqrt{2}x-1)]\cdot\frac{1}{4\sqrt{2}(a^4+1)}+C$$

$$\Omega(a)=\int_0^{\infty}\frac{x}{(ax+1)(x^4+1)}dx=\frac{-4a^2log(a)+a^3\pi\sqrt{2}-a\sqrt{2}\pi+\pi}{4(a^4+1)}$$

**921. Find without softs:**

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log\left(\frac{e^{\sqrt{x^2+1}}}{2-\sqrt{3}+tanx}\right)dx$$

***Proposed by Radu Diaconu-Romania***

***Solution by Ravi Prakash-New Delhi-India***

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log\left(\frac{e^{\sqrt{x^2+1}}}{2-\sqrt{3}+tanx}\right)dx=\Omega_1-\Omega_2$$

$$\Omega_1=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log\left(e^{\sqrt{x^2+1}}\right)dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sqrt{x^2+1}\,dx=\left[\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}log\left(x+\sqrt{x^2+1}\right)\right]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$=\frac{\pi}{8}\sqrt{\frac{\pi^2}{16}+1}+\frac{1}{2}log\left(\frac{\pi}{4}+\sqrt{\frac{\pi^2}{16}+1}\right)-\frac{\pi}{12}\sqrt{\frac{\pi^2}{36}+1}-\frac{1}{2}log\left(\frac{\pi}{6}+\sqrt{\frac{\pi^2}{36}+1}\right)$$

$$\Omega_2=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log(2-\sqrt{3}+tanx)dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log\left((2-\sqrt{3})+tan\left(\frac{\pi}{4}+\frac{\pi}{6}-x\right)\right)dx$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}log\left((2-\sqrt{3})+\frac{2+\sqrt{3}-tanx}{1+(2+\sqrt{3})tanx}\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} log\left(\frac{(2-\sqrt{3}) + tanx + 2 + \sqrt{3} - tanx}{1 + (2+\sqrt{3})tanx}\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} log\left(\frac{\frac{4}{2+\sqrt{3}}}{2-\sqrt{3} + tanx}\right)dx = \left(\frac{\pi}{4} - \frac{\pi}{6}\right)log\left(\frac{4}{2+\sqrt{3}}\right) - \Omega_2$$

$$\Rightarrow \Omega_2 = \frac{\pi}{24}log\left(\frac{4}{2+\sqrt{3}}\right).$$ *Thus*

$$\Omega = \frac{\pi}{32}\sqrt{\pi^2+16} + \frac{1}{2}log\left(\frac{\pi}{4} + \sqrt{\frac{\pi^2}{16}+1}\right) - \frac{\pi}{72}\sqrt{\pi^2+1} - \frac{1}{2}log\left(\frac{\pi}{6} + \sqrt{\frac{\pi^2}{36}+1}\right) - \frac{\pi}{24}log\left(\frac{4}{2+\sqrt{3}}\right)$$

**922. Find without softs:**

$$\Omega = \int_0^1 \frac{log^2(1+x) - logxlog(1+x) - log(1-x)log(1+x)}{1+x^2}dx$$

*Proposed by Precious Itsuokor-Nigeria*

*Solution by Zaharia Burghelea-Romania*

$$\Omega = \int_0^1 \frac{log^2(1+x) - logxlog(1+x) - log(1-x)log(1+x)}{1+x^2}dx$$

$$= -\int_0^1 \frac{log(1+x)\left(log\left(\frac{1-x}{1+x}\right) + logx\right)}{1+x^2}dx \overset{x\to\frac{1-x}{1+x}}{\cong}$$

$$-\int_0^1 \frac{log(1+x)\left(logx + log\left(\frac{1-x}{1+x}\right)\right)}{1+x^2}dx$$

$$2\Omega = -2log2\int_0^1 \frac{logx + log\left(\frac{1-x}{1+x}\right)}{1+x^2}dx \overset{x\to\frac{1-x}{1+x}}{\Longrightarrow}$$

$$\int_0^1 \frac{log\left(\frac{1-x}{1+x}\right)}{1+x^2}dx = \int_0^1 \frac{logx}{1+x^2}dx \Rightarrow$$

$$\Omega = -\frac{log2}{2}\cdot 2\int_0^1 \frac{logx}{1+x^2}dx$$

$$\int_0^1 \frac{logx}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}logx\,dx = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = -G$$

$\Rightarrow \Omega = Glog2$, ***where G is Catalan's constant.***

**923. For any positive integer $n \geq 1$, prove that:**

$$\int_0^{\infty}\frac{sinx\cdot sin^n\left(\frac{x}{n}\right)}{x^{n+1}}dx = \frac{\pi}{2n^n}$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$I_n = \int_0^{\infty}\frac{sinx\cdot sin^n\left(\frac{x}{n}\right)}{x^{n+1}}dx \overset{x=nt}{\triangleq} \frac{1}{n^n}\int_0^{+\infty}\frac{sin(nt)\cdot sin^n t}{t^{n+1}}dt$$

***Put:*** $u_n = \int_0^{+\infty}\frac{sin(nt)\cdot sin^n t}{t^{n+1}}dt \overset{IBP}{\triangleq} \int_0^{+\infty}\frac{sin((n+1)t)\cdot sin^{n-1}t}{t^n}dt$

$$u_{n-1} = \int_0^{+\infty}\frac{sin((n-1)t)\cdot sin^{n-1}t}{t^n}dt$$

$$u_n + u_{n-1} = 2\int_0^{+\infty}\frac{sin(nt)\cdot cost\cdot sin^{n-1}t}{t^n}dt \quad (1)$$

$$u_n - u_{n-1} = 2\int_0^{+\infty}\frac{cos(nt)\cdot sin^n t}{t^n}dt \quad (2)$$

$$(1)+(2) \Leftrightarrow 2u_n = 2\int_0^{+\infty}\frac{sin((n+1)t)\cdot sin^{n-1}t}{t^n}dt + 2\int_0^{+\infty}\frac{cos(nt)\cdot sin^n t}{t^n}dt$$

$$\int_0^{+\infty}\frac{cos(nt)\cdot sin^n t}{t^n}dt = u_n - u_{n-1} = 0$$

So, $u_n = u_{n-1}, \forall n \geq 2$

$$u_n = u_1 = \int_0^{+\infty} \frac{sin(3t)}{t} dt \overset{z=3t}{\cong} \int_0^{+\infty} \frac{sinz}{z} dz = \frac{\pi}{2}$$

So, $I_n = \frac{u_n}{n^n} = \frac{\pi}{2n^n}, \forall n \geq 1$

$$\int_0^{\infty} \frac{sinx \cdot sin^n\left(\frac{x}{n}\right)}{x^{n+1}} dx = \frac{\pi}{2n^n}$$

**924. Prove that:**

$$\int_0^1 \left(\frac{x}{1-x} + \frac{2\sqrt{x}}{1-x^2} + \frac{3\sqrt[3]{x}}{1-x^3}\right) log(x) dx = \frac{1}{3}(12G + 3 - 2\pi^2) - \frac{\psi^{(1)}\left(\frac{4}{9}\right)}{3}$$

$G$ – Catalan Constant

*Proposed by Srinivasa Raghava-AIRMC-India*

*Solution 1 by Kamel Benaicha-Algiers-Algerie*

$$\Omega = \int_0^1 \left(\frac{x}{1-x} + \frac{2\sqrt{x}}{1-x^2} + \frac{3\sqrt[3]{x}}{1-x^3}\right) log(x) dx$$

$$= -\int_0^1 log(x) dx + \int_0^1 \frac{log(x)}{1-x} dx + \frac{1}{2}\int_0^1 \frac{t^{-\frac{1}{4}} log(t)}{1-t} dt + \frac{1}{3}\int_0^1 \frac{t^{-\frac{5}{9}} log(t)}{1-t} dt$$

$$= -xlog(x) + x|_0^1 - Li_2(1) - \frac{1}{2} \cdot \frac{d}{ds}\psi(s+1)\Big|_{s=-\frac{1}{4}} - \frac{1}{3} \cdot \frac{d}{ds}\psi(s+1)\Big|_{s=-\frac{5}{9}} =$$

$$= 1 - \frac{\pi^2}{6} - \frac{1}{2}\psi^{(1)}\left(\frac{3}{4}\right) - \frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right)$$

$$\psi^{(1)}\left(\frac{3}{4}\right) = \sum_{n=0}^{+\infty} \frac{1}{\left(n+\frac{3}{4}\right)^2} = 16\sum_{n=0}^{+\infty} \frac{1}{(4n+3)^2} = 16\left(\sum_{n=0}^{+\infty} \frac{1}{(4n+1)^2} - C\right)$$

*We have:*

$$\psi^{(1)}\left(\frac{1}{4}\right) + \psi^{(1)}\left(\frac{3}{4}\right) = \frac{\pi^2}{sin^2\left(\frac{\pi}{4}\right)} = 2\pi^2$$

***Then:***

$$\sum_{n=0}^{+\infty}\frac{1}{(4n+1)^2}=\frac{\pi^2}{8}-\frac{1}{16}\psi^{(1)}\left(\frac{3}{4}\right)$$

$$2\psi^{(1)}\left(\frac{3}{4}\right)=2\pi^2-16C\Rightarrow\psi^{(1)}\left(\frac{3}{4}\right)=\pi^2-8C$$

$$\Omega=1-\frac{\pi^2}{6}-\frac{\pi^2}{2}+4C-\frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right)=\frac{1}{3}\left(12C-2\pi^2+3-\psi^{(1)}\left(\frac{4}{9}\right)\right)$$

***Solution 2 by Ekpo Samuel-Nigeria***

$$\Omega=\int_0^1\left(\frac{x}{1-x}+\frac{2\sqrt{x}}{1-x^2}+\frac{3\sqrt[3]{x}}{1-x^3}\right)log(x)dx$$

$$=\int_0^1\frac{xlog(x)}{1-x}dx+2\underbrace{\int_0^1\frac{\sqrt{x}log(x)}{1-x^2}dx}_{x^2\to x}+3\underbrace{\int_0^1\frac{\sqrt[3]{x}log(x)}{1-x^3}dx}_{x^3\to x}$$

$$=\int_0^1\frac{xlog(x)}{1-x}dx+\frac{1}{2}\int_0^1\frac{x^{-\frac{1}{4}}log(x)}{1-x^2}dx+\frac{1}{3}\int_0^1\frac{x^{-\frac{5}{9}}log(x)}{1-x^3}dx$$

$$=\sum_{n=0}^{\infty}\int_0^1 x^{n+1}log(x)dx+\frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 x^{n-\frac{1}{4}}log(x)dx+\frac{1}{3}\sum_{n=0}^{\infty}\int_0^1 x^{n-\frac{5}{9}}log(x)dx$$

***Using that:*** $\int_0^1 x^m log^n(x)dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$

$$=-\sum_{n=0}^{\infty}\frac{1}{(n+2)^2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+3/4)^2}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{(n+4/9)^2}$$

***Recall:*** $\psi_n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(k+z)^{n+1}}\overset{n=1}{\Longrightarrow}$

$$\psi_1(z)=\sum_{k=0}^{\infty}\frac{1}{(k+z)^2}=-\psi_1(2)-\frac{1}{2}\psi_1\left(\frac{3}{4}\right)-\frac{1}{3}\psi_1\left(\frac{4}{9}\right)$$

$$=-\left(\frac{\pi^2}{6}-1\right)-\frac{1}{2}(\pi^2-8G)-\frac{1}{3}\psi_1\left(\frac{4}{9}\right)=\frac{1}{3}(12G+3-2\pi^2)-\frac{\psi^{(1)}\left(\frac{4}{9}\right)}{3}$$

**Solution 3 by Mokhtar Khassani-Mostaganem-Algerie**

Since: $\int_0^1 \frac{x^{p-1}\log(x)}{1-x^q}dx = -\frac{1}{q^2}\psi_1\left(\frac{p}{q}\right)$, where $p,q>0$; $\psi_1$ –trigamma function we get:

$$M=\int_0^1\left(\frac{x}{1-x}+\frac{2\sqrt{x}}{1-x^2}+\frac{3\sqrt[3]{x}}{1-x^3}\right)\log(x)dx$$

$$=-\psi_1(2)-\frac{1}{2}\psi_1\left(\frac{3}{4}\right)-\frac{1}{3}\psi_1\left(\frac{4}{9}\right)=4G+1-\frac{2\pi^2}{3}-\frac{1}{3}\psi_1\left(\frac{4}{9}\right)$$

**Solution 4 by Dawid Bialek-Poland**

$$\Omega=\int_0^1\left(\frac{x}{1-x}+\frac{2\sqrt{x}}{1-x^2}+\frac{3\sqrt[3]{x}}{1-x^3}\right)\log(x)dx$$

$$=\underbrace{\int_0^1\frac{x\log(x)}{1-x}dx}_{I_1}+2\underbrace{\int_0^1\frac{\sqrt{x}\log(x)}{1-x^2}dx}_{I_2}+3\underbrace{\int_0^1\frac{\sqrt[3]{x}\log(x)}{1-x^3}dx}_{I_3}\ldots\ldots(1)$$

$$I_1=\int_0^1\frac{x\log(x)}{1-x}dx\overset{(*)}{\triangleq}\sum_{n=0}^{\infty}\int_0^1\log(x)\cdot x^{n+1}\,dx\overset{IBP}{\triangleq}-\sum_{n=0}^{\infty}\frac{1}{n+2}\int_0^1 x^{n+1}dx=-\sum_{n=0}^{\infty}\frac{1}{(n+2)^2}$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n^2}+1=1-\zeta(2)=1-\frac{\pi^2}{6}$$

$$I_2=2\int_0^1\frac{\sqrt{x}\log(x)}{1-x^2}dx\overset{(**)}{\triangleq}2\sum_{n=0}^{\infty}\int_0^1\log(x)\cdot x^{2n+\frac{1}{2}}\,dx\overset{IBP}{\triangleq}-4\sum_{n=0}^{\infty}\frac{1}{4n+3}\int_0^1 x^{2n+\frac{1}{2}}dx$$

$$=-8\sum_{n=0}^{\infty}\frac{1}{(4n+3)^2}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{4}\right)^2}\overset{def}{\triangleq}-\frac{1}{2}\psi^{(1)}\left(\frac{3}{4}\right)=-\frac{1}{2}(\pi^2-8G)=4G-\frac{\pi^2}{2}$$

$$I_3=3\int_0^1\frac{\sqrt{x}\log(x)}{1-x^2}dx\overset{(***)}{\triangleq}3\sum_{n=0}^{\infty}\int_0^1\log(x)\cdot x^{3n+\frac{1}{3}}\,dx\overset{IBP}{\triangleq}-9\sum_{n=0}^{\infty}\frac{1}{9n+4}\int_0^1 x^{3n+\frac{1}{3}}dx$$

$$=-27\sum_{n=0}^{\infty}\frac{1}{(9n+4)^2}=-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{4}{9})^2}\overset{def}{\triangleq}-\frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right)$$

***Where:*** $(*)\,\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ ; $(**)\,\frac{1}{1-x^2}=\sum_{n=0}^{\infty}x^{2n}$ ; $(***)\,\frac{1}{1-x^3}=\sum_{n=0}^{\infty}x^{3n}$ , $|x|<1$

***Note:*** $\psi^{(1)}(x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}$; $\psi^{(1)}(x)$ ***–trigamma function.***

***Rewriting (1) with*** $I_1, I_2, I_3$, ***we get:***

$$\int_0^1\left(\frac{x}{1-x}+\frac{2\sqrt{x}}{1-x^2}+\frac{3\sqrt[3]{x}}{1-x^3}\right)log(x)dx=I_1+I_2+I_3$$

$$=1-\frac{\pi^2}{6}+4G-\frac{\pi^2}{2}-\frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right)=4G+1-\frac{2\pi^2}{3}-\frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right)=\frac{1}{3}(12G+3-2\pi^2)-\frac{\psi^{(1)}\left(\frac{4}{9}\right)}{3}$$

***Solutions 5 by Dawid Bialek-Poland***

***Let's find a closed form of the following integral:***

$$I(a,b)=\int_0^1\frac{x^a}{1-x^b}log(x)dx\overset{(*)}{=}\sum_{n=0}^{\infty}log(x)\cdot\int_0^1 x^{n\cdot b+a}\,dx\overset{IBP}{=}$$

$$=-\sum_{n=0}^{\infty}\frac{1}{n\cdot b+a+1}\int_0^1 x^{n\cdot b+a}\,dx=-\sum_{n=0}^{\infty}\frac{1}{(n\cdot b+a+1)^2}=$$

$$=-\frac{1}{b^2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{a+1}{b}\right)^2}\overset{(**)}{=}-\frac{1}{b^2}\psi^{(1)}\left(\frac{a+1}{b}\right)\ldots..(1)$$

***Where:*** $(*)\ \frac{1}{1-b^2}=\sum_{n=0}^{\infty}x^{n\cdot b}$ , $|x|<1$ $\quad(**)\ \psi^{(1)}(x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}$; $\psi^{(1)}(x)$ ***–trigamma function.***

$$=\int_0^1\left(\frac{x}{1-x}+\frac{2\sqrt{x}}{1-x^2}+\frac{3\sqrt[3]{x}}{1-x^3}\right)log(x)dx$$

$$=\int_0^1\frac{xlog(x)}{1-x}dx+2\int_0^1\frac{\sqrt{x}log(x)}{1-x^2}dx+3\int_0^1\frac{\sqrt[3]{x}log(x)}{1-x^3}dx\overset{(1)}{=}$$

$$=I(1,1)+2I\left(\frac{1}{2},2\right)+3I\left(\frac{1}{3},3\right)=-\frac{1}{1^2}\psi^{(1)}\left(\frac{1+1}{1}\right)-\frac{2}{2^2}\psi^{(1)}\left(\frac{\frac{1}{2}+1}{2}\right)-\frac{3}{3^2}\psi^{(1)}\left(\frac{\frac{1}{3}+1}{3}\right)$$

$$= -\psi^{(1)}(2) - \frac{1}{2}\psi^{(1)}\left(\frac{3}{4}\right) - \frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right) \overset{\psi^{(1)}(2)=\frac{\pi^2}{6}-1}{\underset{\psi^{(1)}\left(\frac{3}{4}\right)=\pi^2-8G}{=}} -\left(\frac{\pi^2}{6}-1\right) - \frac{1}{2}(\pi^2-8G) - \frac{1}{3}\psi^{(1)}\left(\frac{4}{9}\right) =$$

$$= \frac{1}{3}(12G + 3 - 2\pi^2) - \frac{\psi^{(1)}\left(\frac{4}{9}\right)}{3}$$

**925. Find without softs:**

$$\Omega = \int_0^1 \left(\sqrt[2017]{1-x^{2019}} + x^{2020}\left(1-\sqrt{x}\right)^{2021} - \sqrt[2019]{1-x^{2017}}\right) dx$$

*Proposed by Rajeev Rastogi-India*

*Solution by Sagar Kumar-Kolkata-India*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

$$\Omega_1 = \int_0^1 (1-x^{2019})^{\frac{1}{2017}}dx \overset{x^{2019}\to x}{\cong} \frac{1}{2019}\int_0^1 x^{-\frac{2018}{2019}}(1-x)^{\frac{1}{2017}}dx$$

$$= \frac{1}{2019}B\left(\frac{1}{2019}, \frac{2018}{2019}\right) \quad \textit{(1)}$$

$$\Omega_2 = \int_0^1 (1-x^{2017})^{\frac{1}{2019}}dx \overset{x^{2017}\to x}{\cong} \frac{1}{2019}\int_0^1 x^{-\frac{2016}{2017}}(1-x)^{\frac{1}{2019}}dx$$

$$= \frac{1}{2017}B\left(\frac{1}{2017}, \frac{2020}{2019}\right) \quad \textit{(2)}$$

$$\Omega_3 = \int_0^1 x^{2020}(1-\sqrt{x})^{2021}dx \overset{\sqrt{x}\to x}{\cong} 2\int_0^1 x^{4041}(1-x)^{2021}dx = 2B(4042, 2022) \quad \textit{(3)}$$

**926.** **Find a closed form:**

$$\Omega = \int_{-\infty}^{+\infty} \left( \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{\Gamma\left(n + \frac{3}{2}\right)} \right) da$$

*Proposed by Ekpo Samuel-Nigeria*

*Solution 1 by Mokhtar Khassani-Mostaganem-Algerie*

Since: $\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx = t^s \Gamma(s) e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(s+1+n)}, \gamma\left(\frac{1}{2}, t^2\right) = \sqrt{\pi} erf(t)$

$\gamma$: Lower incomplete gammaunction and $erf$: error function. So:

$$\sum_{n=0}^{\infty} \frac{(-a^2)^n}{\Gamma\left(n + \frac{3}{2}\right)} = \frac{e^{-a^2}}{\sqrt{-a^2}\Gamma\left(\frac{1}{2}\right)} \gamma\left(\frac{1}{2}, -a^2\right) = \frac{e^{a^2} erf(ia)}{ai} = \frac{e^{-a^2} erf(ia)}{ai} \Rightarrow$$

$$M = \int_{-\infty}^{+\infty} \frac{e^{-a^2} erf(ia)}{ai} da = \int_{-\infty}^{+\infty} \frac{2}{\sqrt{\pi}} \int_0^a \frac{e^{-a^2} e^{x^2}}{a} dx da \overset{x=ay}{=} \int_{-\infty}^{+\infty} \int_0^1 e^{-(1-x^2)a^2} dx da$$

$$= \int_0^1 \int_{-\infty}^{+\infty} e^{-(1-x^2)a^2} da dx = \frac{2}{\sqrt{\pi}} \int_0^1 \frac{\sqrt{\pi}}{\sqrt{1-x^2}} dx = 2\sin^{-1} x \Big|_0^1 = \pi$$

$$M = \int_{-\infty}^{+\infty} \left( \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{\Gamma\left(n + \frac{3}{2}\right)} \right) da = \pi$$

*Solution 2 by Muhindo Vusangi Martin-Congo*

$$\Omega = \int_{-\infty}^{+\infty} \left( \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{\Gamma\left(n + \frac{3}{2}\right)} \right) da$$

$$\frac{(-1)^n a^{2n}}{\Gamma\left(n + \frac{3}{2}\right)} = \frac{(-1)^n a^{2n} \Gamma(n+1) \Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1) \Gamma\left(\frac{1}{2}\right) \Gamma\left(n + 1 + \frac{1}{2}\right)} = \frac{(-a^2)^n}{n! \sqrt{\pi}} \beta\left(n+1, \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \int_0^1 \frac{(-a^2)^n}{n!} \cdot x^n (1-x)^{-1/2} dx = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{(-a^2 x)^n}{n!} \cdot (1-x)^{-1/2} dx$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{\Gamma\left(n+\frac{3}{2}\right)}=\frac{1}{\sqrt{\pi}}\int_0^1(1-x)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-a^2x)^n}{n!}dx=\frac{1}{\sqrt{\pi}}\int_0^1(1-x)^{-\frac{1}{2}}e^{-a^2x}dx\Rightarrow$$

$$\Omega=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\int_0^1(1-x)^{-\frac{1}{2}}\cdot e^{-a^2x}dx\,da=\frac{1}{\sqrt{\pi}}\int_0^1(1-x)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}e^{-a^2x}dadx$$

$$=\frac{1}{\sqrt{\pi}}\int_0^1(1-x)^{-\frac{1}{2}}\cdot\frac{\sqrt{\pi}}{x}dx=\int_0^1 x^{-\frac{1}{2}}\cdot(1-x)^{-\frac{1}{2}}$$

$$\Rightarrow\Omega=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\frac{\left(\sqrt{\pi}\right)^2}{0!}$$

$$\int_{-\infty}^{+\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{\Gamma\left(n+\frac{3}{2}\right)}\right)da=\pi$$

## Solution 3 by Lucas Paes Barreto-Brazil

$$\Omega=\int_{-\infty}^{+\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{\Gamma\left(n+\frac{3}{2}\right)}\right)da=\int_{-\infty}^{+\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n a^{2n}}{\Gamma\left(n+\frac{1}{2}\right)}\right)\frac{da}{a^2}$$

$$=-\int_{-\infty}^{+\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^n 2^n a^{2n}}{(2n-1)!!\sqrt{\pi}}\right)\frac{da}{a^2}=-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2a^2)^{n+1}}{(2n+1)!!}\right)\frac{da}{a^2}$$

**By:** $erf(x)=\frac{e^{-x^2}}{\sqrt{\pi}}\cdot\sum_{k=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!!}$ **set** $x\to ix$

$$-erf(ix)=\frac{e^{x^2}}{\sqrt{\pi}}\cdot\sum_{k=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!!}=erfi(x)=\frac{e^{x^2}}{\sqrt{\pi}}\cdot\sum_{k=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!!}$$

***Hence***

$$\sum_{k=0}^{\infty}\frac{(-1)^{n+1}\left(2a^2\right)^{2n+1}}{(2n+1)!!}=-a\sqrt{\pi}e^{-a^2}erfi(a)$$

$$-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\sqrt{\pi}(-a)e^{-a^2}erfi(a)\frac{da}{a^2}=\int_{-\infty}^{+\infty}\frac{e^{-a^2}erfi(a)}{a}da$$

***By following de definition to*** $erfi(z)$: $erfi(z)=\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{k!(2k+1)!}$, ***we have:***

$$\int_{-\infty}^{+\infty}\frac{e^{-a^2}}{a}\cdot\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{n!\,(2n+1)}da=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{n!\,(2n+1)}\int_{-\infty}^{+\infty}a^{n-\frac{1}{2}}e^{-a}da$$

$$=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{n!\,(2n+1)}\Gamma\left(n+\frac{1}{2}\right)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\sqrt{\pi}$$

***Note that:*** $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}=sin^{-1}(1)=\frac{\pi}{2}$, ***so***

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(2n+1)}\sqrt{\pi}=2\cdot\frac{\pi}{2}=\pi$$

***Finally, we conclude that:***

$$\Omega=\int_{-\infty}^{+\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{\Gamma\left(n+\frac{3}{2}\right)}\right)da=\pi$$

**927. Evaluate:**

$$\Omega=\int_0^1\frac{\sqrt[3]{x}}{x\sqrt[3]{x}+(1-x)\sqrt[3]{1-x}}dx$$

***Proposed by Jalil Hajimir-Toronto-Canada***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$\Omega=\int_0^1\frac{\sqrt[3]{x}}{x\sqrt[3]{x}+(1-x)\sqrt[3]{1-x}}dx=\int_0^1\frac{dx}{x+(1-x)\sqrt[3]{\frac{1-x}{x}}}$$

***Put:*** $t^3=\frac{1-x}{x}\to x=\frac{1}{1+t^3};1-x=\frac{t^3}{1+t^3};dx=\frac{-3t^2dt}{1+t^3}$

$$\Omega=3\int_0^{+\infty}\frac{t^2dt}{(1+t^3)(1+t^4)}\overset{t=1/z}{=}3\int_0^{+\infty}\frac{t^2dt}{(1+t^3)(1+t^4)}$$

$$2\Omega=3\int_0^{+\infty}\frac{t^2(1+t)dt}{(1+t^3)(1+t^4)}\overset{1+t^3=(1+t)(1-t+t^2)}{=}3\int_0^{+\infty}\frac{t^2}{(1-t+t^2)(1+t^4)}dt$$

$$\Omega=\frac{3}{2}\left(-\int_0^{+\infty}\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}+\int_0^{+\infty}\frac{tdt}{1+t^4}+\int_0^{+\infty}\frac{t^2dt}{1+t^4}+\int_0^{+\infty}\frac{dt}{1+t^4}\right)$$

$$=\frac{3}{2}\left(-\frac{2}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\Big|_0^{+\infty}+\frac{1}{2}\cdot\tan^{-1}t^2\Big|_0^{+\infty}+\frac{1}{4}\int_0^{+\infty}\frac{z^{-\frac{1}{4}}+z^{-\frac{3}{4}}}{1+z}dz\right)$$

$$=\frac{3}{2}\left(-\frac{\pi}{\sqrt{3}}-\frac{3\pi}{\sqrt{3}}+\frac{\pi}{4\sin\frac{\pi}{4}}+\frac{\pi}{4\sin\frac{3\pi}{4}}\right)=\frac{3\pi}{2}\left(\frac{1+2\sqrt{2}}{4}-\frac{4\sqrt{3}}{9}\right)=\frac{\pi}{24}(18\sqrt{2}-16\sqrt{3}+9)$$

$$\int_0^1\frac{\sqrt[3]{x}}{x\sqrt[3]{x}+(1-x)\sqrt[3]{1-x}}dx=\frac{\pi}{24}(18\sqrt{2}-16\sqrt{3}+9)$$

**928. GENERALIZATION OF JALIL HAJIMIR INTEGRAL**

**Find:**

$$\Omega=\int_{\frac{1}{a}}^{a}\frac{x^{k(n+1)-1}\cdot\log x}{(1+x^{2k})^{n+1}}dx,\ a>0, n\in\mathbb{N}, k\in\mathbb{N}-\{0\}$$

***Proposed by Marin Chirciu-Romania***

***Solution 1 by Daniel Sitaru-Romania***

$$\Omega=\int_{\frac{1}{a}}^{a}\frac{x^{k(n+1)-1}\cdot\log x}{(1+x^{2k})^{n+1}}dx\overset{x=\frac{1}{y}}{=}\int_a^{\frac{1}{a}}\frac{\frac{1}{y^{k(n+1)-1}}\cdot\log\frac{1}{y}}{\left(1+\frac{1}{y^{2k}}\right)^{n+1}}\left(-\frac{1}{y^2}\right)dy=$$

$$=\int_{\frac{1}{a}}^{a}\frac{\frac{1}{y^{k(n+1)-1}}\cdot\log\frac{1}{y}}{y^2\left(1+\frac{1}{y^{2k}}\right)^{n+1}}dy=-\int_{\frac{1}{a}}^{a}\frac{\frac{1}{y^{k(n+1)-1}}\cdot\log y}{y^2\left(\frac{y^{2k}+1}{y^{2k}}\right)^{n+1}}dy=$$

$$=-\int_{\frac{1}{a}}^{a}\frac{\frac{1}{y^{k(n+1)-1}}\cdot\log y}{\frac{1}{y^{2k(n+1)-2}}\cdot(1+y^{2k})^{n+1}}dy=-\int_{\frac{1}{a}}^{a}\frac{y^{k(n+1)-1}\cdot\log y}{(1+y^{2k})^{n+1}}dy=$$

$$= -\int_{\frac{1}{a}}^{a} \frac{x^{k(n+1)-1} \cdot \log x}{(1 + x^{2k})^{n+1}} dx = -\Omega \to 2\Omega = 0 \to \Omega = 0$$

*Solution 2 by Kamel Benaicha-Algiers-Algerie*

$$\Omega = \int_{\frac{1}{a}}^{a} \frac{x^{k(n+1)-1} \cdot \log x}{(1 + x^{2k})^{n+1}} dx \overset{t=\frac{1}{x}}{=} -\int_{\frac{1}{a}}^{a} \frac{t^{2k(n+1)} \log t}{(1 + t^{2k})^{n+1} t^{k(n+1)+1}} dt$$

$$= -\int_{\frac{1}{a}}^{a} \frac{t^{2k(n+1)-k(n+1)-1} \log t}{(1 + t^{2k})^{n+1}} dt = -\int_{\frac{1}{a}}^{a} \frac{t^{k(n+1)-1} \log t}{(1 + t^{2k})^{n+1}} dt = -\Omega \Rightarrow \Omega = 0$$

**929. Find without softs:**

$$\Omega = \int_0^{2020} \frac{\sqrt[3]{x^2} + \sqrt[3]{(1010 - x)^2}}{\sqrt[3]{x^2} + 2\sqrt[3]{(1010 - x)^2} + \sqrt[3]{(2020 - x)^2}} dx$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Daniel Sitaru-Romania*

$$\Omega = \int_0^{2020} \frac{\sqrt[3]{x^2} + \sqrt[3]{(1010 - x)^2}}{\sqrt[3]{x^2} + 2\sqrt[3]{(1010 - x)^2} + \sqrt[3]{(2020 - x)^2}} dx \overset{y=2020-x}{=}$$

$$= \int_{2020}^{0} \frac{\sqrt[3]{(2020 - y)^2} + \sqrt[3]{(1010 - 2020 + y)^2}}{\sqrt[3]{(2020 - y)^2} + 2\sqrt[3]{(1010 - 2020 + y)^2} + \sqrt[3]{y^2}} (-dy) =$$

$$= \int_0^{2020} \frac{\sqrt[3]{(2020 - y)^2} + \sqrt[3]{(y - 1010)^2}}{\sqrt[3]{(2020 - y)^2} + 2\sqrt[3]{(y - 1010)^2} + \sqrt[3]{y^2}} dy =$$

$$= \int_0^{2020} \frac{\sqrt[3]{(2020 - x)^2} + \sqrt[3]{(1010 - x)^2}}{\sqrt[3]{(2020 - x)^2} + 2\sqrt[3]{(1010 - x)^2} + \sqrt[3]{x^2}} dx$$

$$2\Omega = \int_0^{2020} \frac{\sqrt[3]{x^2} + 2\sqrt[3]{(1010 - x)^2} + \sqrt[3]{(2020 - x)^2}}{\sqrt[3]{x^2} + 2\sqrt[3]{(1010 - x)^2} + \sqrt[3]{(2020 - x)^2}} dx = \int_0^{2020} dx$$

$$2\Omega = 2020 \to \Omega = 1010$$

**930. Find without softs:**

$$\Omega = \left(\int_0^1 \sqrt{-\log_3 x}\,dx\right)\left(\int_0^1 \frac{\sqrt{-\log x}}{x^3}\,dx\right)^{-1}$$

*Proposed by Abdul Mukhtar-Nigeria*

*Solution by Dawid Bialek-Poland*

$$\Omega = \left(\int_0^1 \sqrt{-\log_3 x}\,dx\right)\left(\int_0^1 \frac{\sqrt{-\log x}}{x^3}\,dx\right)^{-1} = \left(\int_0^1 \sqrt{\frac{-\log x}{\log 3}}\,dx\right)\left(\int_0^1 \frac{\sqrt{-\log x}}{x^3}\,dx\right)^{-1}$$

$$= \frac{1}{\sqrt{\log 3}}\left(\int_0^1 \sqrt{-\log x}\,dx\right)\left(\int_0^1 \frac{\sqrt{-\log x}}{x^3}\,dx\right)^{-1}$$

$$= \frac{1}{\sqrt{\log 3}}\left(\underbrace{\int_0^1 \frac{\sqrt{-\log x}}{x^0}\,dx}_{=I(0)}\right)\left(\underbrace{\int_0^1 \frac{\sqrt{-\log x}}{x^3}\,dx}_{=I(3)}\right)^{-1}$$

***After finding a closed form of the following integral:***

$$I(a) = \int_0^1 \frac{\sqrt{-\log x}}{x^a}\,dx \overset{t=-\log x}{=} \int_0^\infty t^{\frac{1}{2}} \cdot e^{-(1-a)\cdot t}\,dt \overset{def}{=} \frac{\Gamma\left(\frac{1}{2}+1\right)}{(1-a)^{\left(\frac{1}{2}+1\right)}}$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{(1-a)^{\frac{3}{2}}} \overset{\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}}{=} \frac{\sqrt{\pi}}{2\cdot\sqrt{(1-a)^3}}; \quad (*)$$

***We get:***

$$\Omega = \frac{1}{\sqrt{\log 3}} \cdot I(0) \cdot \left(I(3)\right)^{-1} \overset{(*)}{=} \frac{1}{\sqrt{\log 3}} \cdot \frac{\sqrt{\pi}}{2\cdot\sqrt{(1-0)^3}} \cdot \frac{2\cdot\sqrt{(1-3)^3}}{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{\log 3}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{2\cdot\sqrt{-8}}{\sqrt{\pi}} = \frac{2\sqrt{2}i}{\sqrt{\log 3}}$$ **,where i-imaginary unit.**

**931. Evaluate in a closed form:**

$$\int_0^1 log(x)log(1-x)Li_3(1-x)dx$$

*Proposed by Mokhtar Khassani-Mostaganem-Algerie*

*Solution by Dawid Bialek-Poland*

$$\int_0^1 log(x)log(1-x)Li_3(1-x)dx \overset{x\to 1-x}{=} \int_0^1 log(x)log(1-x)Li_3(x)dx \overset{(I)}{=}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^1 log(x)\cdot x^{k+n}dx \overset{IBP}{=} \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^3(k+n+1)}\int_0^1 x^{k+n}dx$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^3(k+n+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{\infty}\frac{n+1}{k^3(k+n+1)^2}$$

$$\overset{PFD}{=} \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left[\frac{3}{(n+1)^3}\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+n+1}\right)}_{H_{n+1}} - \frac{2}{(n+1)^2}\underbrace{\sum_{k=1}^{n}\frac{1}{k^2}}_{\zeta(2)} + \frac{1}{n+1}\underbrace{\sum_{k=1}^{\infty}\frac{1}{k^3}}_{\zeta(3)} + \frac{1}{(n+1)^2}\sum_{k=1}^{\infty}\frac{n+1}{k^3(k+n+1)^2}\right]$$

$$= \sum_{k=1}^{\infty}\frac{1}{n(n+1)}\left[\frac{3H_{n+1}}{(n+1)^3} - \frac{2\zeta(2)}{(n+1)^2} + \frac{\zeta(3)}{n+1} - \frac{1}{(n+1)^2}\left(\underbrace{\sum_{k=1}^{\infty}\frac{1}{(k+n+1)^2}}_{\psi^{(1)}(n+1)} - \frac{1}{(n+1)^2}\right)\right] =$$

$$= \underbrace{3\sum_{n=1}^{\infty}\frac{H_{n+1}}{n(n+1)^4}}_{S_1} + \underbrace{2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3}}_{S_2} + \underbrace{\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}}_{S_3} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)^5}}_{S_4} + \underbrace{\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n(n+1)^3}}_{S_5};(1)$$

*Where* $(I): log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, Li_3 = \sum_{k=1}^{\infty}\frac{x^k}{k^3}, |x| < 1$

$H_{n+1}$ *–harmonic number*

$\psi^{(1)}(n+1)$ *–trigamma function*

$$S_1 = 3\sum_{n=1}^{\infty}\frac{H_{n+1}}{n(n+1)^4} \overset{PFD}{=} -3\sum_{n=1}^{\infty}\frac{H_{n+1}}{n+1} - 3\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^2} - 3\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^3} - 3\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^4} + 3\sum_{n=1}^{\infty}\frac{H_{n+1}}{n}$$

$$= -3\sum_{n=1}^{\infty}\frac{H_n}{n} + 3 - 3\sum_{n=1}^{\infty}\frac{H_n}{n^2} + 3 - 3\sum_{n=1}^{\infty}\frac{H_n}{n^3} + 3 - 3\sum_{n=1}^{\infty}\frac{H_n}{n^4} + 3 + 3\sum_{n=1}^{\infty}\frac{H_n}{n} + 3\underbrace{\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}}_{telescoping\ series=1}$$

$$= 15 - 3\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2} + \sum_{n=1}^{\infty}\frac{H_n}{n^3} + \sum_{n=1}^{\infty}\frac{H_n}{n^4}\right) \overset{(II)}{=} 15 - 3\left(2\zeta(3) + \frac{5}{4}\zeta(4) + 3\zeta(5) - \zeta(2)\zeta(3)\right)$$

$$= 15 - 6\zeta(3) - \frac{15}{4}\zeta(4) - 9\zeta(5) + 3\zeta(2)\zeta(3)$$

***Where:*** $(II): \sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3); \sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{5}{4}\zeta(4); \sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3)$

***Note:*** $H_{n+1} = H_n + \frac{1}{n+1}$

$$S_2 = 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3} \overset{PFD}{=} -2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n+1} - 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{(n+1)^2} - 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{(n+1)^3} + 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n}$$

$$= -2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n} + 2\zeta(2) - 2\zeta(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\zeta(2)} + 2\zeta(2) - 2\zeta(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\zeta(3)} + 2\zeta(2) + 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n}$$

$$= 6\zeta(2) - 2\zeta^2(2) - 2\zeta(2)\zeta(3)$$

$$S_3 = \zeta(3)\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2} \overset{PFD}{=} -\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n+1} - \zeta(3)\sum_{n=1}^{\infty}\frac{1}{(n+1)^2} + \zeta(3)\sum_{n=1}^{\infty}\frac{1}{n}$$

$$= -\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n} + \zeta(3) - \zeta(3)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\zeta(2)} + \zeta(3) + \zeta(3)\sum_{n=1}^{\infty}\frac{1}{n}$$

$$= 2\zeta(3) - \zeta(2)\zeta(3)$$

$$S_4 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)^5} \overset{PFD}{=} -\sum_{n=1}^{\infty}\frac{1}{n+1} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^3} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^4} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^5} + \sum_{n=1}^{\infty}\frac{1}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n} + 1 - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\zeta(2)} + 1 - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\zeta(3)} + 1 - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^4}}_{\zeta(4)} + 1 - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^5}}_{\zeta(5)} + 1 + \sum_{n=1}^{\infty}\frac{1}{n}$$

$$= 5 - \zeta(2) - \zeta(3) - \zeta(4) - \zeta(5)$$

$$S_5 = \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n(n+1)^3} \overset{PFD}{=} -\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n} - \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{(n+1)^3} + \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n}$$

$$\overset{(III)}{=} \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)}{n} - \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\zeta(3)} - \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)}{n} + \zeta(2) - \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{(n+1)^3}$$

$$\overset{(IV)}{=} \zeta(2)-\zeta(3)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2}-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}$$

$$=\zeta(2)-\zeta(3)-\zeta(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\zeta(2)}+\zeta(2)-\zeta(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^3}}_{\zeta(3)}+\zeta(2)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3}$$

$$=3\zeta(2)-\zeta(3)-\zeta^2(2)-\zeta(2)\zeta(3)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3}\overset{H_n^{(2)}=H_{n+1}^{(2)}-\frac{1}{(n+1)^2}}{=}$$

$$=3\zeta(2)-\zeta(3)-\zeta^2(2)-\zeta(2)\zeta(3)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^5}$$

$$=3\zeta(2)-\zeta(3)-\zeta^2(2)-\zeta(2)\zeta(3)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}-1-\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^4}}_{\zeta(4)}+1+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}-1-\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^5}}_{\zeta(5)}+1\overset{(V)}{=}$$

$$=3\zeta(2)-\zeta(3)-\zeta^2(2)-\zeta(2)\zeta(3)+\frac{7}{4}\zeta(4)-\zeta(4)+3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5)-\zeta(5)=$$

$$=3\zeta(2)-\zeta^2(2)-\zeta(3)+\frac{3}{4}\zeta(4)-\frac{11}{2}\zeta(5)+2\zeta(2)\zeta(3)$$

***Where:***

$$(III):\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n+1}=\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)}{n}-\psi^{(1)}(1)=\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)}{n}-\zeta(2)$$

$$\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n+1)}{n}=\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)-\frac{1}{n^2}}{n}=\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)}{n}-\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$(IV):\psi^{(1)}(n+1)=\frac{d}{dn}(H_n-\gamma)=\frac{d}{dn}(H_n):=\zeta(2)-H_n^{(2)},H_n^{(2)}$ ***–second order harmonic number.***

$(V):\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\frac{7}{4}\zeta(4);\ \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=2\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5)$ ***–(ref.Borwein 1995)***

$$\int_0^1 log(x)log(1-x)Li_3(1-x)dx=S_1-S_2+S_3+S_4-S_5$$

$$=15-6\zeta(3)-\frac{15}{4}\zeta(4)-9\zeta(5)+3\zeta(2)\zeta(3)-$$

$$-\left(6\zeta(2)-2\zeta^2(2)-2\zeta(2)\zeta(3)\right)+2\zeta(3)-\zeta(2)\zeta(3)+5-\zeta(2)-\zeta(3)-\zeta(4)-\zeta(5)$$

$$-\left(3\zeta(2)-\zeta^2(2)-\zeta(3)+\frac{3}{4}\zeta(4)-\frac{11}{2}\zeta(5)+2\zeta(2)\zeta(3)\right)$$

$$=20-10\,\zeta(2)-4\zeta(3)-\frac{11}{4}\zeta(4)-\frac{9}{2}\zeta(5)+3\zeta^2(2)+2\zeta(2)\zeta(3)$$

**932. Prove that:**

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\cdot\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$$

***Proposed by Adeleke Daniel-Nigeria***

***Solution by Yen Tung Chung-Taichung-Taiwan***

$$i)\ \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\overset{x=\sqrt[4]{y}}{=}\frac{1}{4}\int_0^1\frac{\sqrt{y}}{\sqrt{1-y}}\cdot\frac{1}{\sqrt[4]{y^3}}dy=\frac{1}{4}\int_0^1 y^{\frac{3}{4}-1}(1-y)^{\frac{1}{2}-1}dy=\frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)$$

$$ii)\ \int_0^1\frac{dx}{\sqrt{1+x^4}}\overset{x=\sqrt{y}}{=}\int_0^1\frac{1}{\sqrt{1+y^2}}\cdot\frac{1}{2\sqrt{y}}dy=\frac{1}{2}\int_0^1\frac{1}{\sqrt{y}\sqrt{1+y^2}}dy$$

$$\overset{y=\tan\theta}{=}\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{\tan\theta}\sqrt{1+\tan^2\theta}}\cdot\sec^2\theta d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sec\theta}{\sqrt{\tan\theta}}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$=\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{\sin 2\theta}}d\theta\overset{\Phi=2\theta}{=}\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{1}{2\sqrt{\sin\Phi}}d\Phi=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}-1}\Phi d\Phi=\frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\cdot\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)\cdot\frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$=\frac{1}{16\sqrt{2}}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}=\frac{\pi}{4\sqrt{2}}$$

**933. Find:**

$$\Omega = \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)(2-cosx)}$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution 1 by Kamel Benaicha-Algiers-Algerie***

$f(x) = \frac{1}{2-cosx}$, $f$ *–is* $2\pi$ *periodic and even function.*

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n cos(nx)$ ; $f$ *–is continuous function.*

***Where***

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{cos(nx)}{2-cosx}dx \overset{z=e^{iz}}{=} \frac{1}{\pi}Re\left(i\oint_{|z|=1} \frac{z^n}{z^2-4z+1}dz\right)$$

$$z^2-4z+1=0 \to \Delta = 12 = \left(2\sqrt{3}\right)^2 \to z_0 = 2-\sqrt{3}, z_1 = 2+\sqrt{3};\ |z_0|<1, |z_1|>1$$

$$a_n = \frac{1}{\pi}Re\left(2i\oint_{|z|=1} \frac{\frac{z^n}{z-(2+\sqrt{3})}}{z-(2-\sqrt{3})}dz\right) = \frac{1}{\pi}Re\left(-4\pi\frac{\left(2-\sqrt{3}\right)^n}{-2\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\left(2-\sqrt{3}\right)^n$$

$$\frac{1}{2-cosx} = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sum_{n=1}^{+\infty}\left(2-\sqrt{3}\right)^n cos(nx)$$

$$\frac{1}{(1+x^2)(4+x^2)} = \frac{1}{3}\left(\frac{1}{1+x^2} - \frac{1}{4+x^2}\right)$$

$$\Omega = \frac{1}{3\sqrt{3}}\int_0^{+\infty}\left(\frac{1}{1+x^2}-\frac{1}{4+x^2}\right)dx + \frac{2}{3\sqrt{3}}\sum_{n=1}^{+\infty}\left(2-\sqrt{3}\right)^n\left(\int_0^{+\infty}\frac{cos(nx)}{1+x^2}dx + \frac{1}{2}\int_0^{+\infty}\frac{cos(2nt)}{1+t^2}dt\right)$$

$$= \frac{\pi}{12\sqrt{3}} + \frac{2}{3\sqrt{3}}\sum_{n=1}^{+\infty}\left(2-\sqrt{3}\right)^n\left(\frac{\pi}{2e^n} - \frac{\pi}{4e^{2n}}\right)$$

$$= \frac{\pi}{12\sqrt{3}} + \frac{\pi}{3\sqrt{3}}\left(\frac{2-\sqrt{3}}{e}\cdot\frac{1}{1-\frac{2-\sqrt{3}}{e}} - \frac{2-\sqrt{3}}{e^2}\cdot\frac{1}{1-\frac{2-\sqrt{3}}{e^2}}\right)$$

$$= \frac{\pi}{12\sqrt{3}} - \frac{\pi}{3\sqrt{3}}\left(\sqrt{3}-2\right)\left(\frac{1}{e+\sqrt{3}-2} - \frac{1}{2\left(e^2+\sqrt{3}-2\right)}\right)$$

$$\Omega=\int_0^\infty \frac{dx}{(1+x^2)(4+x^2)(2-\cos x)}$$

$$=\frac{\pi}{36}\left(\sqrt{3}-4\left(3-2\sqrt{3}\right)\right)\left(\frac{1}{e+\sqrt{3}-2}-\frac{1}{2\left(e^2+\sqrt{3}-2\right)}\right)$$

**Solution 2 by Mokhtar Khassani-Mostaganem-Algerie**

***From fourier series we have the trigonometric series identity for:*** $|a|<1$:

$$\frac{1-a\cos x}{1-2a\cos x}=\sum_{n=0}^{\infty}a^n\cos(nx)\Rightarrow\frac{1}{1-2a\cos x}=\frac{1}{1-a^2}\left(1+2\sum_{n=0}^{\infty}a^n\cos(nx)\right)$$

***Now let find:***

$$\Omega=\int_0^\infty\frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a}-\cos x\right)}=\frac{2a}{1-a^2}\int_0^\infty\frac{\left(1+2\sum_{n=0}^{\infty}a^n\cos(nx)\right)}{(1+x^2)(4+x^2)}dx$$

$$=\frac{2a}{1-a^2}\int_0^\infty\frac{1}{(1+x^2)(4+x^2)}dx+\frac{4a}{1-a^2}\sum_{n=1}^{\infty}a^n\int_0^\infty\frac{\cos(nx)dx}{(1+x^2)(4+x^2)}$$

$$=\frac{2a}{1-a^2}M+\frac{4a}{1-a^2}\sum_{n=1}^{\infty}a^n\varphi(n)$$

$$M=\int_0^\infty\frac{1}{(1+x^2)(4+x^2)}dx=\frac{1}{3}\int_0^\infty\left(\frac{1}{1+x^2}-\frac{1}{4+x^2}\right)dx=\frac{1}{3}\left[\tan^{-1}x-\frac{\tan^{-1}\left(\frac{x}{2}\right)}{2}\right]_0^\infty=\frac{\pi}{12}$$

$$\varphi(n)=\int_0^\infty\frac{\cos(nx)dx}{(1+x^2)(4+x^2)}$$

$$=\pi iRe\left(Res\left(\frac{e^{inz}}{(1+z^2)(4+z^2)},i\right)+Res\left(\frac{e^{inz}}{(1+z^2)(4+z^2)},2i\right)\right)$$

$$=\pi iRe\left(\frac{e^{-n}}{6i}-\frac{e^{-2n}}{12i}\right)=\pi\left(\frac{e^{-n}}{6}-\frac{e^{-2n}}{12}\right)\Rightarrow$$

$$\sum_{n=1}^{\infty}a^n\varphi(n)\overset{|a|<e}{=}\frac{\pi a}{6(e-a)}-\frac{\pi a}{12(e^2-a)}$$

***So, we get:***

$$\Omega = \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a} - cosx\right)} = \frac{2a}{1-a^2}\left(\frac{\pi}{12} + \frac{\pi a}{3(e-a)} - \frac{\pi a}{6(e^2-a)}\right)$$

***But we have:*** $2 - cosx$ ***so*** $\frac{1+a^2}{2a} = 2$ ***and*** $|a| < e \Rightarrow a = 2 - \sqrt{3}$***. Finally:***

$$\Omega = \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a} - cosx\right)}$$

$$= \frac{\pi}{3} \cdot \frac{2(2-\sqrt{3})}{4\sqrt{3}-6}\left(\frac{1}{4} + (2-\sqrt{3})(\frac{1}{e-2+\sqrt{3}} - \frac{1}{2(e^2-2+\sqrt{3})}\right)$$

**934. Evaluate in a closed form**

$$\int_0^1\int_0^1 \frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution 1 by Mokhtar Khassani-Mostaganem-Algerie***

$$\int_0^1\int_0^1 \frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy = \int_0^1\int_0^1 (\sqrt{x}+\sqrt{y})\sum_{n=0}^\infty (xy)^{n-\frac{1}{4}}dydx$$

$$= \sum_{n=0}^\infty \int_0^1\int_0^1 \left(x^{n+\frac{1}{4}}y^{n-\frac{1}{4}} + x^{n-\frac{1}{4}}y^{n+\frac{1}{4}}\right)dydx = \sum_{n=0}^\infty \int_0^1 \left(\frac{x^{n+\frac{1}{4}}}{n+\frac{3}{4}} + \frac{x^{n-\frac{1}{4}}}{n+\frac{5}{4}}\right)dx$$

$$= \sum_{n=0}^\infty \frac{2}{\left(n+\frac{3}{4}\right)\left(n+\frac{5}{4}\right)} = 2\frac{\psi\left(\frac{5}{4}\right) - \psi\left(\frac{3}{4}\right)}{\frac{5}{4} - \frac{3}{4}} = 4\left(\psi\left(1+\frac{1}{4}\right) - \psi\left(1-\frac{1}{4}\right)\right) = 4(4-\pi)$$

***Note:*** $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi cot(\pi z), \psi$ ***–digamma function.***

**Solution 2 by Ngulum George Baite-India**

$$I=\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy=\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt[4]{xy}}dxdy$$

$$=\sum_{k=0}^{\infty}\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(xy)^{\frac{1}{4}}}(xy)^k dxdy=\sum_{k=0}^{\infty}\int_0^1\int_0^1(\sqrt{x}+\sqrt{y})(xy)^{k-\frac{1}{4}}dxdy$$

$$=\sum_{k=0}^{\infty}\int_0^1\int_0^1(\sqrt{x}\cdot x^{k-\frac{1}{4}}\cdot y^{k-\frac{1}{4}}+\sqrt{y}\cdot x^{k-\frac{1}{4}}\cdot y^{k-\frac{1}{4}})dxdy$$

$$=\sum_{k=0}^{\infty}\int_0^1 x^{k-\frac{1}{4}}\left[\sqrt{x}\int_0^1 y^{k-\frac{1}{4}}dy+\int_0^1 y^{k-\frac{1}{4}+\frac{1}{2}}dy\right]dx$$

$$=\sum_{k=0}^{\infty}\int_0^1 x^{k-\frac{1}{4}}\left[\sqrt{x}\cdot\frac{y^{k+\frac{3}{4}}}{k+\frac{3}{4}}+\frac{y^{k+\frac{5}{4}}}{k+\frac{5}{4}}\right]_0^1=\sum_{k=0}^{\infty}\int_0^1 x^{k-\frac{1}{4}}\left[\frac{4\sqrt{x}}{4k+3}+\frac{4}{4k+5}\right]dx$$

$$=4\sum_{k=0}^{\infty}\left[\frac{1}{4k+3}\int_0^1 x^{k-\frac{1}{4}+\frac{1}{2}}dx+\frac{1}{4k+5}\int_0^1 x^{k-\frac{1}{4}}dx\right]$$

$$=4\sum_{k=0}^{\infty}\left[\frac{4}{(4k+3)(4k+5)}+\frac{4}{(4k+5)(4k+3)}\right]=16\sum_{k=0}^{\infty}\frac{2}{(4k+3)(4k+5)}$$

$$=16\left[\sum_{k=0}^{\infty}\frac{1}{4k+3}-\sum_{k=0}^{\infty}\frac{1}{4k+5}\right]=16\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k+1}\right]=16\left[\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{2k+1}-1\right]$$

$$=16-16\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{2k+1}$$

***But*** $\tan^{-1}x=\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{2k+1}x^{2k+1}$, ***put*** $x=1$

$$\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{2k+1}=\frac{\pi}{4}\Rightarrow I=16-16\cdot\frac{\pi}{4}=4(4-\pi)$$

$$I=\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy=4(4-\pi)$$

**Solution 3 by Kamel Benaicha-Algiers-Algerie**

$$\Omega=\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy$$

**Put:** $\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}\Rightarrow$

$$\Omega=\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\theta}}\frac{(\sqrt{\cos\theta}+\sqrt{\sin\theta})rdrd\theta}{(1-r^2\sin\theta\cos\theta)}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\sin\theta}}\frac{(\sqrt{\cos\theta}+\sqrt{\sin\theta})rdrd\theta}{(1-r^2\sin\theta\cos\theta)}$$

$$\Omega=2\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\theta}}\frac{(\sqrt{\cos\theta}+\sqrt{\sin\theta})rdrd\theta}{(1-r^2\sin\theta\cos\theta)}$$

$$=-\int_0^{\frac{\pi}{4}}\log(1-\tan\theta)\left(\frac{1}{\cos^{\frac{3}{4}}\theta\sin^{\frac{5}{4}}\theta}+\frac{1}{\cos^{\frac{5}{4}}\theta\sin^{\frac{3}{4}}\theta}\right)$$

$$\overset{t=\tan\theta}{=}4\left(-\int_0^1\frac{t^{-\frac{1}{4}}-1}{1-t}dt+\int_0^1\frac{t^{\frac{1}{4}}-1}{1-t}dt\right)=4\left(\psi\left(\frac{5}{4}\right)-\psi\left(\frac{3}{4}\right)\right)=4(4-\pi)$$

$$\int_0^1\int_0^1\frac{\sqrt{x}+\sqrt{y}}{(1-xy)\sqrt{\sqrt{xy}}}dxdy=4(4-\pi)$$

**935.** For $n\geq 0$

$$\Lambda(n)=\int_0^\infty\frac{(1+x)e^{-nx}}{\sqrt{1+\cosh(x)}}dx$$

compute the integral in closed-form:

$$\int_0^\infty\Lambda(n)e^{-n}dn$$

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution 1 by Kamel Benaicha-Algiers-Algerie**

$$\Lambda(n) = \int_0^\infty \frac{(1+x)e^{-nx}}{\sqrt{1+\cosh(x)}}dx$$

$$\Omega = \int_0^\infty \Lambda(n)e^{-n}dn = \int_0^\infty \frac{(1+x)}{\sqrt{1+\cosh(x)}}\int_0^\infty e^{-n(x+1)}\,dndx$$

$$= \frac{1}{\sqrt{2}}\int_0^\infty \frac{dx}{\cosh\left(\frac{x}{2}\right)} = \sqrt{2}\int_0^\infty \frac{e^{\frac{x}{2}}}{e^x+1}dx \overset{t=e^{\frac{x}{2}}}{=} 2\sqrt{2}\int_1^\infty \frac{dt}{1+t^2} = \frac{\pi}{\sqrt{2}}$$

**Solution 2 by Ekpo Samuel-Nigeria**

$$\Lambda(n) = \int_0^\infty \frac{(1+x)e^{-nx}}{\sqrt{1+\cosh(x)}}dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{(1+x)e^{-nx}}{\cosh\left(\frac{x}{2}\right)}dx$$

$$\int_0^\infty \Lambda(n)e^{-n}dn = \int_0^\infty \left(\frac{1}{\sqrt{2}}\int_0^\infty \frac{(1+x)e^{-nx}}{\cosh\left(\frac{x}{2}\right)}dx\right)e^{-n}dn$$

$$= \frac{1}{\sqrt{2}}\int_0^\infty\int_0^\infty \frac{(1+x)e^{-n(x+1)}}{\cosh\left(\frac{x}{2}\right)}dxdn \overset{y=x+1}{=} \frac{1}{\sqrt{2}}\int_0^\infty\int_1^\infty \frac{ye^{-ny}}{\cosh\left(\frac{y-1}{2}\right)}dydn$$

$$= \frac{1}{\sqrt{2}}\int_0^\infty e^{-ny}dn\int_1^\infty \frac{ydy}{\cosh\left(\frac{y-1}{2}\right)} = \frac{1}{\sqrt{2}}\int_1^\infty \frac{ydy}{\cosh\left(\frac{y-1}{2}\right)}\int_0^\infty e^{-ny}dn$$

$$= \frac{1}{\sqrt{2}}\int_1^\infty \frac{ydy}{\cosh\left(\frac{y-1}{2}\right)}\cdot\frac{1}{y} = \frac{1}{\sqrt{2}}\int_1^\infty \frac{dy}{\cosh\left(\frac{y-1}{2}\right)} = \frac{1}{\sqrt{2}}\int_1^\infty \operatorname{sech}\left(\frac{y-1}{2}\right)dy$$

$$\overset{x=\frac{y-1}{2}}{=} \frac{2}{\sqrt{2}}\int_0^\infty \operatorname{sech}(x)dx = \frac{2}{\sqrt{2}}\int_0^\infty \frac{\cosh(x)}{\cosh^2(x)}dx = \frac{2}{\sqrt{2}}\int_0^\infty \frac{\cosh(x)}{\sinh^2(x)+1}dx$$

$$\overset{u=\sinh^2(x)}{=} \frac{2}{\sqrt{2}}\int_0^\infty \frac{1}{u^2+1}dx = \frac{2}{\sqrt{2}}\tan^{-1}u\Big|_0^\infty = \frac{2}{\sqrt{2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$$

**936. Find without softs:**

$$\Omega = \int_{\frac{1}{2018}}^{2018} \frac{x^{2019} \log x}{(x^2+1)^{2020}} dx$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Daniel Sitaru-Romania*

$$\Omega = \int_{\frac{1}{2018}}^{2018} \frac{x^{2019} \log x}{(x^2+1)^{2020}} dx \overset{x=\frac{1}{y}}{\cong}$$

$$= \int_{2018}^{\frac{1}{2018}} \frac{\frac{1}{y^{2019}} \log \frac{1}{y}}{\left(\frac{1}{y^2}+1\right)^{2020}} \left(-\frac{1}{y^2}\right) dy = \int_{\frac{1}{2018}}^{2018} \frac{\frac{1}{y^{2019}} \log y}{\left(\frac{1}{y^2}+1\right)^{2020}} \left(-\frac{1}{y^2}\right) dy =$$

$$= -\int_{\frac{1}{2018}}^{2018} \frac{\frac{1}{y^{2019}} \log y}{\frac{(y^2+1)^{2020}}{y^{4040}}} \cdot \frac{1}{y^2} dy = -\int_{\frac{1}{2018}}^{2018} \frac{y^{2019} \log y}{(y^2+1)^{2020}} dx = -\Omega$$

$$\Omega = -\Omega \to 2\Omega = 0 \to \Omega = 0$$

**937. GENERALIZATION FOR FLOREA COSTEL PROBLEM**

**Find:**

$$\Omega(a) = \int_a^{a+1} \left( \sqrt[n]{\frac{x^{n-1}}{x+1}} + \sqrt[n]{\frac{(x+1)^{n-1}}{x}} \right) dx, a > 0, n \in \mathbb{N}, n \geq 2$$

*Proposed by Marin Chirciu-Romania*

*Solution 1 by Soumitra Mandal-Chandar Nagore-India*

$$\Omega(a) = \int_a^{a+1} \left( \sqrt[n]{\frac{x^{n-1}}{x+1}} + \sqrt[n]{\frac{(x+1)^{n-1}}{x}} \right) dx = \int_a^{a+1} \frac{2x+1}{\sqrt[n]{x^2+x}} dx$$

$$=\int_a^{a+1}\frac{d(x^2+x)}{\sqrt[n]{x^2+x}}=\left[\frac{(x^2+x)^{1-\frac{1}{n}}}{1-\frac{1}{n}}\right]_a^{a+1}=\frac{n\sqrt[n]{(a+1)^{n-1}}}{n-1}\left(\sqrt[n]{(a+2)^{n-1}}-\sqrt[n]{a^{n-1}}\right)$$

**Solution 2 by Nassim Nicholas Taleb-New York-USA**

$$f=x^{\frac{-1+n}{n}}(1+x)^{-\frac{1}{n}}+x^{-\frac{1}{n}}(1+x)^{\frac{-1+n}{n}}$$

**Let:** $m=\frac{1}{n}$ **to factorize** $\Rightarrow f=(2x+1)x^{-m}(x+1)^m$, **with antiderivative**

$$f^{(1)}=\frac{x^{1-m}(x+1)^{1-m}}{1-m}$$

$$\Omega(a)=f^{(1)}\Big|_{m=a}^{m=a+1}=\frac{a^{-m}(a+1)^{1-m}(a+2)^{-m}(a(a+2)^m-(a+2)a^m)}{m-1}$$

$$=\frac{n(a+1)^{\frac{n-1}{n}}\left((a+2)^{\frac{n-1}{n}}-a^{\frac{n-1}{n}}\right)}{n-1}$$

**938. Find:**

$$\Omega(a)=\int_0^{\infty}\left(\frac{x^2}{(1-x^2+x^4)(1+ax)}\right)dx,\ a>0$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution by Kamel Benaicha-Algiers-Algerie***

***We have:*** $\frac{1}{x^4-x^2+1}=\frac{x^2+1}{x^6+1}$

$$\Omega(a)=\int_0^{\infty}\left(\frac{x^4+x^2}{(x^6+1)(1+ax)}\right)dx$$

$$\frac{x^4+x^2}{(x^6+1)(1+ax)}=\frac{a_1x^5+a_2x^4+a_3x^3+a_2x^2+a_1x+a_0}{1+x^6}+\frac{a_7}{1+ax}\ \dots(E)$$

$$=\frac{(a_7+a_1a)x^6+(aa_2+a_1)x^5+(aa_3+a_2)x^4+(aa_4+a_3)x^3+(aa_5+a_4)x^2+(aa_6+a_5)x+a_7+a_6}{(1+x^6)(1+ax)}$$

$$(E)\cdot(1+ax),x=-\frac{1}{a}\Rightarrow a_7=\frac{a^2}{1-a^2+a^4}$$

$$a_1=-\frac{a_7}{a}=-\frac{a}{1-a^2+a^4};\ a_2=-\frac{a_1}{a}=\frac{1}{1-a^2+a^4};$$

$$a_3=\frac{1-a_2}{a}=\frac{a^3-a}{1-a^2+a^4};\ a_4=-\frac{a_3}{a}=\frac{1-a^2}{1-a^2+a^4}$$

$$a_5=\frac{1-a_4}{a}=\frac{a^3}{1-a^2+a^4}\ldots(A)$$

$$a_6=-a_7=-\frac{a^2}{1-a^2+a^4}$$

$$a_5=-aa_6=\frac{a^3}{1-a^2+a^4}\ldots(B)$$

$$\Omega(a)=\frac{1}{1-a^2+a^4}\int_0^\infty\left(\frac{-ax^5+x^4+(a^3-a)x^3+(1-a^2)x^2+a^3x-a^2}{1+x^6}+\frac{a^2}{1+ax}\right)$$

$$=\frac{1}{1-a^2+a^4}\left(a\log\left(\frac{1+ax}{\sqrt[6]{1+x^6}}\right)\Big|_0^\infty-\int_0^\infty\frac{x^4+(a^3-a)x^3+(1-a^2)x^2+a^3x-a^2}{1+x^6}dx\right)$$

***Using:*** $\int_0^\infty\frac{x^p}{1+x^q}dx\overset{t=x^q}{=}\frac{1}{q}\int_0^\infty\frac{t^{\frac{p+q}{q}-1}}{1+t}dt=\frac{\pi}{q\sin\left(\frac{p+1}{q}\cdot\pi\right)};\ (p+1<q)$

$$\Omega(a)=\frac{1}{1-a^2+a^4}\left[a\log a+\frac{\pi}{6}\left(\frac{1}{\sin\frac{5\pi}{6}}+\frac{a^3-a}{\sin\frac{4\pi}{6}}+\frac{1-a^2}{\sin\frac{3\pi}{6}}+\frac{a^3}{\sin\frac{2\pi}{6}}-\frac{a^2}{\sin\frac{\pi}{6}}\right)\right]$$

$$=\frac{1}{1-a^2+a^4}\left[a\log a+\frac{\pi}{6}\left(2+\frac{2}{\sqrt{3}}(a^3-a)+1-a^2+\frac{2}{\sqrt{3}}a^3-2a^2\right)\right]$$

$$=\frac{a\log a}{1-a^2+a^4}+\frac{\pi}{6\sqrt{3}}\cdot\frac{3\sqrt{3}-2a-3\sqrt{3}a^2+4a^3}{1-a^2+a^4}$$

$$\Omega(a)=\int_0^\infty\left(\frac{x^2}{(1-x^2+x^4)(1+ax)}\right)dx=$$

$$=\frac{a\log a}{1-a^2+a^4}+\frac{\pi}{6\sqrt{3}}\cdot\frac{3\sqrt{3}-2a-3\sqrt{3}a^2+4a^3}{1-a^2+a^4}$$

**939. Evaluate:**

$$\int_0^1 \frac{x^4 dx}{x^3 + (1-x)^3}$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution 1 by Lucas Paes Barreto-Brazil*

$$\int_0^1 \frac{x^4 dx}{x^3 + (1-x)^3} = \int_0^1 \frac{x^4}{1 - 3x + 3x^2} dx = \int_0^1 \left(\frac{x^2}{3} + \frac{3x-2}{9(3x^2 - 3x + 1)} + \frac{x}{3} + \frac{2}{9}\right) dx$$

$$= \frac{1}{9} + \frac{1}{9}\int_0^1 \frac{3x-2}{3x^2 - 3x + 1} dx + \frac{1}{6} + \frac{2}{9} = \frac{1}{2} + \frac{1}{18}\int_0^1 \frac{6x - 4 + 1 - 1}{3x^2 - 3x + 1} dx$$

$$= \frac{1}{2} + \frac{1}{18}\int_0^1 \frac{6x-3}{3x^2 - 3x + 1} dx - \frac{1}{18}\int_0^1 \frac{dx}{3\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}}$$

$$= \frac{1}{2} + \frac{1}{18}\left[\log(3x^2 - 3x + 1)\right]_0^1 = \frac{1}{9\sqrt{3}} \tan^{-1}\left[\sqrt{3}(2x-1)\right]_0^1 = \frac{1}{2} - \frac{2\pi}{27\sqrt{3}}$$

$$\int_0^1 \frac{x^4 dx}{x^3 + (1-x)^3} = \frac{1}{2} - \frac{2\pi}{27\sqrt{3}}$$

*Solution 2 by Yen Tung Chung-Taichung-Taiwan*

$$\int_0^1 \frac{x^4 dx}{x^3 + (1-x)^3} = \int_0^1 \frac{x}{1 + \left(\frac{1-x}{x}\right)^3} dx \overset{y=\frac{1-x}{x} \to x = \frac{1}{1+y}}{=} \int_\infty^0 \frac{\frac{1}{1+y}}{1+y^3} \cdot \left(-\frac{1}{(1+y)^2} dy\right)$$

$$= \int_0^\infty \frac{1}{(1+y^3)(1+y)^3} dy = \int_0^\infty \frac{1}{(1 - y + y^2)(1+y)^4} dy$$

$$= \int_0^\infty \left(\frac{\frac{1}{9}}{1+y} + \frac{\frac{2}{9}}{(1+y)^2} + \frac{\frac{1}{3}}{(1+y)^3} + \frac{\frac{1}{3}}{(1+y)^4} - \frac{\frac{1}{9}y}{1 - y + y^2}\right) dy$$

$$= \left(\frac{1}{9}\log\left|\frac{1+y}{\sqrt{y^2 - y + 1}}\right| - \frac{2}{9} \cdot \frac{1}{1+y} - \frac{1}{6} \cdot \frac{1}{(1+y)^2} - \frac{1}{9} \cdot \frac{1}{(1+y)^3} - \frac{1}{9\sqrt{3}} \tan^{-1}\left(\frac{2y-1}{\sqrt{3}}\right)\right)_0^\infty$$

$$=\frac{1}{2}-\frac{2\pi}{27\sqrt{3}}$$

**940.** **Find without softs:**

$$\Omega=\int_0^{\frac{2}{5}}\frac{\cos^2 x}{\cos^2\left(x-\frac{1}{5}\right)}dx$$

*Proposed by Radu Diaconu-Romania*

*Solution 1 by Kamel Benaicha-Algiers-Algerie*

$$\Omega=\int_0^{\frac{2}{5}}\frac{\cos^2 x}{\cos^2\left(x-\frac{1}{5}\right)}dx \overset{t=\frac{1}{5}-x}{=} \int_{-\frac{1}{5}}^{\frac{1}{5}}\frac{\cos^2\left(\frac{1}{5}-t\right)}{\cos^2 t}dt=(*)$$

$$\cos^2\left(\frac{1}{5}-t\right)=\frac{1+\cos\left(\frac{2}{5}-2t\right)}{2}=\frac{1+\cos\left(\frac{2}{5}\right)\cos(2t)+\sin\left(\frac{2}{5}\right)\sin(2t)}{2}$$

$$(*)=\frac{1}{2}\left(\int_{-\frac{1}{5}}^{\frac{1}{5}}\frac{dt}{\cos^2 t}+\cos\left(\frac{2}{5}\right)\int_{-\frac{1}{5}}^{\frac{1}{5}}\frac{2\cos^2 t-1}{\cos^2 t}dt+2\sin\left(\frac{2}{5}\right)\int_{-\frac{1}{5}}^{\frac{1}{5}}\tan t dt\right)$$

$$=\tan\left(\frac{1}{5}\right)+\frac{2}{5}\cos\left(\frac{2}{5}\right)-\cos\left(\frac{2}{5}\right)\tan\left(\frac{1}{5}\right)=2\sin^2\left(\frac{1}{5}\right)\tan\left(\frac{1}{5}\right)+\frac{2}{5}\cos\left(\frac{2}{5}\right)$$

$$=\frac{2}{5}\cos\left(\frac{2}{5}\right)+2\sin\left(\frac{1}{5}\right)\cos\left(\frac{1}{5}\right)\frac{\sin\left(\frac{1}{5}\right)}{\cos\left(\frac{1}{5}\right)}\tan\left(\frac{1}{5}\right)$$

$$=\frac{2}{5}\cos\left(\frac{2}{5}\right)+\sin\left(\frac{2}{5}\right)\tan^2\left(\frac{1}{5}\right)$$

*Solution 2 by Rovsen Pirguliev-Sumgait-Azerbaijan*

***Denote:*** $x-\frac{1}{5}=t\Rightarrow x=t+\frac{1}{5},dx=dt$

$$\Omega=\int_0^{\frac{2}{5}}\frac{\cos^2 x}{\cos^2\left(x-\frac{1}{5}\right)}dx=\int_{-\frac{1}{5}}^{\frac{1}{5}}\frac{\cos^2\left(t+\frac{1}{5}\right)}{\cos^2 t}dt$$

$$=\int_{-\frac{1}{5}}^{\frac{1}{5}}\left(\frac{cost cos\frac{1}{5}-sint sin\frac{1}{5}}{cost}\right)^2 dt=\int_{-\frac{1}{5}}^{\frac{1}{5}}\left(cos\frac{1}{5}-sin\frac{1}{5}tan\,t\right)^2 dt$$

$$=\int_{-\frac{1}{5}}^{\frac{1}{5}}\left(cos^2\frac{1}{5}-sin\frac{2}{5}tan\,t+sin^2\frac{1}{5}tan^2t\right)dt$$

$$=\int_{-\frac{1}{5}}^{\frac{1}{5}}cos^2\frac{1}{5}dt+sin\frac{2}{5}\cdot\int_{-\frac{1}{5}}^{\frac{1}{5}}tan\,t\,dt+sin^2\frac{1}{5}\cdot\int_{-\frac{1}{5}}^{\frac{1}{5}}tan^2t dt$$

$$\overset{\int_{-\frac{1}{5}}^{\frac{1}{5}}tan\,t\,dt=0}{=}cos^2\frac{1}{5}\cdot t\Big|_{-\frac{1}{5}}^{\frac{1}{5}}+sin^2\frac{1}{5}(tan\,t-t)\Big|_{-\frac{1}{5}}^{\frac{1}{5}}$$

$$=\frac{2}{5}\cdot cos^2\frac{1}{5}+2\cdot sin^2\frac{1}{5}\cdot\left(tan\frac{1}{5}-\frac{1}{5}\right)$$

$$=\frac{2}{5}\left(cos^2\frac{1}{5}-sin^2\frac{1}{5}\right)+2sin^2\frac{1}{5}\cdot tan\frac{1}{5}=\frac{2}{5}\cdot cos\frac{2}{5}+2sin^2\frac{1}{5}\cdot tan\frac{1}{5}$$

**941. Find without softs:**

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{sin2x}dx$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Abner Chinga Bazo-Lima-Peru***

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{sin2x}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{\frac{\pi}{2}-x}{sin2\left(\frac{\pi}{2}-x\right)}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{\frac{\pi}{2}-x}{sin2x}dx$$

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{\sin 2x}dx=\frac{\pi}{2}\left(\frac{1}{2}\log(\tan x)\right)\Big|_{\frac{\pi}{5}}^{\frac{3\pi}{10}}$$

$$=\frac{\pi}{8}\left(\log\left(\tan\left(\frac{3\pi}{10}\right)\right)-\log\left(\tan\left(\frac{\pi}{5}\right)\right)\right)=\frac{\pi}{8}\log\left(\frac{\log\left(\tan\left(\frac{3\pi}{10}\right)\right)}{\log\left(\tan\left(\frac{\pi}{5}\right)\right)}\right)$$

***Solution 2 by Avishek Mitra-West Bengal-India***

$$I=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{\sin 2x}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}x\csc 2x\,dx$$

$$=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\left(\frac{3\pi}{10}+\frac{\pi}{5}-x\right)\csc 2\left(\frac{3\pi}{10}+\frac{\pi}{5}-2x\right)dx$$

$$=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\left(\frac{\pi}{2}-x\right)\csc(\pi-2x)\,dx=\frac{\pi}{2}\cdot\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\csc 2x\,dx-I$$

$$2I=\frac{\pi}{2}\cdot\frac{1}{2}\log\left(\tan\frac{2x}{2}\right)\Big|_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\Rightarrow I=\frac{\pi}{8}\left(\log\left(\tan\frac{3\pi}{10}\right)-\log\left(\tan\frac{\pi}{5}\right)\right)$$

$$=\frac{\pi}{8}\left(\log(\tan 54^\circ)-\log(\tan 18^\circ)\right)$$

$$\theta=18^\circ\Rightarrow 5\theta=90^\circ\Rightarrow 2\theta+3\theta=90^\circ\Rightarrow 2\theta=90^\circ-3\theta$$

$$\Rightarrow \sin 2\theta=\sin(90^\circ-3\theta)=\cos 3\theta\Rightarrow 2\sin\theta\cos\theta=\cos 3\theta=4\cos^3\theta-3\cos\theta$$

$$\Rightarrow 2\sin\theta\cos\theta-4\cos^3\theta+3\cos\theta=0\Rightarrow \cos\theta(2\sin\theta-4\cos^2\theta+3)=0;\ \ (\cos\theta\neq 0)$$

$$\Rightarrow 4\sin^2\theta+2\sin\theta+1=0\Rightarrow \sin\theta=\frac{-1\pm\sqrt{5}}{4}$$

$$\Rightarrow \sin 18^\circ=\frac{\sqrt{5}-1}{4}\Rightarrow \cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}\Rightarrow \tan 18^\circ=\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$$

Now, $sin54^\circ = cos36^\circ = 1 - 2sin^2 18^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$

$$cos54^\circ = sin36^\circ = \sqrt{1 - cos^2 36^\circ} = \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$\Rightarrow tan54^\circ = \frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$$

$$I = \frac{\pi}{8}\left(log\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}} - log\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right)$$

**Solution 3 by Kamel Benaicha-Algiers-Algerie**

$$\Omega = \int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{x}{sin2x}dx = \int_{\frac{\pi}{5}}^{\frac{\pi}{2}-\frac{\pi}{5}} \frac{x}{sin2x}dx \overset{x=\frac{\pi}{2}-t}{=} \int_{\frac{\pi}{5}}^{\frac{\pi}{2}-\frac{\pi}{5}} \frac{\frac{\pi}{2}-t}{sin2t}dt$$

$$\Omega = \frac{\pi}{4}\int_{\frac{\pi}{5}}^{\frac{\pi}{2}-\frac{\pi}{5}} \frac{dt}{sin2t} = \frac{\pi}{8}\int_{\frac{\pi}{5}}^{\frac{\pi}{2}-\frac{\pi}{5}} \frac{d(tanx)}{tanx} = \frac{\pi}{8}log(tanx)|_{\frac{\pi}{5}}^{\frac{\pi}{2}-\frac{\pi}{5}} = -\frac{\pi}{4}log\left(tan\frac{\pi}{5}\right)$$

$$\frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

$$sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = sin\frac{3\pi}{10} = sin\left(\frac{\pi}{10} + \frac{\pi}{5}\right) \dots (E)$$

$$X = cos\frac{\pi}{5}$$

$$\therefore (E) \Rightarrow X = \sqrt{1-X^2}cos\frac{\pi}{10} + Xsin\frac{\pi}{10}$$

$$\Rightarrow X^2 = (1-X^2)cos^2\frac{\pi}{10} + 2X\sqrt{1-X^2}sin\frac{\pi}{10}cos\frac{\pi}{10} + X^2sin^2\frac{\pi}{10}$$

$$X^2 = -X^2\left(cos^2\frac{\pi}{10} - sin^2\frac{\pi}{10}\right) + \frac{1+cos\frac{\pi}{5}}{2} + X\sqrt{1-X^2}sin\frac{\pi}{10}$$

$$\therefore (E) \Rightarrow X^2 = -X^3 + \frac{1+X}{2} + X - X^3$$

$$\Rightarrow 4X^3 + 2X^2 - 3X - 1 = 0 \overset{X=\frac{x}{2}}{\Longrightarrow} x^3 + x^2 - 3x - 2 = 0$$

$$\Leftrightarrow (x+2)(x^2-x-1)=0 \Rightarrow x_1=-2; x_2=\frac{1-\sqrt{5}}{2}; x_3=\frac{1+\sqrt{5}}{2}$$

$$\Rightarrow X_1=-1; X_2=\frac{1-\sqrt{5}}{4}; X_3=\frac{1+\sqrt{5}}{4}$$

$$0<\frac{\pi}{5}<\frac{\pi}{2}\Rightarrow 1>\cos\frac{\pi}{5}>0\Rightarrow X=\frac{1+\sqrt{5}}{4}$$

$$\tan\frac{\pi}{5}=\frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}}=\frac{\sqrt{1-X^2}}{X}=\frac{\sqrt{10-2\sqrt{5}}}{1+\sqrt{5}}=\sqrt{\frac{10-2\sqrt{5}}{6+2\sqrt{5}}}=\sqrt{5-2\sqrt{5}}$$

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{5}}\frac{xdx}{\sin 2x}=-\frac{\pi}{4}\log\left(\tan\frac{\pi}{5}\right)=-\frac{\pi}{8}\log(5-2\sqrt{5})=\frac{\pi}{8}\log\left(1+\frac{2}{\sqrt{5}}\right)$$

**Solution 4 by Sagar Kumar-Kolkata-India**

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{\sin 2x}dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}x\csc 2x\,dx=\left[\frac{x}{2}\log(\tan x)\right]_{\frac{\pi}{5}}^{\frac{3\pi}{10}}-\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\log(\tan x)\,dx$$

$$=-\frac{\pi}{10}\log\left(\tan\frac{\pi}{5}\right)+\frac{3\pi}{20}\log\left(\tan\frac{3\pi}{10}\right)-\Omega_1$$

$$\Omega_1=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\log(\tan x)\,dx=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\log(\cot x)\,dx\Rightarrow 2\Omega_1=0\Rightarrow\Omega_1=0$$

$$\Omega=\frac{\pi}{20}\log\left(\frac{\left(\tan\frac{3\pi}{10}\right)^3}{\left(\tan\frac{\pi}{5}\right)^2}\right)=\frac{\pi}{20}\log\left(\frac{(\tan 18^\circ)^3}{(\cot 18^\circ)^2}\right)=\frac{\pi}{20}\log\left(1+\frac{2}{\sqrt{5}}\right)^{\frac{5}{2}}=\frac{\pi}{8}\log\left(1+\frac{2}{\sqrt{5}}\right)$$

**942. If $0<a\le b$ then:**

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz\le\frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

**Proposed by Daniel Sitaru-Romania**

**Solution 1 by Cao Mai Tanh Tam-Vietnam**

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz=\int_a^b\int_a^b\int_a^b\left(\frac{x^2y+x^2z+xy^2+3xyz+xz^2+y^2z+yz^2}{xyz}\right)dxdydz$$

$$=\int_a^b\int_a^b\int_a^b\left[\frac{x}{z}+\frac{x}{y}+\frac{y}{z}+3+\frac{z}{y}+\frac{y}{x}+\frac{z}{x}\right]dxdydz$$

$$=\int_a^b\int_a^b\left[\frac{x^2}{2z}+\frac{x^2}{2y}+\frac{xy}{z}+3x+\frac{xz}{y}+(y+z)\log|x|\right]\Big|_a^b dydz$$

$$=\int_a^b\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{1}{y}+\frac{1}{z}\right)+(b-a)\left(\frac{y}{z}+3+\frac{y}{z}\right)+(y+z)\log\left(\frac{b}{a}\right)\right]dydz$$

$$=\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{y}{z}+\log|y|\right)+(b-a)\left(\frac{y^2}{2z}+3y+\frac{y^2}{2z}\right)+\left(zy+\frac{y^2}{2}\right)\log\left(\frac{b}{a}\right)\right]\Big|_a^b$$

$$=\int_a^b\left[\frac{b^2-a^2}{2}\left(\frac{b-a}{z}+\log\left(\frac{b}{a}\right)\right)+(b-a)\left(\frac{b^2-a^2}{2z}+3(b-a)+\frac{b^2-a^2}{2z}\right)+\left((b-a)z+\frac{b^2-a^2}{2}\right)\log\left(\frac{b}{a}\right)\right]dz$$

$$=\left[\frac{b^2-a^2}{2}\left((b-a)\log|z|+z\log\left(\frac{b}{a}\right)\right)+(b-a)\left(\frac{b^2-a^2}{2}\log|z|+3z(b-a)+\frac{b^2-a^2}{2}\log|z|\right)+\left((b-a)\frac{z^2}{2}+\frac{b^2-a^2}{2}z\right)\log\left(\frac{b}{a}\right)\right]\Big|_a^b$$

$$=3(b-a)^2(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]$$

***Let prove:***

$$3(b-a)^2(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]\le\frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

$$\Leftrightarrow 3ab(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]\le(2a^2+5ab+2b^2)(b-a)^3$$

$$\Leftrightarrow [2(a+b)^2+ab](b-a)-3ab(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]\ge 0$$

$$\Leftrightarrow 2(a+b)^2(b-a)+ab(b-a)-2ab(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]-ab(b+a)\left[\log\left(\frac{b}{a}\right)+1\right]\ge 0$$

$$\Leftrightarrow 2(a+b)\left[b^2-a^2-ab\log\left(\frac{b}{a}\right)-1\right]+ab\left(b-a-b\log\left(\frac{b}{a}\right)-a\log\left(\frac{b}{a}\right)-b-a\right)\ge 0$$

***Solution 2 by George Florin Serban***

$$\frac{(x+y+z)(xy+yz+zx)}{xyz}=\frac{x^2y+x^2z+xy^2+3xyz+xz^2+y^2z+yz^2}{xyz}$$

$$=\frac{\sum(x^2y+xy^2)+3xyz}{xyz}=\sum\left(\frac{x}{z}+\frac{z}{x}\right)+3=f(x,y,z)$$

$$I=\int_a^b f(x,y,z)\,dz=\int_a^b\frac{x}{z}dz+\int_a^b\frac{z}{x}dz+\left(\frac{x}{y}+\frac{y}{x}\right)(b-a)+\int_a^b\frac{y}{z}dz+\int_a^b\frac{z}{y}dz+3(b-a)$$

$$=x(\log b-\log a)+\frac{b^2-a^2}{2x}+\left(\frac{x}{y}+\frac{y}{x}\right)(b-a)+y(\log b-\log a)+\frac{b^2-a^2}{2y}+3(b-a)$$

$$\int_a^b\left(\int_a^b f(x,y,z)dz\right)dy=x(b-a)\log\left(\frac{b}{a}\right)+\frac{(b^2-a^2)(b-a)}{2x}+x(b-a)(\log b-\log a)+$$

$$+\frac{(b-a)(b^2-a^2)}{2x}+\frac{b^2-a^2}{2}\log\left(\frac{b}{a}\right)+\frac{b^2-a^2}{2}(\log b-\log a)+3(b-a)^2$$

$$\int_a^b\left(\int_a^b\left(\int_a^b f(x,y,z)dz\right)dy\right)dx=6\cdot\frac{(b^2-a^2)(b-a)}{2}\log\left(\frac{b}{a}\right)+3(b-a)^3$$

$$I=3(b-a)^2(b+a)\log\left(\frac{b}{a}\right)+3(b-a)^3$$

$$I\overset{?}{\leq}\frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

$$\Leftrightarrow 3(b+a)\log\left(\frac{b}{a}\right)\leq\frac{2(b-a)(a^2+ab+b^2)}{ab}\Leftrightarrow\log\left(\frac{b}{a}\right)\leq\frac{2(b-a)(a^2+ab+b^2)}{3ab(a+b)}$$

$$\Leftrightarrow\log\left(\frac{b}{a}\right)\leq\frac{2(b^3-a^3)}{3ab(a+b)}=\frac{2\left[\left(\frac{b}{a}\right)^3-1\right]}{3\frac{b}{a}\left(\frac{b}{a}+1\right)}\ \overset{a\leq b}{\underset{t=\frac{b}{a}\geq 1}{\Rightarrow}}$$

$$\log(t)\leq\frac{2(t^3-1)}{3t(t+1)},t\geq 1$$

$$\textit{Let } f:[1,\infty)\to\mathbb{R}, f(t)=\frac{2(t^3-1)}{3t(t+1)}-\log(t)$$

$$f'(t)=\frac{(t-1)^2(2t^2+5t+2)}{3t^2(t+1)^2}\geq 0,\forall t\geq 1$$

$$f'(t)\geq 0,\forall t\geq 1\Rightarrow f\nearrow \text{on}[1,\infty)\Rightarrow f(t)\geq f(1),\forall t\geq 1\Rightarrow f(t)\geq 0,\forall t\geq 1$$

$$\Rightarrow\log(t)\leq\frac{2(t^3-1)}{3t(t+1)},t\geq 1$$

**Solution 3 by Soumitra Mandal-Chandar Nagore-India**

$x \to \frac{1}{x}$ ***is convex function, hence Hermite Hadamard Inequality***

$$\frac{2}{a+b} \le \frac{\log b - \log a}{b-a} \le \frac{a+b}{2ab}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz = \int_a^b\int_a^b\int_a^b\sum_{cyc}\left(1+\frac{y}{x}+\frac{z}{x}\right)dxdydz$$

$$= 3(b-a)^3 + \int_a^b\int_a^b\int_a^b\sum_{cyc}\frac{y}{x}dxdydz + \int_a^b\int_a^b\int_a^b\sum_{cyc}\frac{z}{x}dxdydz$$

$$= 3(b-a)^3 + 3(b-a)(b^2-a^2)\log\left(\frac{b}{a}\right)$$

$$= 3(b-a)^3 + 3(b-a)^3(a+b)\cdot\frac{\log b - \log a}{b-a}$$

$$\le 3(b-a)^3\left(1+(a+b)\cdot\frac{a+b}{2ab}\right) = 3(b-a)^3\cdot\frac{a^2+4ab+b^2}{2ab}$$

***We need to prove:***

$$3(b-a)^3\cdot\frac{a^2+4ab+b^2}{2ab} \le \frac{(2a^2+5ab+2b^2)(b-a)^3}{ab} \Leftrightarrow$$

$$(b-a)^3\left[\frac{(2a^2+5ab+2b^2)}{ab} - \frac{3(a^2+4ab+b^2)}{2ab}\right] \ge 0 \Leftrightarrow$$

$$(b-a)^3\frac{(b-a)^2}{2ab} \ge 0, \textit{ which is true.}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right)dxdydz \le \frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

**943. If $1 < a \leq b$ then:**

$$log\left(\frac{\sqrt{b}\cdot\Gamma(b)}{\sqrt{a}\cdot\Gamma(a)}\right) \leq \int_a^b log(x)dx \leq log\left(\frac{b\cdot\Gamma(b)}{a\cdot\Gamma(a)}\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution by Soumitra Mandal-Chandar Nagore-India***

*Let:* $f(x) = \frac{e^{x-1}\Gamma(x)}{x^{x-\frac{1}{2}}}$ *for all* $x \geq 1$

$$logf(x) = x - 1 + log\Gamma(x) - \left(x - \frac{1}{2}\right)log(x) \Rightarrow$$

$$\frac{f'(x)}{f(x)} = 1 + \frac{\Gamma'(x)}{\Gamma(x)} - \frac{x}{x - \frac{1}{2}} - \frac{x - \frac{1}{2}}{x}$$

$\frac{f'(x)}{f(x)} = \frac{\Gamma'(x)}{\Gamma(x)} - \frac{1}{2\left(x-\frac{1}{2}\right)} + \frac{1}{2x} - 1 = \int_0^\infty \left(\frac{1}{2} - \delta(y)\right)e^{-xy}dy$ *where* $\delta(y) = \frac{1}{1-e^{-y}} - \frac{1}{y}$

*and we have* $\int_0^\infty e^{-xy}dy = \frac{1}{x}$. *So* $\frac{f'(x)}{f(x)} < 0 \Rightarrow f'(x) < 0$. *Hence $f$ is decreasing function*

*so for* $b \geq a$ *we have* $f(a) \geq f(b) \Rightarrow \frac{e^{a-1}\Gamma(a)}{a^{a-\frac{1}{2}}} \geq \frac{e^{b-1}\Gamma(b)}{b^{b-\frac{1}{2}}} \Rightarrow$

$$\frac{e^{-b}b^b}{e^{-a}a^a} \geq \sqrt{\frac{b}{a}}\cdot\frac{\Gamma(b)}{\Gamma(a)} \Rightarrow b(log(b) - 1) - a(log(a) - 1) \geq log\left(\sqrt{\frac{b}{a}}\cdot\frac{\Gamma(b)}{\Gamma(a)}\right)$$

$$\int_a^b log(x)dx \geq log\left(\sqrt{\frac{b}{a}}\cdot\frac{\Gamma(b)}{\Gamma(a)}\right)$$

*Let:* $g(x) = \frac{e^x\Gamma(x)}{x^{x-1}}$ *for all* $x \geq 1$ *now*

$$log(g(x)) = x + log(\Gamma(x)) - (x-1)log(x) = x + log\Gamma(x) - \left(x - \frac{1}{2}\right)log(x) + \frac{log(x)}{2}$$

$$= log\sqrt{2\pi} + o(x^{-1}) + \frac{log(x)}{2}$$

*Since* $log\Gamma(x) = \left(x - \frac{1}{2}\right)log(x) - x + log\sqrt{2\pi} + o(x^{-1})$

$\frac{g'(x)}{g(x)} = \frac{1}{2x} + o\left(\frac{1}{x^2}\right) > 0$ *for all* $x \geq 1$ *so* $g'(x) > 0$ *hence* $g(x)$ *is increasing*

For all $b \geq a$ we have $g(b) \geq g(a) \Rightarrow \frac{e^{b-1}\Gamma(b)}{b^{b-1}} \geq \frac{e^{a-1}\Gamma(a)}{a^{a-1}} \Rightarrow$

$$log\left(\frac{a \cdot \Gamma(b)}{b \cdot \Gamma(a)}\right) \geq \int_a^b log(x)dx$$

$$log\left(\frac{\sqrt{b} \cdot \Gamma(b)}{\sqrt{a} \cdot \Gamma(a)}\right) \leq \int_a^b log(x)dx \leq log\left(\frac{b \cdot \Gamma(b)}{a \cdot \Gamma(a)}\right)$$

**944. If $f: [a, b] \to \left(0, \frac{\pi}{2}\right)$, $a \leq b$, $f$ –continuous, then:**

$$\frac{2\sqrt{2}}{5}\int_a^b sinf(x)dx + \frac{1}{10}\int_a^b tanf(x)dx + \frac{2\sqrt{2}}{5}\int_a^b cosf(x)dx + \frac{1}{10}\int_a^b cotf(x)dx \geq b - a$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Khaled Abd Imouti-Damascus-Syria*

$$l = \frac{2\sqrt{2}}{5}\int_a^b sinf(x)dx + \frac{1}{10}\int_a^b tanf(x)dx + \frac{2\sqrt{2}}{5}\int_a^b cosf(x)dx + \frac{1}{10}\int_a^b cotf(x)dx \overset{?}{\gtrsim} b - a$$

$$\int_a^b \left[\frac{2\sqrt{2}}{5}\left(sinf(x) + cosf(x)\right) + \frac{1}{10}\left(tanf(x) + cotf(x)\right)\right]dx \overset{?}{\gtrsim} b - a$$

$$g(\theta) = \frac{2\sqrt{2}}{5}(sin\theta + cos\theta) + \frac{1}{10}(tan\theta + cot\theta)$$

$$\lim_{\theta \to 0} g(\theta) = 0; \lim_{\theta \to \infty} g(\theta) = +\infty$$

$$g'(\theta) = \frac{2\sqrt{2}}{5}(cos\theta - sin\theta) + \frac{1}{10} \cdot \frac{cos^2\theta - sin^2\theta}{sin\theta \cdot cos\theta}$$

$$g'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}$$

$$g\left(\frac{\pi}{4}\right) = 1 \Rightarrow g(\theta) \geq 1, \forall \theta \in \left(0, \frac{\pi}{2}\right)$$

So: $l \geq \int_a^b 1dx = b - a$

| $\theta$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ |
|---|---|---|---|
| $g'(\theta)$ | $-\ -\ -\ -\ -$ | $0$ $+\ +\ +\ +\ +$ | |
| $g(\theta)$ | $+\infty$ ↘↘↘↘ | $1$ ↗↗↗↗ | $+\infty$ |

**945. Prove that:**

$$\int_1^2 x^x dx + \int_4^5 x^x dx > \int_2^4 x^x dx$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Daniel Sitaru-Romania*

$$g:[1,\infty)\to\mathbb{R},\ g(t)=t^t,\ g'(t)=(1+\log t)t^t>0,$$

$$g''(t)=t^{t-1}+(1+\log t)^2 t^t>0,\ g'-increasing$$

$$g:[x,x+1]\to\mathbb{R},\ g(x+1)-g(x)\overset{MVT}{\cong} g'^{(c_1)}(x+1-x),$$

$$c_1\in(x,x+1)$$

$$g:[x+2,x+3]\to\mathbb{R},\ g(x+3)-g(x+2)\overset{MVT}{\cong} g'(c_2)(x+3-x-2),$$

$$c_2\in(x+2,x+3)$$

$$x<c_1<x+1<x+2<c_2<x+3,\ g-increasing$$

$$g'(x)<g'(c_1)<g'(x+1)<g'(x+2)<g'(c_2)<g'(x+3)$$

$$g(x+1)-g(x)<g(x+3)-g(x+2)\ (1)$$

$$F:[1,\infty)\to\mathbb{R},\ F(x)=\int_x^{x+1} t^t\,dt-\int_{x+2}^{x+3} t^t dt$$

$$F'(x)=g(x+1)-g(x)-g(x+3)+g(x+2)\overset{(1)}{<}0,\ F-decreasing$$

$$F(1)>F(2)\to\int_1^2 t^t\,dt-\int_3^4 t^t dt>\int_2^3 t^t\,dt-\int_4^5 t^t dt$$

$$\int_1^2 t^t\,dt+\int_4^5 t^t dt>\int_2^3 t^t\,dt+\int_3^4 t^t dt=\int_2^4 t^t dt$$

**946. If $x, y, z, a > 0$ then:**

$$\int_0^a\int_0^a\int_0^a \frac{(x+y+z)^3}{(x+y)^2}dxdydz \geq \frac{27a^3}{8}$$

***Proposed by Daniel Sitaru-Romania***

***Solution by Rahim Shahbazov-Baku-Azerbaijan***

$$x+y=k \Rightarrow (x+y+z)^3 = (k+z)^3 = \left(\frac{k}{2}+\frac{k}{2}+2\right)^3 \geq 27\cdot\frac{k^2}{4}\cdot z \Rightarrow$$

$$(x+y+z)^3 \geq \frac{27}{4}\cdot(x+y)^2\cdot z \Rightarrow \frac{(x+y+z)^3}{(x+y)^2} \geq \frac{27}{4}\cdot z \text{ then}$$

$$\int_0^a\int_0^a\int_0^a \frac{(x+y+z)^3}{(x+y)^2}dxdydz \geq \frac{27}{4}\int_0^a dx\int_0^a dy\int_0^a zdz = \frac{27a^3}{8}$$

**947.**

$$\Omega = \int_0^1 erf^3(x)dx + 36\int_0^1 erf(x)dx - 12\left(\int_0^1 erf(x)dx\right)^2$$

$A.\ \Omega < 0$ $\qquad B.\ \Omega = 0$ $\qquad C.\ \Omega > 0$

***Proposed by Daniel Sitaru-Romania***

***Solution by Avishek Mitra-West Bengal-India***

$$erf^3(x) + 36erf(x) \overset{AM-GM}{\geq} 2\sqrt{36erf^4(x)} = 12erf^2(x)$$

$$\Rightarrow \int_0^1 erf^3(x)dx + 36\int_0^1 erf(x)dx \geq 12\int_0^1 erf^2(x)dx$$

$$\Rightarrow \int_0^1 erf^3(x)dx + 36\int_0^1 erf(x)dx - 12\left(\int_0^1 erf(x)dx\right)^2 \geq 12\left[\int_0^1 erf^2(x)dx - \left(\int_0^1 erf(x)dx\right)^2\right] \geq 0$$

$$\therefore \left[\int_0^1 1\cdot erf(x)dx \overset{HOLDER}{\leq} \sqrt{\int_0^1 1^2 dx\cdot\int_0^1 erf^2(x)dx}\right.$$

$$\left.\Rightarrow \int_0^1 erf^2(x)dx \geq \left(\int_0^1 erf(x)dx\right)^2\right]$$

**948. If $x_i, y_i > 0, i = \overline{1,n}, n \in \mathbb{N}-\{0\}, 0 < a \leq b$ then:**

$$\underbrace{\int_a^b\int_a^b\dots\int_a^b}_{for\ "2n"times}\dots\prod_{i=1}^{n}\frac{dx_i dy_i}{x_i+y_i} \leq \left(\frac{b-a}{2}\log\frac{b}{a}\right)^n$$

***Proposed by Daniel Sitaru-Romania***

***Solution by Soumitra Mandal-Chandahar Nagore-India***

$$\underbrace{\int_a^b\int_a^b\dots\int_a^b}_{for\ "2n"times}\dots\prod_{i=1}^{n}\frac{dx_i dy_i}{x_i+y_i} \overset{Am-Gm}{\leq} \frac{1}{2^n}\left(\prod_{i=1}^{n}\int_a^b\frac{dx_i}{\sqrt{x_i}}\right)\left(\prod_{i=1}^{n}\int_a^b\frac{dy_i}{\sqrt{y_i}}\right)$$

$$= \frac{1}{2^n}\left(\prod_{i=1}^{n}\frac{\sqrt{b}-\sqrt{a}}{1-\frac{1}{2}}\right)\left(\prod_{i=1}^{n}\frac{\sqrt{b}-\sqrt{a}}{1-\frac{1}{2}}\right) = 2^n\left(\sqrt{b}-\sqrt{a}\right)^{2n}$$

***We need to prove:***

$$\left(\frac{b-a}{2}\log\frac{b}{a}\right)^n \geq 2^n\left(\sqrt{b}-\sqrt{a}\right)^{2n} \Leftrightarrow \frac{b-a}{2}\log\frac{b}{a} \geq 2\left(\sqrt{b}-\sqrt{a}\right)^2$$

$$\Leftrightarrow \log\frac{b}{a} \geq \frac{4\left(\sqrt{b}-\sqrt{a}\right)^2}{b-a} = \frac{4\left(\sqrt{b}-\sqrt{a}\right)}{\sqrt{b}+\sqrt{a}}$$

***Let:*** $f(x) = \log x - \frac{4(\sqrt{x}-1)}{\sqrt{x}+1}$ ***for all*** $x \geq 1$

$$f'(x) = \frac{1}{x} - \frac{1}{x(\sqrt{x}+1)^2} > 0\ for\ all\ x \geq 1.$$

***Hence f is an increasing function*** $\therefore f(x) \geq f(1) = 0$ ***then:***

$\log x - \frac{4(\sqrt{x}-1)}{\sqrt{x}+1}$ ***for all*** $x \geq 1$

*Let:* $x = \frac{b}{a}$ *hence* $\log\frac{b}{a} \geq \frac{4(\sqrt{b}-\sqrt{a})}{\sqrt{b}+\sqrt{a}}$ *is proved. So:*

$$\underbrace{\int_a^b\int_a^b\cdots\int_a^b\cdots}_{for\ "2n"times}\prod_{i=1}^{n}\frac{dx_idy_i}{x_i+y_i} \leq \left(\frac{b-a}{2}\log\frac{b}{a}\right)^n$$

**949. If $0 < a \leq b$ then:**

$$\int_a^b\int_a^b\frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b\frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\left(\frac{b-a}{b+a}\right)^2$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Adrian Popa-Romania***

$$\frac{1^3}{(3x+2y)^2}+\frac{1^3}{(2x+3y)^2} \overset{Radon}{\geq} \frac{(1+1)^3}{(2x+3y+3x+2y)^2} = \frac{8}{25(x+y)^2}$$

$$\int_a^b\int_a^b\frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b\frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

$$\exists c \in (c_1,c_2), a < c_1, c_2 < b: \int_a^b\int_a^b f(x,y)dxdy = f(c_1,c_2)\cdot(b-a)^2$$

$$\Rightarrow \frac{8}{25}\int_a^b\int_a^b\frac{dxdy}{(x+y)^2} = \frac{8}{25}\cdot\frac{(b-a)^2}{(b+a)^2}$$

***The domain is a square with sides:*** $c_1 = c_2 = \frac{a+b}{2}$

***Solution 2 by Soumitra Mandal-Chandar Nagore-India***

$$\frac{1^3}{(3x+2y)^2}+\frac{1^3}{(2x+3y)^2} \overset{Radon}{\geq} \frac{(1+1)^3}{(2x+3y+3x+2y)^2} = \frac{8}{25(x+y)^2}$$

$$\int_a^b\int_a^b\frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b\frac{dxdy}{(2x+3y)^2} \geq \frac{8}{25}\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

$$= \frac{8}{25}\int_a^b\left(-\frac{1}{x+y}\right)\Big|_a^b dy = \frac{8}{25}\log\frac{\left(1+\frac{b}{a}\right)^2}{4\frac{a}{b}}$$

*We need to prove:* $\frac{8}{25}log\frac{\left(1+\frac{b}{a}\right)^2}{4\frac{a}{b}}\ge\frac{8}{25}\left(\frac{b-a}{b+a}\right)^2=\frac{8}{25}\left(\frac{\frac{b}{a}-1}{\frac{b}{a}+1}\right)^2$

$$\Rightarrow log\frac{(1+x)^2}{4x}\ge\left(\frac{x-1}{x+1}\right)^2 \quad (1)$$

*Let* $f(x)=2log(1+x)-log(4x)-\left(\frac{x-1}{x+1}\right)^2; x\ge 1$

$$f'(x)=\frac{(x-1)^2(x+2)}{x(x+1)^3}\ge 0, x\ge 0$$

*Hence f is increasing fuction* $f(x)\ge f(1)=0\Rightarrow \quad (1)$

$$\int_a^b\int_a^b\frac{dxdy}{(3x+2y)^2}+\int_a^b\int_a^b\frac{dxdy}{(2x+3y)^2}\ge\frac{8}{25}\left(\frac{b-a}{b+a}\right)^2$$

**950. In $\Delta ABC$ the following relationship holds:**

$$\frac{1}{a}\int_{h_a}^{m_a}\frac{cosx}{x}dx+\frac{1}{b}\int_{h_b}^{m_b}\frac{cosx}{x}dx+\frac{1}{c}\int_{h_c}^{m_c}\frac{cosx}{x}dx\le\frac{3R-6r}{S}$$

*Proposed by Mokhtar Khassani-Mostaganem-Algerie*

*Solution by Daniel Sitaru-Romania*

$$\frac{1}{a}\int_{h_a}^{m_a}\frac{cosx}{x}dx+\frac{1}{b}\int_{h_b}^{m_b}\frac{cosx}{x}dx+\frac{1}{c}\int_{h_c}^{m_c}\frac{cosx}{x}dx=$$

$$=\frac{1}{a}\cdot\frac{cos\alpha}{\alpha}\cdot(m_a-h_a)+\frac{1}{b}\cdot\frac{cos\beta}{\beta}\cdot(m_b-h_b)+\frac{1}{c}\cdot\frac{cos\gamma}{\gamma}\cdot(m_c-h_c)\le$$

$$\alpha\in(h_a,m_a),\beta\in(h_b,m_b),\gamma\in(h_c,m_c)$$

$$\le\frac{1}{a}\cdot\frac{1}{\alpha}\cdot(m_a-h_a)+\frac{1}{b}\cdot\frac{1}{\beta}\cdot(m_b-h_b)+\frac{1}{c}\cdot\frac{1}{\gamma}\cdot(m_c-h_c)\le$$

$$\le\frac{1}{a}\cdot\frac{1}{h_a}\cdot(m_a-h_a)+\frac{1}{b}\cdot\frac{1}{h_b}\cdot(m_b-h_b)+\frac{1}{c}\cdot\frac{1}{h_c}\cdot(m_c-h_c)=$$

$$=\frac{1}{2S}(m_a-h_a)+\frac{1}{2S}(m_b-h_b)+\frac{1}{2S}(m_c-h_c)\overset{ROTARU\ INEQ\ 1991}{\le}$$

$$\leq \frac{1}{2S}2(R-2r)) + \frac{1}{2S}2(R-2r) + \frac{1}{2S}2(R-2r) = \frac{3R-6r}{S}$$

*Equality holds for $a = b = c$.*

**951. If $0 < a \leq b$ then:**

$$\int_a^b \int_a^b \frac{dxdy}{(x+y)^4} \leq \frac{(b-a)^2(a^2+ab+b^2)}{48a^3b^3}$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Rahim Shahbazov-Baku-Azerbaijan*

$$(x+y)^4 \geq 16x^2y^2$$

$$\int_a^b \int_a^b \frac{dxdy}{(x+y)^4} \leq \int_a^b \int_a^b \frac{dxdy}{16x^2y^2} = \frac{1}{16}\int_a^b \frac{dx}{x^2} \cdot \int_a^b \frac{dy}{y^2} = \frac{1}{16} \cdot \frac{(b-a)^2}{a^2b^2}$$

$$\frac{1}{16} \cdot \frac{(b-a)^2}{a^2b^2} \leq \frac{(b-a)^2(a^2+ab+b^2)}{48a^3b^3} \Leftrightarrow$$

$$a^2+ab+b^2 \geq 3ab \Leftrightarrow (a-b)^2 \geq 0$$

**952. Let $f: [1, 13] \to \mathbb{R}$ be a convexe function.Prove:**

$$\int_1^3 f(x)dx + \int_{11}^{13} f(x)dx \geq \int_5^9 f(x)dx$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Marian Dincă-Romania*

*By Hermite-Hadamard inequality:*

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\left(\frac{f(a)+f(b)}{2}\right);\ f\text{ −convex.}$$

$$\int_1^3 f(x)dx \geq (3-1)f\left(\frac{3+1}{2}\right) = 2f(2)$$

$$\int_{11}^{13} f(x)dx \geq (13-11)f\left(\frac{13+11}{2}\right) = 2f(12)$$

$$\int_{1}^{3} f(x)dx + \int_{11}^{13} f(x)dx \geq 2f(2) + 2f(12)$$

***because:***

$$\int_{5}^{9} f(x)dx \leq (9-5)\left(\frac{f(9)+f(5)}{2}\right) = 2\big(f(9)+f(5)\big)$$

***and*** $2f(2) + 2f(12) \geq 2\big(f(9)+f(5)\big)$ ***Hardy-Litlewood-Polya inequality.***

**953.**

$$\Omega(m) = \int_{0}^{\pi/2} \sqrt[m]{tanx}dx, m \in \mathbb{N}, m \geq 2$$

**If** $m, n, p \in \mathbb{N}, m, n, p \geq 2$ **then:**

$$\Omega(m)\cdot\Omega(n)\cdot\Omega(p) \geq \left(\frac{3\pi}{2\left(cos\frac{\pi}{2m} + cos\frac{\pi}{2n} + cos\frac{\pi}{2p}\right)}\right)^3$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Rahim Shahbazov-Baku-Azerbaijan***

$$\Omega(m) = \int_0^{\frac{\pi}{2}} (sin^2x)^{\frac{1}{2m}}\cdot(cos^2x)^{-\frac{1}{2m}}dx \overset{sin^2x=t}{\cong} \frac{1}{2}\int_0^1 t^{\frac{1}{2}-\frac{1}{2m}-1}\cdot(1-t)^{\frac{1}{2}+\frac{1}{2m}-1}dt$$

$$= \frac{1}{2}B\left(\frac{1}{2}-\frac{1}{2m}, \frac{1}{2}+\frac{1}{2m}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}+\frac{1}{2m}, \frac{1}{2}-\frac{1}{2m}\right)$$

$$\Gamma(x)\Gamma(1-x) = \frac{1}{2}\cdot\frac{\pi}{cos\pi x} \xrightarrow{x=\frac{1}{2}+\frac{1}{2m}} \Omega(m) = \frac{1}{2}\cdot\frac{\pi}{cos\frac{\pi}{2m}} \Rightarrow$$

$$\Omega(m)\cdot\Omega(n)\cdot\Omega(p)=\frac{1}{8}\cdot\frac{\pi^3}{\cos\frac{\pi}{2m}\cdot\cos\frac{\pi}{2n}\cdot\cos\frac{\pi}{2p}}\ge\left(\frac{3\pi}{2\left(\cos\frac{\pi}{2m}+\cos\frac{\pi}{2n}+\cos\frac{\pi}{2p}\right)}\right)^3$$

**Solution 2 by Adrian Popa-Romania**

$$\left(\frac{3\pi}{2\left(\cos\frac{\pi}{2m}+\cos\frac{\pi}{2n}+\cos\frac{\pi}{2p}\right)}\right)^3\overset{Am-Gm}{\le}\left(\frac{3\pi}{2\left(3\cdot\sqrt[3]{\cos\frac{\pi}{2m}\cdot\cos\frac{\pi}{2n}\cdot\cos\frac{\pi}{2p}}\right)}\right)^3$$

$$=\frac{\pi^3}{2^3\cdot\cos\frac{\pi}{2m}\cdot\cos\frac{\pi}{2n}\cdot\cos\frac{\pi}{2p}}$$

*We must show that:* $\int_0^{\pi/2}\sqrt[m]{\tan x}\,dx\ge\frac{\pi}{2\cos\frac{\pi}{2m}}$

$$I=\int_0^{\frac{\pi}{2}}\sqrt[m]{\tan x}\,dx=\int_0^{\frac{\pi}{2}}(\tan x)^{\frac{1}{m}}dx=\int_0^{\frac{\pi}{2}}(\tan x)^n dx$$

$$\underbrace{\overset{\frac{1}{m}=n\in\left(0,\frac{1}{2}\right)}{=}}_{\tan x=t}\int_0^{\infty}\frac{t^n}{1+t^2}dt\overset{t^2=y}{=}\frac{1}{2}\int_0^{\infty}\frac{y^{\frac{n-1}{2}}}{1+y}dy=\frac{1}{2}\int_0^{\infty}\frac{y^{\frac{n+1}{2}-1}}{(1+y)^{\frac{n+1}{2}-\frac{1-n}{2}}}dy=B\left(\frac{n+1}{2},\frac{1-n}{2}\right)$$

$$=\frac{\pi}{2\sin\frac{\pi(n+1)}{2}}=\frac{\pi}{2\cos\frac{\pi}{2m}}$$

*So:* $I=\int_0^{\frac{\pi}{2}}\sqrt[m]{\tan x}\,dx=\frac{\pi}{2\cos\frac{\pi}{2m}}$ *then:*

$$\Omega(m)\cdot\Omega(n)\cdot\Omega(p)\ge\left(\frac{3\pi}{2\left(\cos\frac{\pi}{2m}+\cos\frac{\pi}{2n}+\cos\frac{\pi}{2p}\right)}\right)^3$$

**954. If $f:[a,b]\to\left(0,\frac{\pi}{2}\right)$, $f$ –continuous, $a\le b$ then:**

$$\int_a^b \sin f(x)dx+\frac{1}{2}\int_a^b \tan f(y)dy+\int_a^b \cos f(t)dt+\frac{1}{2}\int_a^b \cot f(z)dz\ge\left(\sqrt{2}+1\right)(b-a)$$

*Proposed by Daniel Sitaru-Romania*

***Solution by Khaled Abd Imouti-Damascus-Syria***

$$\int_a^b sinf(x)dx+\frac{1}{2}\int_a^b tanf(y)dy+\int_a^b cosf(t)dt+\frac{1}{2}\int_a^b cotf(z)dz\overset{?}{\gtreqless}(\sqrt{2}+1)(b-a)$$

$$\int_a^b [sinf(x)+cosf(x)]dx+\frac{1}{2}\int_a^b [tanf(y)+cotf(y)]dy\overset{?}{\gtreqless}(\sqrt{2}+1)(b-a)$$

***Let be the function:*** $g(\theta)=sin\theta+cos\theta+\frac{1}{2}(tan\theta+cot\theta),\theta\in\left(0,\frac{\pi}{2}\right)$

$$\lim_{\theta\to 0_+}[g(\theta)]=+\infty;\ \lim_{\theta\to\frac{\pi}{2}}[g(\theta)]=+\infty$$

$$g'(\theta)=cos\theta-sin\theta+\frac{1}{2}(tan^2\theta-cot^2\theta)$$

$$g'(\theta)=0\Leftrightarrow\theta=\frac{\pi}{4};g\left(\frac{\pi}{4}\right)=\sqrt{2}+1$$

| $\theta$ | $0$ $\qquad\qquad\qquad \frac{\pi}{4} \qquad\qquad\qquad \frac{\pi}{2}$ |
|---|---|
| $g'(\theta)$ | $-----------\ 0\ +++++++$ |
| $g(\theta$ | $+\infty$ ↘↘↘↘↘ $\sqrt{2}+1$ ↗↗↗↗↗ $+\infty$ |

***So,*** $g(\theta)\ge\sqrt{2}+1,\forall\theta\in\left(0,\frac{\pi}{2}\right)$

$$\int_a^b sinf(x)dx+\frac{1}{2}\int_a^b tanf(y)dy+\int_a^b cosf(t)dt+\frac{1}{2}\int_a^b cotf(z)dz\ge(\sqrt{2}+1)(b-a)$$

**955. Prove:**

$$\int_a^b \frac{(1+\sqrt{x})(1+3\sqrt{x})\dots(1+(2n-1)\sqrt{x})}{(1+n\sqrt{x})^n}dx<b-a;0<a<b$$

***Proposed by Jalil Hajimir-Toronto-Canada***

***Solution 1 by Rahim Shahbazov-Baku-Azerbaijan***

$$1)\ x_1x_2\dots x_n\le\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^n$$

$$2)\ 1+3+5+\dots+(2n-1)=n^2$$

***We have:***

$$(1+\sqrt{x})(1+3\sqrt{x})\dots(1+(2n-1)\sqrt{x}) \le \left(\frac{n+(1+3+5+\dots+(2n-1))\sqrt{x}}{n}\right)^n = (1+n\sqrt{x})^n$$

$$\int_a^b \frac{(1+\sqrt{x})(1+3\sqrt{x})\dots(1+(2n-1)\sqrt{x})}{(1+n\sqrt{x})^n}dx < b-a$$

***Solution 2 by Max Wong-Hong Kong***

$$\int_a^b \frac{(1+\sqrt{x})(1+3\sqrt{x})\dots(1+(2n-1)\sqrt{x})}{(1+n\sqrt{x})^n}dx$$

$$\overset{Am-Gm}{\le} \int_a^b \left(\frac{(1+\sqrt{x})+(1+3\sqrt{x})+\dots+(1+(2n-1)\sqrt{x})}{n(1+n\sqrt{x})}\right)^n dx$$

$$= \int_a^b \left(\frac{n(1+n\sqrt{x})}{n(1+n\sqrt{x})}\right)^n dx = \int_a^b 1dx = b-a$$

***So:***

$$\int_a^b \frac{(1+\sqrt{x})(1+3\sqrt{x})\dots(1+(2n-1)\sqrt{x})}{(1+n\sqrt{x})^n}dx < b-a$$

**956. Prove:**

$$\int_0^{\frac{1}{2}} (1+x)^{\frac{1}{2}+x}\cdot(1-x)^{\frac{1}{2}-x}dx > \frac{1}{2}$$

***Proposed by Jalil Hajimir-Toronto-Canada***

***Solution 1 by Florică Anastase-Romania***

$$\text{Let: } E(x) = (1+x)^{\frac{1}{2}+x}\cdot(1-x)^{\frac{1}{2}-x}, \forall x \in \left[0,\frac{1}{2}\right]$$

$$logE(x) = \left(\frac{1}{2}+x\right)log(1+x) + \left(\frac{1}{2}-x\right)log(1-x) =$$

$$= \left(\frac{1}{2}+x\right)log\left(\frac{1}{2}+\left(\frac{1}{2}+x\right)\right) + \left(\frac{1}{2}-x\right)log\left(\frac{1}{2}+\left(\frac{1}{2}-x\right)\right) \overset{(*)}{\ge}$$

$$\geq 2\left(\frac{1}{2}+x+\frac{1}{2}-x\right)log\left(\frac{1}{2}+\frac{\frac{1}{2}+x+\frac{1}{2}-x}{2}\right)=2log1=0\to$$

$$E(x)>1\to\int_0^{\frac{1}{2}}(1+x)^{\frac{1}{2}+x}\cdot(1-x)^{\frac{1}{2}-x}dx>\frac{1}{2}$$

$$(*)f:\left[0,\frac{1}{2}\right]\to\mathbb{R},f(t)=tlog\left(\frac{1}{2}+t\right)=tlog(1+2t)-tlog2$$

$$f'(x)=log(1+2t)+\frac{2t}{1+2t}-log2$$

$$f''(t)=\frac{2}{1+2t}+\frac{2}{(1+2t)^2}>0,\forall t\in\left[0,\frac{1}{2}\right]\to f-\text{convexe}$$

***Solution 2 by Ravi Prakash-New Delhi-India***

***As $1+x,1-x$ are distinct real numbers, using Gm-Hm, we get:***

$$\left[(1+x)^{\frac{1}{2}+x}(1-x)^{\frac{1}{2}-x}\right]^{\left(\frac{1}{2}+x\right)+\left(\frac{1}{2}-x\right)}>\frac{\frac{1}{2}+x+\frac{1}{2}-x}{\frac{\frac{1}{2}+x}{1+x}+\frac{\frac{1}{2}-x}{1-x}}=\frac{1-x^2}{1-2x^2}\geq 1,$$

$$\forall x\in\left[0,\frac{1}{2}\right]\to(1+x)^{\frac{1}{2}+x}\cdot(1-x)^{\frac{1}{2}-x},\forall x\in\left[0,\frac{1}{2}\right]$$

$$\int_0^{\frac{1}{2}}(1+x)^{\frac{1}{2}+x}\cdot(1-x)^{\frac{1}{2}-x}dx>\frac{1}{2}$$

**957. Prove without softs:**

$$\int_0^1\frac{25+sinx}{25+sin(1-x)}dx>1$$

***Proposed by Radu Diaconu-Romania***

***Solution by Adrian Popa-Romania***

$$\therefore\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

$$I=\int_0^1\frac{25+sinx}{25+sin(1-x)}dx=\int_0^1\frac{25+sin(1-x)}{25+sinx}dx,$$

$$2I=\int_0^1\left(\frac{25+sinx}{25+sin(1-x)}+\frac{25+sin(1-x)}{25+sinx}\right)dx$$

Denote: $\frac{25+sinx}{25+sin(1-x)}=a\Rightarrow\frac{25+sin(1-x)}{25+sinx}=\frac{1}{a}$ and from $a+\frac{1}{a}\geq 2,\forall a>0$

we get: $2I\geq 2\Rightarrow I\geq 2$

**958. If $a\geq 1$ then:**

$$\frac{8}{\pi-2}\int_1^a\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}dx+\frac{16}{\pi^2}\geq\frac{1}{(tan^{-1}a)^2}$$

***Proposed by Daniel Sitaru-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

Let: $\varphi(a)=\frac{8}{\pi-2}\int_1^a\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}dx+\frac{16}{\pi^2}-\frac{1}{(tan^{-1}a)^2}$

$$\varphi'(a)=\frac{8}{\pi-2}\left[\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}\right]+\frac{2}{(1+a^2)(tan^{-1}a)^3}$$

$$=\frac{2}{(1+a^2)(tan^{-1}a)^2}\left[\frac{4(a-tan^{-1}a)}{\pi-2}+\frac{1}{tan^{-1}a}\right]\overset{(*)}{>}0;\ \forall a\geq 1$$

***(*) is true, because:*** $\forall a\geq 1\Rightarrow 1+a^2\geq 2a\geq 2>0;\Rightarrow tan^{-1}a\geq\frac{\pi}{4}>0$

$$\pi-2>0$$

$$\forall a\geq 1\Rightarrow tan(a)\geq a\Rightarrow tan^{-1}(tana)>tan^{-1}a\Rightarrow a>tan^{-1}a$$

$$\Rightarrow 4(a-tan^{-1}a)>0$$

So, $\varphi'(a)>0,\forall a\geq 1\Rightarrow\varphi(a)\uparrow[1,\infty)\Rightarrow\varphi(a)\geq\varphi(1)=0$

$$\Rightarrow\frac{8}{\pi-2}\int_1^a\frac{x-tan^{-1}x}{(1+x^2)^2(tan^{-1}x)^2}dx+\frac{16}{\pi^2}-\frac{1}{(tan^{-1}a)^2}\geq 0$$

**959.** $a,b,c>0, abc=1$

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx$$

**Prove that:** $\Omega(a)+\Omega(b)+\Omega(c)\geq 3e^{a^2+b^2+c^2}$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx$$

$$=\int_{-a}^{0}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx+\int_{0}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx$$

$$=\int_{0}^{a}\left(\frac{e^{3(-x)^2}}{1+e^{-x}}+6(-x)e^{3(-x)^2}log(1+e^{-x})\right)dx+\int_{0}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx$$

$$=\int_{0}^{a}\left(\frac{e^x\cdot e^{3x^2}}{1+e^x}-6xe^{3x^2}(log(1+e^x)-x)\right)dx+\int_{0}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx$$

$$=\int_{0}^{a}e^{3x^2}dx+2\left[\frac{ae^{3a^2}}{2}-\frac{1}{2}\int_{0}^{a}e^{3x^2}dx\right]=ae^{3a^2}$$

$$\Omega(a)+\Omega(b)+\Omega(c)=ae^{3a^2}+be^{3b^2}+ce^{3c^2}\overset{Am-Gm}{\geq}3\sqrt[3]{abc\left(e^{a^2}e^{b^2}e^{c^2}\right)^3}$$

$$\overset{abc=1}{=}3e^{a^2}e^{b^2}e^{c^2}=3e^{a^2+b^2+c^2}$$

***Solution 2 by Ravi Prakash-New Delhi-India***

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx;\quad (1)$$

$$\Omega(a)=\int_{-a}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}log(1+e^x)\right)dx\overset{x=-t}{=}$$

$$= \int_{-a}^{a} \left( \frac{e^{3t^2}}{1+e^{-t}} - 6te^{3t^2} log(1+e^{-t}) \right) dt$$

$$= \int_{-a}^{a} \left( \frac{e^{3x^2}}{1+e^{x}} - 6xe^{3x^2} (log(1+e^{x}) - x) \right) dx; \quad (2)$$

***Adding (1),(2) we get:***

$$2\Omega(a) = \int_{-a}^{a} (e^{3x^2} + 6x^2 e^{3x^2}) dx = 2\int_{0}^{a} (e^{3x^2} + 6xe^{3x^2} \cdot x) dx =$$

$$= 2\int_{0}^{a} e^{3x^2} dx + 2xe^{3x^2} \Big|_{0}^{a} = 2ae^{3a^2}$$

$$\Omega(a) + \Omega(b) + \Omega(c) \ge 3\sqrt[3]{abc} \cdot \sqrt[3]{e^{3a^2+3b^2+3c^2}} = 3e^{a^2+b^2+c^2}$$ ***. Proved.***

**960. If $a, b, c > 0, a + b + c = 3$ then:**

$$\int_{0}^{\frac{\pi}{2}} a^{sinx} dx + \int_{0}^{\frac{\pi}{2}} b^{sinx} dx + \int_{0}^{\frac{\pi}{2}} c^{sinx} dx \le \frac{3\pi}{2}$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Ali Jaffal-Lebanon***

***Let:*** $x \in \left[0, \frac{\pi}{2}\right] \to sinx \in [0, 1]$

***Consider the function*** $f(t) = t^{sinx}, t \in [0, \infty)$

$f''(t) = sinx(sinx - 1)t^{sinx-2} < 0 \Rightarrow f$ ***–concave on*** $[0, \infty)$

***Let:*** $a, b, c > 0, a + b + c = 3$

$$f\left(\frac{a+b+c}{3}\right) \ge \frac{f(a) + f(b) + f(c)}{3} \Rightarrow f(a) + f(b) + f(c) \le 3f(1) \le 3$$

***So,*** $a^{sinx} + b^{sinx} + c^{sinx} \le 3, \forall x \in \left[0, \frac{\pi}{2}\right]$

***So:***

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx \le \frac{3\pi}{2}$$

**Solution 2 by Khaled Abd Imouti-Damascus-Syria**

**Let**: $f(x) = x^{\alpha} = e^{\alpha log(x)}, x \in \left[0, \frac{\pi}{2}\right], \alpha \in [0, 1]$

$$\lim_{x\to 0_+} f(x) = 0;\ \lim_{x\to\frac{\pi}{2}} f(x) = e^{\alpha log\left(\frac{\pi}{2}\right)}$$

$f'(x) = \frac{\alpha}{x} e^{\alpha log(x)}, f''(x) = \frac{\alpha}{x^2}(\alpha - 1)e^{\alpha log(x)} < 0, \alpha \le 1 \Rightarrow f$ −**concave.**

**So:** $a^{\alpha} + b^{\alpha} + c^{\alpha} \le 3\left(\frac{a+b+c}{3}\right)^{\alpha} \Rightarrow a^{\alpha} + b^{\alpha} + c^{\alpha} \le 3$

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx \le \frac{3\pi}{2}$$

**Solution 3 by Florentin Vişescu-Romania**

$$a, b, c > 0, a + b + c = 3$$

**Let**: $f: (0; 3) \to \mathbb{R};\ f(t) = t^{sinx},\ x \in \left[0, \frac{\pi}{2}\right]$

$f'(t) = sinx \cdot t^{sinx-1}, f''(x) = \underbrace{sinx}_{>0}\underbrace{(sinx - 1)}_{<0} t^{sinx-2} \le 0 \to f$ −**concave.**

$$f(a) + f(b) + f(c) \le 3f\left(\frac{a + b + c}{3}\right)$$

$$a^{sinx} + b^{sinx} + c^{sinx} \le 3\left(\frac{a + b + c}{3}\right)^{sinx}$$

**So,** $a^{sinx} + b^{sinx} + c^{sinx} \le 3, \forall x \in \left[0, \frac{\pi}{2}\right]$

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx \le \frac{3\pi}{2}$$

**Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand**

**For** $a, b, c > 0, a + b + c = 3$ **we have:**

$$\int_0^{\frac{\pi}{2}} a^{sinx}dx + \int_0^{\frac{\pi}{2}} b^{sinx}dx + \int_0^{\frac{\pi}{2}} c^{sinx}dx = \int_0^{\frac{\pi}{2}} (a^{sinx} + b^{sinx} + c^{sinx})dx$$

$$\leq \int_0^{\frac{\pi}{2}} \frac{(a+b+c)^{sinx}}{3^{sinx-1}}dx = \int_0^{\frac{\pi}{2}} \frac{3^{sinx}}{3^{sinx-1}}dx = 3x\Big|_0^{\frac{\pi}{2}} = \frac{3\pi}{2}$$

**961.** If $0 < a < b$ then:

$$\int_a^{4a}\int_b^{4b} \frac{x^2+y^2}{(2x+y)(x+2y)}dxdy > 2ab$$

***Proposed by Jalil Hajimir-Toronto-Canada***

***Solution 1 by Daniel Sitaru-Romania***

$$(x-y)^2 \geq 0 \to x^2 - 2xy + y^2 \geq 0 \to 5x^2 - 10xy + 5y^2 \geq 0 \to$$

$$9x^2 - 10xy + 9y^2 \geq 4x^2 + 4y^2 \to 9x^2 + 9y^2 \geq 4x^2 + 4y^2 + 10xy \to$$

$$9(x^2+y^2) \geq 2(2x^2 + 2y^2 + 5xy) \to 9(x^2+y^2) \geq 2(2x+y)(x+2y) \to$$

$$\frac{x^2+y^2}{(2x+y)(x+2y)} \geq \frac{2}{9}$$

$$\int_a^{4a}\int_b^{4b} \frac{x^2+y^2}{(2x+y)(x+2y)}dxdy > \int_a^{4a}\int_b^{4b} \frac{2}{9}dxdy =$$

$$= \frac{2}{9}\int_a^{4a} dx \int_b^{4b} dy = \frac{2}{9}(4a-a)(4b-b) = 2ab$$

***Solution 2 by Khaled Abd Imouti-Damascus Syria***

$$\begin{cases} a < x < 4a \\ 2a < x < 8a \end{cases} \Rightarrow 4a < 2x < 16a$$

$$b < y < 4b \to 4a + b < 2x + y < 16a + 4b$$

$$\begin{cases} 2b < 2y < 8b \\ a < x < 4a \end{cases} \Rightarrow a + 2b < 2y + x < 4a + 8b$$

$$(2x+y)(2y+x) \leq (16a+4b)(4a+8b) \Leftrightarrow$$

$$(2x+y)(2y+x) \leq 16(4a+b)(a+2b)$$

***But:*** $\frac{1}{(2x+y)(2y+x)} \geq \frac{1}{16(4a+b)(a+2b)}$ ***and*** $x^2 + y^2 \geq a^2 + b^2$ ***then:***

$$\frac{x^2 + y^2}{(2x + y)(2y + x)} \geq \frac{a^2 + b^2}{16(4a + b)(a + 2b)}$$

$$I \geq \frac{9(a^2 + b^2)}{16(4a + b)(a + 2b)} \cdot ab \Rightarrow I \geq \frac{9(a^2 + b^2)}{32(4a + b)(a + 2b)} \cdot 2ab$$

$$9(a^2 + b^2) \overset{?}{\leq} 32(4a + b)(a + 2b) \Leftrightarrow 32(4a + b)(a + 2b) \geq 5a \cdot 3a \cdot 32$$

$$32(4a + b)(a + 2b) \geq 15 \cdot 32a^2 \Leftrightarrow (4a + b)(a + 2b) \geq 15a^2$$

$$4a^2 + 9ab + 2b^2 \geq 15a^2$$

$$4a^2 + 9ab + 2b^2 \geq 4a^2 + 9a^2 + 2a^2 = 15a^2 \text{ true.}$$

**962.**

$$a, b, c: 1, (a + 1)(b + 1)(c + 1) = 8e^{2020}, \Omega(a, b) = bloga \int_0^1 \frac{dt}{a^t + b}$$

**Prove that:**

$$\Omega(a, b) + \Omega(b, c) + \Omega(c, a) \leq 2020$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Tran Hong-Dong Thap-Vietnam***

- $\Omega(a, b) = b\log a . \int_0^1 \frac{dx}{a^x+b} = b \log a . \left[\frac{\log a^x - \log(a^x+b)}{b \log a}\right]\Big|_0^1 = \left(\log \frac{a^x}{a^x+b}\right)\Big|_0^1 = \log\left(\frac{a}{a+b}\right) - \log\left(\frac{1}{1+b}\right) = \log\left(\frac{a(1+b)}{a+b}\right);$

***Similary***:

$$\Omega(b, c) = \log\left(\frac{b(1 + c)}{b + c}\right); \ \Omega(c, a) = \log\left(\frac{c(1 + a)}{c + a}\right)$$

$$\rightarrow \Omega(a, b) + \Omega(b, c) + \Omega(c, a) = \log\left(\frac{a(1 + b)}{a + b} . \frac{b(1 + c)}{b + c} . \frac{c(1 + a)}{c + a}\right) \overset{(*)}{\lesseqqgtr} 2020;$$

$$(*) \leftrightarrow \frac{abc.(1 + a)(1 + b)(1 + c)}{(a + b)(b + c)(c + a)} \leq e^{2020} = \frac{(a + 1)(b + 1)(c + 1)}{8}$$

$$\leftrightarrow (a + b)(b + c)(c + a) \geq 8abc;$$

***Which is true because :*** $a + b \geq 2\sqrt{ab}; b + c \geq 2\sqrt{bc}; c + a \geq 2\sqrt{ca}$

$$\rightarrow (a + b)(b + c)(c + a) \geq 8abc \text{ . Proved.}$$

**Solution 2 by Florică Anastase-Romania**

$$a,b>1\to y=a^t+b>1\to y-b=a^t\ (t\to a^t-bijective)$$

$$\begin{cases} t=\log_a(y-b) \\ dt=\dfrac{1}{y-b}\cdot\dfrac{1}{loga}\end{cases}$$

$$\Omega(a,b)=bloga\int_0^1\frac{dt}{a^t+b}=\int_{1+b}^{a+b}\left(\frac{1}{y-b}-\frac{1}{y}\right)dy=log\frac{a(1+b)}{a+b}$$

$$\Omega(a,b)+\Omega(b,c)+\Omega(c,a)=log\frac{abc(1+a)(1+b)(1+c)}{(a+b)(b+c)(c+a)}\overset{Am-Gm}{\leq}$$

$$\leq log\frac{abc(1+a)(1+b)(1+c)}{8abc}=log\frac{8e^{2020}}{8}=2020$$

**963.**

$$a,b,c:1,(a+1)(b+1)(c+1)=8e^{2020},\Omega(a,b)=bloga\int_0^1\frac{dt}{a^t+b}$$

**Prove that:**

$$\Omega(a,b)+\Omega(b,c)+\Omega(c,a)\leq 2020$$

***Proposed by Daniel Sitaru-Romania***

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

- $\Omega(a,b)=b\log a.\int_0^1\frac{dx}{a^x+b}=b\log a.\left[\frac{\log a^x-\log(a^x+b)}{b\log a}\right]\Big|_0^1=\left(\log\frac{a^x}{a^x+b}\right)\Big|_0^1=$

  **log*aa+b*–log*11+b*=log*a(1+b)a+b;***

***Similary:***

$$\Omega(b,c)=\log\left(\frac{b(1+c)}{b+c}\right);\ \Omega(c,a)=\log\left(\frac{c(1+a)}{c+a}\right)$$

$$\to\Omega(a,b)+\Omega(b,c)+\Omega(c,a)=\log\left(\frac{a(1+b)}{a+b}.\frac{b(1+c)}{b+c}.\frac{c(1+a)}{c+a}\right)\overset{(*)}{\lesseqgtr}2020;$$

$$(*)\leftrightarrow\frac{abc.(1+a)(1+b)(1+c)}{(a+b)(b+c)(c+a)}\leq e^{2020}=\frac{(a+1)(b+1)(c+1)}{8}$$

$$\leftrightarrow\ (a+b)(b+c)(c+a)\geq 8abc;$$

***Which is true because:*** $a+b\geq 2\sqrt{ab};\ b+c\geq 2\sqrt{bc};\ c+a\geq 2\sqrt{ca}$

$$\to (a+b)(b+c)(c+a)\geq 8abc.$$ ***Proved.***

***Solution 2 by Florică Anastase-Romania***

$$a,b>1\to y=a^t+b>1\to y-b=a^t\ (t\to a^t-bijective)$$

$$\begin{cases} t=\log_a(y-b)\\ dt=\dfrac{1}{y-b}\cdot\dfrac{1}{loga}\end{cases}$$

$$\Omega(a,b)=bloga\int_0^1\frac{dt}{a^t+b}=\int_{1+b}^{a+b}\left(\frac{1}{y-b}-\frac{1}{y}\right)dy=log\frac{a(1+b)}{a+b}$$

$$\Omega(a,b)+\Omega(b,c)+\Omega(c,a)=log\frac{abc(1+a)(1+b)(1+c)}{(a+b)(b+c)(c+a)}\overset{Am-Gm}{\leq}$$

$$\leq log\frac{abc(1+a)(1+b)(1+c)}{8abc}=log\frac{8e^{2020}}{8}=2020$$

**964. If $0<a\leq b<\frac{\pi}{2}$ then:**

$$\left(\int_0^{\frac{4a+b}{5}}sin(xsinx)dx\right)\left(\int_0^{\frac{a+4b}{5}}cos(xsinx)dx\right)\leq\left(\int_0^{\frac{a+4b}{5}}sin(xsinx)dx\right)\left(\int_0^{\frac{4a+b}{5}}cos(xsinx)dx\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution by Tran Hong-Dong Thap-Vietnam***

***Let*** $f(x)=\sin(x\sin x),\ g(x)=\cos(x\sin x),\ \ \forall x\in\left(0;\frac{\pi}{2}\right)$

$$\to f(x),g(x)>0,\forall x\in\left(0;\frac{\pi}{2}\right)$$

***Now, put:*** $\varphi(t)=\frac{\int_0^t f(x)dx}{\int_0^t g(x)dx},\ \forall t\in\left(0;\frac{\pi}{2}\right)$

$$\to \varphi'(t) = \frac{f(t)\int_0^t g(x)dx - g(t)\int_0^t g(x)dx}{\left(\int_0^t g(x)dx\right)^2};$$

- $h(t) = f(t)\int_0^t g(x)dx - g(t)\int_0^t f(x)dx, \quad \forall t \in \left(0; \frac{\pi}{2}\right)$

$$\to h'(t) = \left[f'(t)\int_0^t g(x)dx + f(t)g(t)\right] - \left[g'(t)\int_0^t f(x)dx + g(t)f(t)\right]$$

$$= (\sin t + t\cos t)\cos(t\sin t)\int_0^t g(x)dx + (\sin t + t\cos t)\sin(t\sin t)\int_0^t f(x)dx$$

$$= (\sin t + t\cos t)\left(\cos(t\sin t)\int_0^t g(x)dx + \sin(t\sin t)\int_0^t f(x)dx\right) > 0$$

$$\forall t \in \left(0; \frac{\pi}{2}\right)$$

$$\to h(t) \uparrow \left(0; \frac{\pi}{2}\right) \to h(t) > h(0) = 0, \ \forall t \in \left(0; \frac{\pi}{2}\right)$$

$$\to \varphi'(t) > 0, \forall t \in \left(0; \frac{\pi}{2}\right) \to \varphi(t) \uparrow \left(0; \frac{\pi}{2}\right)$$

$$\overset{0<a\le b<\frac{\pi}{2}}{\Rightarrow} 0 < \frac{4a+b}{5} \le \frac{a+4b}{5} < \frac{\pi}{2} \to \varphi\left(\frac{4a+b}{5}\right) \le \varphi\left(\frac{a+4b}{5}\right)$$

$$\to \frac{\int_0^{\frac{4a+b}{5}} f(x)dx}{\int_0^{\frac{4a+b}{5}} g(x)dx} \le \frac{\int_0^{\frac{a+4b}{5}} f(x)dx}{\int_0^{\frac{a+4b}{5}} g(x)dx}$$

**965.**

$$n, k \in \mathbb{N}, n \ge k, a_n = (n+1)\sum_{k=0}^{n}\int_0^1 (1-x)^{n-k}x^k dx$$

**Find:**

$$\Omega = \lim_{n\to\infty} n(a_n - 2)$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by Izumi Ainsworth-Lima-Peru**

$$By\ Beta\ function \rightarrow B(x,y) = \int_0^1 (1-u)^{x-1} \cdot u^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$Then\ in\ the\ problem: \int_0^1 (1-x)^{n-k} \cdot x^k dx = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \rightarrow$$

$$\sum_{k=0}^{n} \int_0^1 (1-x)^{n-k} \cdot x^k dx = 2 \sum_{k=0}^{\frac{n}{2}-1} \int_0^1 (1-x)^{n-k} \cdot x^k dx + \frac{\Gamma^2\left(\frac{n}{2}+1\right)}{\Gamma(n+2)} =$$

$$= 2\left[\frac{0!\,n!}{(n+1)!} + \frac{1!\,(n-1)!}{(n+1)!} + \dots + \frac{\left(\frac{n}{2}-1\right)!\left(\frac{n}{2}+1\right)!}{(n+1)!}\right] + \frac{\left(\frac{n}{2}\right)!^2}{(n+1)!}$$

***Multiply (n+1):***

$$\rightarrow a_n = 2 + 2\left[\frac{1!\,(n-1)!}{n!} + \frac{2!\,(n-2)!}{n!} \dots + \frac{\left(\frac{n}{2}-1\right)!\left(\frac{n}{2}+1\right)!}{n!}\right] + \frac{\left(\frac{n}{2}\right)!^2}{n!}$$

$$\rightarrow S = n(a_n - 1) = 2\left[1 + \frac{2!\,(n-2)!}{(n-1)!} \dots + \frac{\left(\frac{n}{2}-1\right)!\left(\frac{n}{2}+1\right)!}{(n-1)!}\right] + \frac{\left(\frac{n}{2}\right)!^2}{(n-1)!}$$

$$\Omega = \lim_{n\to\infty} n(a_n - 2) = 2$$

**Solution 2 by Adrian Popa-Romania**

$$\int_0^1 (1-x)^{n-k} \cdot x^k dx = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{k!\,(n-k)!}{(n+1)!} = \frac{n!}{(n+1)!\binom{n}{k}} = \frac{1}{(n+1)\binom{n}{k}}$$

$$a_n = (n+1)\sum_{k=0}^{n} \frac{1}{(n+1)\binom{n}{k}} = \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} < 1 + \frac{1}{n} + \frac{2}{n(n-1)} + \underbrace{\frac{6}{n(n-1)(n-2)} + \dots + \frac{6}{n(n-1)(n-2)}}_{n-5\ times} + \frac{2}{n(n-1)} + \frac{1}{n} + 1 =$$

$$= 2 + \frac{2}{n} + \frac{4}{n(n-1)} + \frac{6(n-5)}{n(n-1)(n-2)} \rightarrow 2 \quad (i)$$

$$\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} > 2+\frac{2}{n}\to 2\quad (ii)$$

***From (i),(ii) we have:*** $\Omega = lim_{n\to\infty} n(a_n-2) = 2$

***Solution 3 by proposer***

$$I=\int_0^1 (tx+1-x)^n dx=\int_0^1((t-1)x+1)^n dx=\left(\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right)\Bigg|_0^1=\frac{1}{n+1}(t^n+t^{n-1}+\dots+t+1)\quad (i)$$

$$I=\int_0^1 (tx+1-x)^n dx=\int_0^1((1-x)+tx)^n dx=\sum_{k=0}^{n}\binom{n}{0}t^k\int_0^1(1-x)^{n-k}x^k dx\quad (ii)$$

***From (i),(ii) we have:***

$$\sum_{k=0}^{n}\binom{n}{0}t^k\int_0^1(1-x)^{n-k}x^k dx=\frac{1}{n+1}(t^n+t^{n-1}+\dots+t+1)$$

$$and\ identifyng\ the\ coefficients$$

$$\sum_{k=0}^{n}\int_0^1(1-x)^{n-k}x^k dx=\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}\to a_n=\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}$$

$$Let\ n\ge 6\to\binom{n}{k}\ge\binom{n}{3},3\le k\le n-3\to$$

$$\begin{cases} a_n > \left(\frac{1}{\binom{n}{0}}+\frac{1}{\binom{n}{n}}\right)+\left(\frac{1}{\binom{n}{1}}+\frac{1}{\binom{n}{n-1}}\right) > 2+\frac{2}{n} \\ a_n < \left(\frac{1}{\binom{n}{0}}+\frac{1}{\binom{n}{n}}\right)+\left(\frac{1}{\binom{n}{1}}+\frac{1}{\binom{n}{n-1}}\right)+\left(\frac{1}{\binom{n}{2}}+\frac{1}{\binom{n}{n-2}}\right)+\frac{n-5}{\binom{n}{3}} \end{cases}\to$$

$$2+\frac{2}{n}<a_n<2+\frac{2}{n}+\frac{4}{n(n-1)}+\frac{6(n-5)}{(n-1)(n-2)}\to\Omega=\lim_{n\to\infty}n(a_n-2)=2$$

**966. Find:**

$$\Omega=\lim_{n\to\infty}\left(\frac{n\cdot\sqrt[n]{\prod_{k=1}^{n}\sin^2\left(\frac{k}{n}\right)}}{\sum_{1\le i<j\le n}\sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)}\right)$$

***Proposed by Daniel Sitaru – Romania***

***Solution by Ali Jaffal-Lebanon***

$$\text{Let } U_n = \sqrt[n]{\prod_{k=1}^{k=n} \sin^2\left(\frac{k}{n}\right)}$$

$$\log U_n = \frac{1}{n}\sum_{k=1}^{k=n} \sin^2\left(\frac{k}{n}\right)$$

$$\text{So, } \lim_{n\to+\infty} \log U_n = \int_0^1 \sin^2 x\,dx = \frac{1}{2}x - \frac{\sin(2x)}{4}\Big]_0^1$$

$$= \frac{1}{2} - \frac{\sin(2)}{4}$$

$$\text{then } \lim_{n\to+\infty} V_n = e^{\frac{1}{2}-\frac{\sin(2)}{4}}$$

***We know that***

$$2\sum_{1\le i<j\le n} \sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right) + \sum_{i=1}^{i=n} \sin\left(\frac{i}{n}\right)\sin\left(\frac{i}{n}\right) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right)$$

$$\text{then } \frac{1}{n}\sum_{1\le i<j<n} \sin\left(\frac{i}{n}\right)\sin\left(\frac{j}{n}\right) = \frac{1}{2n}\sum_{i=1}^{i=n} \sin^2\left(\frac{1}{n}\right) + \left(\frac{1}{n}\sum_{i=1}^{i=n} \sin\left(\frac{i}{n}\right)\right) \times \frac{1}{2}\sum_{i=1}^{i=n} \sin\left(\frac{i}{n}\right)$$

$$\text{but } \lim_{n\to+\infty} \sin\left(\frac{i}{n}\right) = \lim_{n\to+\infty} n \times \frac{1}{n} 8 \sum_{i=1}^{i=n} \sin\left(\frac{i}{n}\right)$$

$$= +\infty \times \int_0^1 \sin x\,dx = +\infty$$

$$\text{So, } \lim_{n\to+\infty} \frac{1}{n}\sum_{1\le i<j\le n} \sin\left(\frac{1}{n}\right)\sin\left(\frac{i}{n}\right) = \frac{1}{2}\int_0^1 \sin^2(x)\,dx + \infty = +\infty$$

$$\text{Therefore } \Omega = \frac{e^{\frac{1}{2}-\frac{\sin(2)}{4}}}{+\infty} = 0$$

**967. Find:**

$$\Omega = \lim_{n\to\infty} \exp\left(\frac{1}{n^3}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\sin k \cdot \sin j}{\sin k + \sin j} - 2 \cdot \sum_{1\le k<j\le n} \frac{\sin k \cdot \sin j}{\sin k + \sin j}\right)\right)$$

***Proposed by Daniel Sitaru – Romania***

**Solution 1 by Naren Bhandari-Bajura-Nepal**

*Denote*

$$S(n)=\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\sin k\sin j}{\sin j+\sin k}=\sin 1\sum_{j=1}^{n}\frac{\sin j}{\sin 1+\sin j}+$$

$$+\sin 2\sum_{j=1}^{n}\frac{\sin j}{\sin 2+\sin j}+\cdots+\sin k\sum_{j=1}^{n}\frac{\sin j}{\sin k+\sin j}$$

$$=\frac{1}{2}\sum_{k=1}^{n}\frac{\sin^2 k}{\sin k}+2\sum_{1\le k<l\le n}\frac{\sin k\sin j}{\sin j+\sin k}=\frac{1}{2}\sum_{k=1}^{n}\sin k+a$$

*Thus we have*

$$\Omega=\lim_{n\to\infty}\exp\left(\frac{S(n)}{n^3}-\frac{2}{n^3}\sum_{1\le k<j\le n}\frac{\sin i\sin j}{\sin i+\sin j}\right)$$

$$=\lim_{n\to\infty}\exp\left(\frac{1}{2n^3}\sum_{k=1}^{n}\sin k\right)\quad (1)$$

*Here*

$$\sum_{k=1}^{n}\sin k=\mathcal{T}\sum_{k=1}^{n}e^{ik}=\mathcal{T}\left(\frac{e^{i}(e^{in}-1)}{e^{i}-1}\right)$$

$$=\mathcal{T}\left(\frac{e^{i}e^{\frac{in}{2}}\left(e^{\frac{ni}{2}}-e^{-\frac{ni}{2}}\right)}{e^{\frac{i}{2}}\left(e^{\frac{i}{2}}-e^{-\frac{i}{2}}\right)}\right)=\mathcal{T}\left(e^{\frac{(n+1)i}{2}}\cdot\frac{\sin\frac{n}{2}}{\sin\frac{1}{2}}\right)=\frac{\sin\frac{n+1}{2}\sin\frac{n}{2}}{\sin\frac{1}{2}}$$

*Plugging in 1 we have:*

$$\lim_{n\to\infty}\exp\left(\frac{1}{2n^3}\cdot\frac{\sin\frac{n+1}{2}\sin\frac{n}{2}}{\sin\frac{1}{2}}\right)=e^0=1$$

**Solution 2 by Ali Jaffal-Lebanon**

*We know that:*

$$2\sum_{1\le k<j\le n}\frac{\sin k\cdot\sin j}{\sin k+\sin j}+\sum_{i=1}^{i=n}\frac{\sin i\cdot\sin i}{\sin i+\sin i}=\sum_{k=1}^{k=n}\sum_{j=1}^{j=n}\frac{\sin k+\sin j}{\sin k+\sin j}$$

*then* $\sum_{k=1}^{k=n}\sum_{j=1}^{i=n}\frac{\sin k\cdot\sin j}{\sin k+\sin j}-2\sum_{1\le k<j\le n}\frac{\sin k\cdot\sin j}{\sin k+\sin j}=\sum_{i=1}^{i=n}\frac{\sin i}{2}$

$$\left|\frac{1}{n^3}\sum_{i=1}^{i=n}\frac{\sin i}{2}\right| \le \frac{1}{2n^3}\sum_{i=1}^{i=n}|\sin i| \le \frac{1}{2n^3}\sum_{i=1}^{i=n} i\sin k\,|\sin i| \le i$$

$$\le \frac{1}{2n^3}\times\left(\frac{n}{2}(n+1)\right) \le \frac{n+1}{4n^2} \textit{ then } \lim_{n\to+\infty}\left|\frac{1}{n^3}\sum_{i=1}^{i=n}\frac{\sin i}{2}\right| = 0$$

$$\textit{then for } \lim_{n\to+\infty}\exp\left(\frac{1}{n^3}\sum_{i=1}^{i=n}\frac{\sin i}{2}\right) = 1 \textit{ and } \Omega = 1$$

**968. Find:**

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(k+1)e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}}}{n(n+1)(n+2)}$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Remus Florin Stanca-Romania*

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(k+1)e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}}}{n(n+1)(n+2)}$$

$$\textit{Let } x_k = \frac{k(k+1)(2k+1)}{n(n+1)(n+2)} \Rightarrow x_{k+1} - x_k = \frac{6k(k+1)}{n(n+1)(n+2)} \Rightarrow ||\Delta_n|| \overset{\max}{\underset{k\le n}{=}} \frac{6}{n+2} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}||\Delta_n|| = 0 \Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)f(\zeta_k) = \int_a^b f(x)\,dx, \zeta_k \in [x_k; x_{k+1}]$$

$$\textit{let } \zeta_k = x_k \textit{and } a = \lim_{n\to\infty} x_1 \textit{ and}$$

$$b = \lim_{n\to\infty} x_n \Rightarrow \lim_{n\to\infty}\frac{1}{6}\sum_{k=1}^{n}\frac{6(k+1)k}{n(n+1)(n+2)}e^{\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}} = \frac{1}{6}\int_0^2 e^x\,dx = \frac{e^2-1}{6} \Rightarrow$$

$$\Rightarrow \Omega = \frac{e^2-1}{6}$$

**969.**

$$a_n = \sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}, n, k \in \mathbb{N}, n \ge k$$

**Find:**

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$$\Omega = \lim_{n\to\infty}\left(1+\frac{a_n}{n^n}\right)^{\frac{a_n}{n^{n-2}}}$$

**Proposed by Florică Anastase-Romania**

**Solution1 by Kamel Benaicha-Algiers-Algerie**

$$a_n = \sum_{p=0}^{n}\binom{n}{p}\frac{n^p}{p+1}$$

$$Put\ f(x) = \sum_{k=0}^{n}\binom{n}{p}x^p = (1+x)^n$$

$$So: \sum_{p=0}^{n}\binom{n}{p}\int_0^n x^p dx = \sum_{p=0}^{n}\binom{n}{p}\frac{n^{p+1}}{p+1} = \frac{(1+n)^{n+1}-1}{n+1}$$

$$\therefore a_n = \frac{(1+n)^n}{n} - \frac{1}{n(n+1)} \ldots\ldots(1)$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{a_n}{n^n}\right)^{\frac{a_n}{n^{n-2}}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\left(\frac{n+1}{n}\right)^n\right)^{\left(n\left(\frac{n+1}{n}\right)^n - \frac{1}{(n+1)n^{n-1}}\right)}$$

$$\Omega = \lim_{n\to\infty} e^{\left(n\left(\frac{n+1}{n}\right)^n - \frac{1}{(n+1)n^{n-1}}\right)\log\left(1+\frac{1}{n}\left(\frac{n+1}{n}\right)^n\right)} \ldots\ldots(2)$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e^{\lim_{\varepsilon\to 0}\frac{\log(1+\varepsilon)}{\varepsilon}} = e \ldots.(3) \to \lim_{n\to\infty}\frac{1}{n}\left(\frac{n+1}{n}\right)^n = 0$$

$$\therefore \log\left(1+\frac{1}{n}\left(\frac{n+1}{n}\right)^n\right) \sim \frac{1}{n}\left(\frac{n+1}{n}\right)^n$$

$$\therefore (2) \leftrightarrow \Omega = \lim_{n\to\infty} e^{\left(n\left(\frac{n+1}{n}\right)^n - \frac{1}{(n+1)n^{n-1}}\right)\frac{1}{n}\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} e^{\left[\left(\left(\frac{n+1}{n}\right)^n\right)^2 - \frac{1}{n+1}\left(\frac{n+1}{n^2}\right)^n\right]}$$

$$Or: \lim_{n\to\infty}\left(\frac{n+1}{n^2}\right)^n = e^{\lim_{n\to\infty} n\log\left(1+\frac{1}{n^2}\right)} = 1$$

$$\therefore \lim_{n\to\infty}\frac{1}{n+1}\left(\frac{n+1}{n^2}\right)^n = 0, using\ \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e \ldots.(3)$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{a_n}{n^n}\right)^{\frac{1}{n^{n-2}}a_n} = e^{e^2}$$

**Solution 2 by Hemn Hsain-Iraq**

$$a_n=\frac{(n+1)^n}{n}-\frac{1}{n(n+1)}\ \therefore\ \lim_{x\to\infty}(f(x))^{g(x)}=e^{\lim_{x\to\infty}g(x)(f(x)-1)}$$

$$\lim_{n\to\infty}\left(1+\frac{a_n}{n^n}\right)^{\frac{1}{n^{n-2}}a_n}=e^{\lim_{n\to\infty}\left(1+\frac{a_n}{n^n}-1\right)\left(\frac{a_n}{n^{n-2}}\right)}$$

$$\Omega=e^{\lim_{n\to\infty}\left(\frac{na_n}{n^n}\right)^2}=e^{\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n-\frac{1}{n^n(n+1)}\right)^2}=e^{e^2}$$

**Solution 3 by proposer**

$$Let\ f:[0,1]\to R,f(x)=(1+nx)^n,n\in\mathbb{N},then\ we\ have:$$

$$I=\int_0^1 f(x)dx=\int_0^1(1+nx)^n dx=\int_0^1\left(\sum_{k=0}^{n}n^kx^k\binom{n}{k}\right)dx=\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}\quad(i)$$

$$I=\int_0^1(1+nx)^ndx\overset{t=1+nx,dx=\frac{dt}{n}}{\triangleq}\int_1^{n+1}\frac{t^n}{n}dt=\frac{(n+1)^{n+1}-1}{n(n+1)}\quad(ii)$$

$$From\ (i),(ii)\ we\ obtain:\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}=\frac{(n+1)^{n+1}-1}{n(n+1)}$$

$$Let\ x_n=\frac{1}{n^{n-1}}a_n=\left(1+\frac{1}{n}\right)^n-\frac{1}{(n+1)n^n}\ then:\ \lim_{n\to\infty}x_n=e$$

$$So:\Omega=\lim_{n\to\infty}\left(1+\frac{a_n}{n^n}\right)^{\frac{1}{n^{n-2}}a_n}=\lim_{n\to\infty}\left(1+\frac{x_n}{n}\right)^{nx_n}=e^{\lim_{n\to\infty}x_n^2}=e^{e^2}$$

***

**970.** If $n\in\mathbb{N},n\geq 2\ and\ x>0$ then find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\int_{\frac{1}{n}}^{1}\frac{dx}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}\right)}$$

*Proposed by Florică Anastase-Romania*

## Solution1 by Kamel Benaicha-Algiers-Algerie

$$x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}=x\cdot x^{\frac{1}{2}}\cdot x^{\frac{1}{2\cdot3}}\cdot x^{\frac{1}{2\cdot3\cdot4}}\cdot\ldots\cdot x^{\frac{1}{2\cdot3\cdot4\cdot\ldots\cdot(n-1)\cdot n}}=x^{1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}}=x^{e+E(n)}/\ \lim_{n\to\infty}E(n)=0$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\int_{\frac{1}{n}}^{1}\frac{dx}{x\left(1+x^{1+e+E(n)}\right)}$$

$$\frac{1}{x\left(1+x^{1+e+E(n)}\right)}=\frac{1}{x}-\frac{x^{1+e+E(n)}}{\left(1+x^{1+e+E(n)}\right)}$$

$$\int_{\frac{1}{n}}^{1}\frac{dx}{x\left(1+x^{1+e+E(n)}\right)}=\log(x)\Big|_{\frac{1}{n}}^{1}-\frac{1}{1+e+E(n)}log\left(1+x^{1+e+E(n)}\right)\Big|_{\frac{1}{n}}^{1}=$$

$$=\log(n)-\frac{\log(2)-log\left(1+\frac{1}{n^{1+e+E(n)}}\right)}{1+e+E(n)}$$

$$\therefore\Omega=\left(\frac{\log(n)}{n}-\frac{\log(2)-log\left(1+\frac{1}{n^{1+e+E(n)}}\right)}{1+e+E(n)}\right)=0$$

## Solution 2 by Ali Jaffal-Lebanon

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\int_{\frac{1}{n}}^{1}\frac{dx}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}\right)}$$

***We have***:

$$1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}\geq n,\forall x\geq0\ and\ n\geq2$$

$$So:\ \frac{1}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}}\right)}\leq\frac{1}{x},for\ x>0$$

$$0 \le I_n \le \int_{\frac{1}{n}}^{1} \frac{dx}{x} = \frac{1}{n}\log(n) \to 0 \Rightarrow \lim_{n\to\infty} I_n = 0$$

**Solution 3 by proposer**

$$x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdot\ldots\cdot x\sqrt[n]{x}}}} = x^{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}} = x^{E_n}, undo\ E_n := \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \to e-1$$

$$I = \int_{\frac{1}{n}}^{1} \frac{dx}{x(1+x^{E_n})} = \int_{\frac{1}{n}}^{1}\left(\frac{1}{x} - \frac{x^{E_n-1}}{1+x^{E_n}}\right)dx = \left(logx - \frac{log(1+x^{E_n})}{E_n}\right)\Big|_{1/n}^{1} = \frac{log\left(1+\left(\frac{1}{n}\right)^{E_n}\right)}{E_n} + log\left(\frac{1}{n}\right) - \frac{log2}{E_n}$$

$$\Omega = \lim_{n\to\infty} \frac{\frac{log\left(1+\left(\frac{1}{n}\right)^{E_n}\right)}{E_n} + log\left(\frac{1}{n}\right) - \frac{log2}{E_n}}{n} = \lim_{n\to\infty} \frac{E_n}{n^{E_n-1}}\, log\left(1+\frac{1}{n^{E_n}}\right)^{n^{E_n}} = 0$$

**971. Find:**

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\int_e^n \frac{x^{n-1}e^x(x\log x + n\log x - n)}{\log^{n+1} x}dx}$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Kamel Benaicha-Algiers-Algerie*

$$\Omega = \lim_{n\to+\infty} \sqrt[n]{\int_e^n \frac{x^{n-1}e^x(x\ln(x) + n\ln(x) - n)}{\ln^{n+1}(x)}dx}$$

*Put* $I(n) = \int_e^n \frac{x^{n-1}e^x(x\ln(x)+n\ln(x)-n)}{\ln^{n+1}(x)}dx$

$$= \int_e^n \frac{x^n e^x + nx^{n-1}e^x}{\ln^n(x)}dx - n\int_e^n \frac{x^n e^x}{x\ln^{n+1}(x)}dx$$

$$n\int_e^n \frac{x^n e^x}{x\ln^n(x)}dx \overset{IBP}{=} -\left(\frac{n^n e^n}{\ln^n(n)} - e^{n+e}\right) + \int_e^n \frac{x^n e^x + nx^{n-1}e^x}{\ln^n(x)}dx$$

$$\therefore I(n) = \left(\frac{ne}{\ln(n)}\right)^n - e^{n+e}$$

$$\therefore \Omega = \lim_{n\to+\infty}\left(I(n)\right)^{\frac{1}{n}} = \lim_{n\to+\infty}\frac{ne}{\ln(n)}\left(1 - e^e\left(\frac{\ln(n)}{n}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to+\infty}\left(\frac{ne}{\ln(n)}\right)e^{\frac{1}{n}\ln\left(1-e^e\left(\frac{\ln(n)}{n}\right)^n\right)}$$

$$\lim_{n\to+\infty}\left(\frac{\ln(n)}{n}\right)^n = \lim_{n\to+\infty} e^{n\ln\left(\frac{\ln(n)}{n}\right)} = 0\left(\lim_{n\to+\infty}\left(\frac{\ln(n)}{n}\right) = 0_+\right)$$

$$\therefore \Omega = \lim_{n\to+\infty}\left(\frac{ne}{\ln(n)}\right)e^{-\frac{e^e}{n}\left(\frac{\ln(n)}{n}\right)^n} = +\infty\left(\lim_{n\to+\infty}\frac{n}{\ln(n)} = +\infty, \lim_{n\to+\infty}\frac{e^e}{n}\left(\frac{\ln(n)}{n}\right)^n = 0\right)$$

$$\therefore \lim_{n\to+\infty}\sqrt[n]{\int_e^n \frac{x^{n-1}e^x(x\ln(x) + n\ln(x) - n)}{\ln^{n+1}(x)}dx} = +\infty$$

**972. Find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\sqrt{\frac{1}{n}\sum_{k=0}^{n} H_k H_{n+k}}\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Ali Jaffal-Lebanon***

$$\Omega_n = \frac{1}{n^3}\sum_{k=1}^{n} H_k H_{n+k}$$

***We know that*** $(H_n)_{n\geq 1}$ ***is increasing for***

***So,*** $H_k \leq H_n$ ***and*** $H_{n+k} \leq H_{2n}$ **for all** $1 \leq k \leq n$ ***then:***

$$\Omega_n \leq \frac{1}{n^3}\sum_{k=1}^{n} H_k H_{2n} \leq \frac{1}{n^2}H_n H_{2n}$$

***But*** $H_n \le H_{2n}$ then $0 < \Omega \le \frac{(H_{2n})^2}{n^2}$

***We have:*** $H_{2n} = \gamma + \log(2n) + \zeta(2n)$

$$Where: \lim_{n\to\infty} \zeta(2n) = 0$$

$$\Rightarrow \frac{H_{2n}}{n} = \frac{\gamma}{n} + \frac{\log(2n)}{n} + \frac{\zeta(2n)}{n} \Rightarrow \lim_{n\to\infty} \frac{H_{2n}}{n} = 0 \text{ since } \lim_{n\to\infty} \frac{\log(2n)}{n} = 0$$

$$\text{So, } 0 \le \lim_{n\to\infty} \Omega_n \le 0 \Rightarrow \lim_{n\to\infty} \Omega_n = 0. \text{ Then: } \lim_{n\to\infty} \sqrt{\Omega_n} = 0 \text{ and } \Omega = 0$$

## Solution 2 by Naren Bhandari-Bajura-Nepal

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{k=1}^{n} H_k H_{n+k}\right)^{\frac{1}{2}} = \lim_{n\to\infty} \left(\underbrace{\frac{\sum_{k=1}^{n+1} H_k H_{n+k} - \sum_{k=1}^{n} H_k H_{n+k}}{(n+1)^3 - n^3}}_{Stolz-Cesaro\ theorem}\right)^{\frac{1}{2}}$$

$$= \left(\lim_{n\to\infty} \frac{H_{n+1} H_{2n+1}}{(n+1)^3 - n^3}\right)^{\frac{1}{2}}$$

$$\sim \left(\lim_{n\to\infty} \frac{(\log(n+1) + \gamma)(\log(2n+1) + \gamma)}{(n+1)^2 + n(n+1) + n^2}\right)^{\frac{1}{2}} = 0$$

## Solution 3 by Sergio Esteban-Argentina

$$f(1) + f(2) + \cdots + f(n-1) \le \int_1^n f(x)dx \le f(2) + f(3) + \cdots + f(n)$$

$$\Rightarrow \frac{\sum_{k=1}^n f(k)}{n} - \frac{f(n)}{n} \le \frac{\int_1^n f(x)dx}{n} \le \frac{\sum_{k=1}^n f(k)}{n} - \frac{f(1)}{n}$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n f(k)}{n} = \lim_{n\to\infty} \frac{\int_1^n f(x)dx}{n}$$

***Similar for decreasing functions. Now, in the problem***

$$\Omega = \lim_{n\to\infty} \sqrt{\frac{1}{n^3} \sum_{k=0}^{n} H_k H_{n+k}} = \lim_{n\to\infty} \sqrt{\frac{1}{n} \sum_{k=0}^{n} \frac{H_k}{n} \cdot \frac{H_{n+k}}{n}}$$

$$i)\ We\ calculate\ \frac{H_k}{n}$$

$$\frac{\sum_{i=1}^{k}\frac{1}{i}}{n}\underbrace{\approx}_{n\to\infty}\frac{\int_1^n\frac{1}{x}dx}{n}=\frac{\log(k)}{n}$$

$$ii)\ \textit{We calculate } \frac{H_{n+k}}{n}$$

$$\frac{\sum_{i=1}^{k}\frac{1}{i+n}}{n}\underbrace{\approx}_{n\to\infty}\frac{\int_1^n\frac{1}{n+x}dx}{n}=\frac{\log(k+n)}{n}-\frac{\log(n+1)}{n}$$

$$\Omega=\lim_{n\to\infty}\sqrt{\frac{1}{n}\left[\sum_{k=0}^{n}\frac{\log(k)\log(k+n)}{n^2}-\sum_{k=0}^{n}\frac{\log(k)\log(n+1)}{n^2}\right]}$$

$$=\lim_{n\to\infty}\sqrt{\frac{1}{n^3}\sum_{k=0}^{n}\log(k)\log(k+n)-\frac{\log(n+1)}{n^3}\sum_{k=0}^{n}\log(k)}$$

$$iii)\ \textit{We calculate } \sum_{k=0}^{n}\frac{\log(k)\log(k+n)}{n^3}\underbrace{\approx}_{n\to\infty}\frac{\int_0^n\log(x)\log(x+n)}{n^3}$$

$$\textit{Notice that}: I=\int\log(x)\log(x+n)\,dx$$

***By integration by parts:***

$$I=\log(x)\big((x+n)\log(x+n)-x-n\big)-\int\frac{(x+n)\log(x+n)}{x}dx+n\int\frac{1}{x}dx+\int 1dx$$

$$I=\log(x)\big((x+n)\log(x+n)-x-n\big)-n\underbrace{\int\frac{\log(x+n)}{x}dx}_{I_1}-\int\log(x+n)\,dx+n\int\frac{1}{x}dx+\int 1dx$$

$$I_1=\int\frac{\log(x+n)}{x}dx=\int\frac{\log(\frac{x}{n}+1)}{x}dx+\log(n)\int\frac{1}{x}dx$$

$$\textit{Let}: u=-\frac{x}{n}\to du=-\frac{1}{n}dx$$

$$I_1=-\int\frac{-\log(1-u)}{u}du+\log(u)\int\frac{1}{x}dx=-\mathrm{Li}_2\left(-\frac{x}{n}\right)+\log(n)\log(x)$$

***Replacing and simplifying***

$$I=(x+n)(\log(x-1))(\log(x+n)+n\mathrm{Li}_2\left(-\frac{x}{n}\right)-\big(x+n\log(n)\big)\log(x)+2x+\zeta.\textit{Then}$$

$$\int_0^n \log(x)\log(x+n)\,dx = \frac{n}{12}\left[12\log^2(n) + (24\log(2) - 24)\log(n) - 24\log(2) - \pi^2 + 24\right]$$

*Now, it's easy to see that:* $\frac{\int_0^1 \log(x)\log(x+n)}{n^2}dx \underbrace{\to}_{n\to 0} 0$

*iv) We calculate:* $\frac{\log(n+1)\int_0^n \log(x)dx}{n^3} \underbrace{\approx}_{n\to\infty} \frac{\log(n+1)\sum_{k=0}^n \log(k)}{n^3}$

*Then* $\frac{\log(n+1)(\log(n)-1)n}{n^3} \underbrace{\to}_{n\to\infty} 0$. *By iii)+iv)* $\Rightarrow \Omega = 0$

**973. Prove that:**

$$2\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(mk^n + nk^m)} = \zeta(2) + 2\zeta(3) + \zeta(4)$$

*Where ζ(.) denotes Riemann Zeta function.*

*Proposed by Naren Bhandari-Bajura-Nepal*

*Solution by Khalef Ruhemi Jarash-Jordan*

$$I \coloneqq \sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(mk^n + nk^m)} \dots\dots (*)$$

$$Let\ I_k \coloneqq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(mk^n + nk^m)};\ make\ intercharging\ m \leftrightarrow n$$

$$I_k = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm^2}{k^m(mk^n + nk^m)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn^2}{k^n(mk^n + nk^m)}$$

$$2I_k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mk^n + nk^m)}\left(\frac{nm^2}{k^m} + \frac{mn^2}{k^n}\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{nm}{mk^n + nk^m}\frac{mk^n + nk^m}{k^n k^m} =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{nm}{k^{m+n}} = \sum_{m=1}^{\infty}\frac{m}{k^m}\sum_{n=1}^{\infty}\frac{n}{k^n} = \left(\sum_{k=1}^{\infty}\frac{n}{k^n}\right)^2 \to 2I_k = \left(\sum_{k=1}^{\infty}\frac{n}{k^n}\right)^2$$

$$But\ \sum_{n=1}^{\infty}x^n = \frac{1}{1-x} - 1 \to \sum_{n=1}^{\infty}nx^{n-1} = \frac{1}{(1-x)^2}, |x| < 1$$

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, |x| < 1$$

$$\sum_{n=1}^{\infty} \frac{n}{k^n} = \frac{\frac{1}{k}}{\left(1-\frac{1}{k}\right)^2} = \frac{k}{(k-1)^2} = \frac{k-1+1}{(k-1)^2} = \frac{1}{k-1} + \frac{1}{(k-1)^2}$$

$$\left(\sum_{n=1}^{\infty} \frac{n}{k^n}\right)^2 = \frac{1}{(k-1)^2} + \frac{1}{(k-1)^4} + \frac{2}{(k-1)^3} = 2I_k$$

$$2I = \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)^2} + \frac{1}{(k-1)^4} + \frac{2}{(k-1)^3}\right)$$

$$2I = \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{2}{k^3} + \frac{1}{k^4}\right) = \zeta(2) + 2\zeta(3) + \zeta(4)$$

**974. Find:**

$$\Omega = \lim_{n\to\infty} \left(n\left(\left(\left(1+\frac{1}{n}\right)^n - e - 1\right)^n - e^{-\frac{e}{2}}\right)\right)$$

***Proposed by Rahim Shahbazov-Baku-Azerbaijan***

***Solution by Kamel Benaicha-Algiers-Algerie***

$$Put: t = \frac{1}{n} \therefore \Omega = \lim_{t\to 0} \frac{e^{\frac{1}{t}\left(log\left(1+(1+t)^{\frac{1}{t}}-e\right)\right)} - e^{-\frac{e}{2}}}{t}$$

***We have:*** $(1+t)^{\frac{1}{t}} = e^{\frac{1}{t}\log(1+t)} = e^{\frac{1}{t}\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)+o(t^3)}$

$$= e^{\left(1-\frac{t}{2}+\frac{t^2}{3}\right)+o(t^3)} = e^{\left(1-\frac{t}{2}+\frac{t^2}{3}+\frac{t^2}{8}\right)+o(t^2)} = e^{\left(1-\frac{t}{2}+\frac{11}{24}t^2\right)+o(t^2)}$$

$$\therefore log\left(1+(1+t)^{\frac{1}{t}}-e\right) \sim \left(\frac{11}{24}t^2 - \frac{t}{2}\right)e - \frac{t^2}{8}e^2 + o(t^2)$$

$$\sim \left(\frac{11}{24}e - \frac{e^2}{8}\right)t^2 - \frac{e}{2}t + o(t^2)$$

***So:*** $e^{\frac{1}{t}\left(log\left(1+(1+t)^{\frac{1}{t}}-e\right)\right)} \sim e^{-\frac{e}{2}}\left(\left(\frac{11}{24}e - \frac{e^2}{8}\right)t + 1\right) + o(t)$

$$\Omega = \lim_{t\to 0}\frac{e^{-\frac{e}{2}}+\frac{e^{1-\frac{e}{2}}}{11}(11-3e)t-e^{-\frac{e}{2}}}{t}=\frac{11-3e}{24}e^{1-\frac{e}{2}}$$

$$\Omega = \lim_{n\to\infty}\left(n\left(\left(\left(1+\frac{1}{n}\right)^n-e-1\right)^n-e^{-\frac{e}{2}}\right)\right)=\frac{11-3e}{24}e^{1-\frac{e}{2}}$$

**975. Find:**

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{2}H_n+\log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)\right)$$

***Proposed By Daniel Sitaru-Romania***

***Solution 1 by Ali Jaffal-Lebanon***

$$Let\ u_n = n\ and\ v_n = \frac{1}{2}H_n+log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)$$

***We know that:***

$$v_{n+1}-v_n = \frac{1}{2}H_{n+1}+log\left(\prod_{k=1}^{n+1}\frac{2k}{2k-1}\right)-\frac{1}{2}H_n-log\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)=$$

$$=\frac{1}{2(n+1)}+log\left(\frac{2(n+1)}{2n}\right)=\frac{1}{2(n+1)}+log\left(1+\frac{1}{n}\right)$$

$$Then:\ \lim_{n\to\infty}\frac{v_{n+1}-v_n}{u_{n+1}-u_n}=\lim_{n\to\infty}\left(\frac{1}{2(n+1)}+log\left(1+\frac{1}{n}\right)\right)$$

***So, by Cesaro-Stolz we have:***

$$\lim_{n\to\infty}\frac{v_n}{u_n}=0\ \ then\ \Omega=0.$$

***Solution 2 by Ali Jaffal-Lebanon***

***By Stiriling's approximation we have***

$$n!\sim\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n$$

$$v_n=\prod_{k=1}^{n}\frac{2k}{2k-1}=\frac{2}{1}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{2n}{2n-1}=\frac{(2^n\cdot n!)^2}{(2n)!}=\frac{4^n\cdot(n!)^2}{(2n)!}$$

***We have :*** $4^n \cdot (n!)^2 \sim 2\pi n \cdot \left(\frac{n}{e}\right)^{2n} \cdot 4^n$

$$(2n)! \sim \sqrt{4\pi n} \cdot \left(\frac{2n}{e}\right)^{2n}$$

***Then*** $\frac{4^n(n!)^2}{(2n)!} \sim \sqrt{\pi n}$. ***Then***

$$v_n = \sqrt{\pi n} \cdot (1 + \varphi(n)) \text{ where } \lim_{n\to\infty} \varphi(n) = 0$$

***So,*** $\log v_n = \frac{1}{2}\log(\pi n) + \log(1 + \varphi(n))$

$$\frac{1}{n}\log v_n = \frac{1}{2n}\log(\pi n) + \frac{1}{n}\log(1 + \varphi(n))$$

***We know that:***

$$\lim_{n\to\infty} \frac{\log(\pi n)}{n} = 0$$

$$\text{and } \frac{1}{n}\log(1 + \varphi(n)) \underset{n\to\infty}{\sim} \frac{\varphi(n)}{n}$$

***Then***

$$\lim_{n\to\infty} \frac{1}{n}\log(1 + \varphi(n)) = 0$$

***So,***

$$\lim_{n\to\infty} \frac{1}{n}\log v_n = 0$$

***But***

$$\lim_{n\to\infty} \frac{H_n}{2n} = \lim_{n\to\infty} \frac{\log(n) + \gamma + \Psi(n)}{n} = 0, \text{ since } \lim_{n\to\infty} \Psi(n) = 0$$

$$\text{Therefore } \Omega = 0$$

## ***Solution 3 by Ali Jaffal-Lebanon***

***We have***

$$\log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) = \log\left(\prod_{k=1}^{n}\left(1 + \frac{1}{2k-1}\right)\right)$$

$$= \sum_{k=1}^{n} \log\left(1 + \frac{1}{2k-1}\right) < \sum_{k=1}^{n} \frac{1}{2k-1}$$

*Since* $0 < \log(1+x) < x$, *for all* $x > 0$ *but* $\frac{1}{2k-1} \le \frac{1}{2k-2}$ *for* $k \ge 2$. **Then**

$$\sum_{k=1}^{n} \frac{1}{2k-1} \le 1 + \sum_{k=2}^{n} \frac{1}{2k-1} \le 1 + \sum_{k=2}^{n} \frac{1}{2(k-1)} \le 1 + \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k} \le 1 + \frac{1}{2}H_{n-1}$$

$$\text{Let } \Omega_n = \frac{1}{n}\left(\frac{1}{2}H_n + \log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)\right)$$

**We have** $0 < \Omega_n < \frac{1}{n}\left(\frac{1}{2}H_n + 1 + \frac{1}{2}H_{n-1}\right)$. **We know**

$$H_n = \gamma + \log(n) + \varphi(n), \text{ where } \lim_{n\to\infty} \varphi(n) = 0$$

**So,** $0 < \Omega_n \le \frac{1}{n}\left(\gamma + \frac{1}{2}\log(n) + \frac{1}{2}\log(n-1) + 1 + \Psi(n)\right)$

*where* $\Psi(n) = \varphi(n) + \varphi(n-1)$. **By sandwich theorem we obtain**

$$0 \le \lim_{n\to\infty} \Omega_n \le 0. \text{ Then: } \lim_{n\to\infty} \Omega_n = 0$$

## Solution 4 by Naren Bhandari-Bajura-Nepal

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{2}H_n + \log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)\right)$$

$$\sim \lim_{n\to\infty} \frac{\log(n)+\gamma}{2n} + \lim_{n\to\infty} \frac{1}{n}\log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{(2n)!!}{(2n-1)!!}\right) = \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{4^n(n!)^2}{(2n)!}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\log\left(4^n \underbrace{\binom{2n}{n}^{-1}}_{\text{central binomial coeff.}}\right) \sim \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{4^n\sqrt{\pi n}}{4^n}\right) = \lim_{n\to\infty} \frac{\log(\pi n)}{2n} = 0$$

## Solution 5 by Sergio Esteban-Argentina

$$\Omega = \lim_{n\to\infty}\left(\frac{H_n}{2n} + \frac{1}{n}\log\left[\prod_{k=1}^{n} \frac{2k}{2k-1}\right]\right) = \lim_{n\to\infty}\left(\frac{H_n}{2n} + \frac{1}{n}\sum_{k=1}^{n} \log\left(1 + \frac{1}{2k-1}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\int_1^n \frac{1}{x}dx + \frac{1}{n}\int_1^n \log\left(1 + \frac{1}{2k-1}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{\log(x)|_1^n}{n} + \frac{1}{n}\left[\frac{\log(2x-1)}{2}\Big|_1^n + x\log\left(\frac{1}{2x-1}+1\right)\Big|_1^n\right]\right)$$

$$= \lim_{n\to\infty}\left(\frac{\log(n)}{n} + \frac{1}{n}\left[\frac{\log(2n-1)}{2} + n\log\left(\frac{1}{2n-1}+1\right) - \log 2\right]\right) = 0$$

**976.** **If $f: R \to R$ is continuous function with: $f(x^2) + f(x) = x^2 + x, \forall x \in R$.**

$$\text{Find: } \Omega = \lim_{\substack{x\to 0\\ x>0}} \frac{f^{2n}(\tan x) - f^{2n}(x)}{x^{2n+1}}, n \in \mathbb{N}$$

*Proposed by Florică Anastase-Romania*

*Solution by Marian Ursărescu-Romania*

*Let function* $g: R \to R, g(x) = f(x) - x$ *continuous*.

$$f(x^2) + f(x) = x^2 + x, \forall x \in R \leftrightarrow g(x^2) = -g(x), \forall x \in \mathbb{N}$$

$$g(-x) = g(x), \forall x \in \mathbb{N} \text{ and } g(0) = g(1) = 0$$

$$g(x) = -g(\sqrt{x}) = g(\sqrt[4]{x}) = \cdots = (-1)^n g(\sqrt[2n]{x}), \forall x \in (0,\infty) \to$$

$$|g(x)| = |g(\sqrt[2n]{x})|,, \forall x \in (0,\infty), n \in \mathbb{N}, n \ge 1 \text{ and } f \text{ continuous} \to$$

$$|g(x)| = \lim_{n\to\infty}|g(\sqrt[2n]{x})| = \left|g\left(\lim_{n\to\infty}\sqrt[2n]{x}\right)\right| = |g(0)| = 0, \forall x \in (0,\infty) \to$$

$$g(x) = 0, \forall x \in (0,\infty) \to f(x) = x, \forall x \in (0,\infty)$$

*For x>0 we have:*

$$\lim_{x\to 0}\frac{f^{2n}(tgx) - f^{2n}(x)}{x^{2n+1}} = \lim_{x\to 0}\frac{\tan^{2n}x - x^{2n}}{x^{2n+1}}$$

$$= \lim_{x\to 0}\frac{\tan x - x}{x^3}\cdot\frac{\tan^{2n-1}x + \tan^{2n-2}x + \cdots + x^{2n-1}}{x^{2n-1}} \ldots (1)$$

$$\lim_{\substack{x\to 0\\ x>0}}\left(\left(\frac{\tan x}{x}\right)^{2n-1} + \left(\frac{\tan x}{x}\right)^{2n-1} + \cdots 1\right) = 2n \ldots (2)$$

$$= \lim_{x\to 0}\frac{\tan x - x}{x^3} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1-\cos^2 x}{\cos^2 x}}{3x^2} = \lim_{x\to 0}\frac{\tan^2 x}{3x^2} = \frac{1}{3} \ldots (3)$$

*From (1)+(2)+(3) we have:*

$$\Omega = \lim_{\substack{x\to 0\\ x>0}} \frac{f^{2n}(\tan x) - f^{2n}(x)}{x^{2n+1}} = \frac{2n}{3}$$

**977. Find:**

$$\Omega = \lim_{n\to\infty}\left(n\sum_{k=1}^{n}\frac{k(k+1)}{(k^2+n^2)(k^2+2k+1+n^2)}\right)$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Ravi Prakash-New Delhi-India*

$$Let\ f(x) = \frac{x}{x^2+1}, x \in [0,2]$$

$$Now, let\ a_n = n\sum_{k=1}^{n}\frac{k(k+1)}{(k^2+n^2)(k^2+2k+1+n^2)}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)\cdot\left(\frac{k+1}{n}\right)}{\left(\left(\frac{k}{n}\right)^2+1\right)\left(\left(\frac{k+1}{n}\right)^2+1\right)} = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot f\left(\frac{k+1}{n}\right)$$

*As f is uniforly continuous on* $[0,2]$ *given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that:*

$$|f(x+h) - f(x)| < \varepsilon, \forall x \in [0,2]\ whenever\ |h| < \delta\ and\ x+h \in [0,2]$$

*Choose n sufficiently large so that* $n\delta > 1$,

$$\left|f\left(\frac{k}{n}+\frac{1}{n}\right) - f\left(\frac{k}{n}\right)\right| < \varepsilon, \quad f\left(\frac{k}{n}\right) - \varepsilon < f\left(\frac{k+1}{n}\right) < f\left(\frac{k}{n}\right) + \varepsilon$$

$$\frac{1}{n}\sum_{k=1}^{n}\left(f\left(\frac{k}{n}\right)\right)^2 - \varepsilon\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right) < \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) f\left(\frac{k+1}{n}\right)$$

$$< \frac{1}{n}\sum_{k=1}^{n}\left(f\left(\frac{k}{n}\right)\right)^2 + \varepsilon\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)$$

*Taking limit as* $n \to \infty$, *we get*

$$\int_0^1 f^2(x)dx - \varepsilon\int_0^1 f(x)dx \le \Omega \le \int_0^1 f^2(x)dx + \varepsilon\int_0^1 f(x)dx$$

*Make* $\varepsilon \to 0_+$, *so that*

$$\Omega = \int_0^1 f^2(x)dx = \int_0^1 \frac{x^2}{(x^2+1)^2}dx \overset{x=\tan\theta}{=} \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^4\theta}\cdot\sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta d\theta = \frac{1}{2}\int_0^{\pi/4}(1-\cos 2\theta)d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

**978. Find:**

$$\Omega = \lim_{n\to\infty} \frac{\int_1^n \left(\frac{\tan^{-1}x}{\tan^{-1}x - x}\right)^2 dx}{2\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right) - n}$$

*Proposed by Florică Anastase-Romania*

*Solution 1 by Igor Soposki-Skopje-Macedonia*

$$I = \int\left(\frac{\tan^{-1}x}{\tan^{-1}x - x}\right)^2 dx = \int\left(\frac{\tan^{-1}x - x + x}{\tan^{-1}x - x}\right)^2 dx$$

$$= \int\left(1 - \frac{x}{\tan^{-1}x - x}\right)^2 dx = x + \int \frac{2x\tan^{-1}x - 2x^2 + x^2}{(\tan^{-1}x - x)^2}dx$$

$$= x + \int \frac{2x\tan^{-1}x - x^2}{(\tan^{-1}x - x)^2}dx = x + I_1$$

$$I_1 = \int \frac{2x\tan^{-1}x - x^2}{(\tan^{-1}x - x)^2}dx = (*)$$

$$u = \frac{x^2+1}{\tan^{-1}x - x};\ du = \frac{2x\tan^{-1}x - x^2}{(\tan^{-1}x - x)^2}dx$$

$$(*) = \int du = u = \frac{2x\tan^{-1}x - x^2}{(\tan^{-1}x - x)^2}$$

$$I = x + I_1 = \left(\frac{2x\tan^{-1}x - x^2}{(\tan^{-1}x - x)^2}\right)\Big|_0^n = n + \frac{n^2+1}{\tan^{-1}n - n} + \frac{4+\pi}{4-\pi}$$

$$= A + \frac{n\tan^{-1}n + 1}{\tan^{-1}n - n}$$

$$2\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right) = 2\tan^{-1}\left(\frac{n}{1+\sqrt{n^2+1}}\right) = \tan^{-1}n$$

$$\Omega = \lim_{n\to\infty}\frac{A + \frac{ntan^{-1}n + 1}{tan^{-1}n - n}}{tan^{-1}n - n}$$

$$= \lim_{n\to\infty}\frac{A}{tan^{-1}n - n} + \lim_{n\to\infty}\frac{ntan^{-1}n}{(tan^{-1}n - n)^2} + \lim_{n\to\infty}\frac{1}{(tan^{-1}n - n)^2} = L_1 + L_2 + L_3$$

$$L_1 = \lim_{n\to\infty}\frac{A}{tan^{-1}n - n} = 0$$

$$L_2 = \lim_{n\to\infty}\frac{ntan^{-1}n}{(tan^{-1}n - n)^2} = \lim_{n\to\infty}\frac{\frac{n\pi}{2}}{\left(\frac{\pi}{2} - n\right)^2} =$$

$$= \frac{\pi}{2}\lim_{n\to\infty}\frac{n}{\left(\frac{\pi}{2} - n\right)^2} = \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{2\left(\frac{\pi}{2} - n\right)(-1)} = 0$$

$$L_3 = \lim_{n\to\infty}\frac{1}{(tan^{-1}n - n)^2} = 0$$

So, $\Omega = L_1 + L_2 + L_3 = 0$

## Solution 2 by Kamel Samp Benaicha-Algeirs-Algerie

***Put:***

$$I = \int_1^\infty\left(\frac{tan^{-1}x}{tan^{-1}x - x}\right)^2 dx$$

***Singular point of I is*** $(+\infty)$***. We have:*** $\left(\frac{tan^{-1}x}{tan^{-1}x - x}\right)^2 \underbrace{\sim}_{n\to\infty} \frac{4}{\pi^2 x^2}$

$$\int_1^\infty \frac{dx}{x^2} = 1\ converge$$

***So I is convergent*** $\therefore I = C / C \in \mathbb{R}_+$

$$\lim_{n\to\infty}\left(2tan^{-1}\left(\frac{\sqrt{1+n^2}}{n}\right) - n\right) = -\infty;\ \therefore \Omega = 0$$

$$\Omega = \lim_{n\to\infty}\frac{\int_1^n\left(\frac{tan^{-1}x}{tan^{-1}x - x}\right)^2 dx}{2tan^{-1}\left(\frac{\sqrt{1+n^2} - 1}{n}\right) - n} = 0$$

**Solution 3 by Ali Jaffal-Lebanon**

*Let:*

$$I_n=\int_1^n\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2dx$$

$$I_n=\underbrace{\int_1^3\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2dx}_{A}+\underbrace{\int_3^n\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2dx}_{J_n}$$

$$J_n=\int_3^n\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2dx$$

*If* $3\le x\le n$ *then* $\frac{\pi}{4}\le\tan^{-1}x\le\frac{\pi}{2}$

*So,* $\left(\frac{\frac{\pi}{4}}{x-\frac{\pi}{4}}\right)^2\le\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2\le\left(\frac{\frac{\pi}{2}}{x-\frac{\pi}{2}}\right)^2\le\frac{\frac{\pi^2}{4}}{(x-2)^2}$

$$\int_3^n\frac{dx}{(x-2)^2}=\frac{-1}{x-2}\Big|_3^n+1\le 1$$

*So,* $0\le J_n\le\frac{\pi}{4}$*. Then:* $0\le I_n\le\frac{\pi^2}{4}+A$*. We know that:*

$$\lim_{n\to\infty}\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)=\lim_{n\to\infty}\tan^{-1}\left(\sqrt{1+\frac{1}{n^2}}-\frac{1}{n}\right)=\frac{\pi}{4}$$

*But:* $0\le\left|\frac{I_n}{\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n}\right|\le\frac{\frac{\pi^2}{4}+A}{\left|\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n\right|}\xrightarrow[n\to\infty]{}0$*. Therefore*

$$\Omega=\lim_{n\to\infty}\frac{\int_1^n\left(\frac{\tan^{-1}x}{\tan^{-1}x-x}\right)^2dx}{2\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n}=0$$

**Solution 4 by Sergio Esteban-Argentina**

*Observe that:* $2\tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)\xrightarrow{n\to\infty}\frac{\pi}{2}$*. Then:*

$$\Omega = \lim_{n\to\infty}\left[\frac{\sum_{k=1}^{n}\left(\frac{tan^{-1}k}{tan^{-1}k-k}\right)^2}{n-\frac{\pi}{2}}\right]$$

**Let:**

$$a_n = \sum_{k=1}^{n}\left(\frac{tan^{-1}k}{tan^{-1}k-k}\right)^2 \text{ and } b_n = n-\frac{\pi}{2}$$

**By Stolz Cesaro**

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\left(\frac{tan^{-1}(n+1)}{tan^{-1}(n+1)-(n+1)}\right)^2\cdot\frac{1}{n+1-\frac{\pi}{2}+n-\frac{\pi}{2}} = 0$$

$$\Omega = \lim_{n\to\infty}\frac{\int_1^n\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx}{2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n} = 0$$

## Solution 5 by proposer

$$F(x) = \int\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx = \int\left(1+\frac{x}{tan^{-1}x-x}\right)^2dx$$

$$= \int dx + 2\int\frac{x}{tan^{-1}x-x}dx + \int\left(\frac{x}{tan^{-1}x-x}\right)^2dt = (*)$$

$$\left(\frac{x}{tan^{-1}x-x}\right)' = \left(\frac{x}{tan^{-1}x-x}\right)^2\cdot\frac{1}{1+x^2}\to$$

$$\int\left(\frac{x}{tan^{-1}x-x}\right)^2dt = \int\left(\frac{x}{tan^{-1}x-x}\right)'(1+x^2)dx = \frac{1+x^2}{tan^{-1}x-x}-2\int\frac{x}{tan^{-1}x-x}dx$$

$$(*) = x+\frac{1+x^2}{tan^{-1}x-x}+C$$

$$2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n \overset{n=tany}{\cong} 2tan^{-1}\left(\frac{\sqrt{1+tan^2y}-1}{tany}\right)-tany =$$

$$= 2tan^{-1}\left(\frac{1-cosy}{siny}\right)-tany = 2tan^{-1}\left(\frac{2sin^2\frac{y}{2}}{2sin\frac{y}{2}cos\frac{y}{2}}\right)-tany =$$

$$= 2tan^{-1}\left(tan\frac{y}{2}\right)-tany = y-tany = tan^{-1}n-n$$

$$\Omega=\lim_{n\to\infty}\frac{\int_1^n\left(\frac{tan^{-1}x}{tan^{-1}x-x}\right)^2dx}{2tan^{-1}\left(\frac{\sqrt{1+n^2}-1}{n}\right)-n}=\lim_{n\to\infty}\frac{\frac{1+ntan^{-1}n}{tan^{-1}n-n}-\frac{\pi+4}{\pi-4}}{tan^{-1}n-n}$$

$$=\lim_{n\to\infty}\frac{(\pi-4)(1+ntan^{-1}n)-(\pi+4)(tan^{-1}n-n)}{(\pi-4)(tan^{-1}n-n)^2}$$

$$=\lim_{n\to\infty}\frac{(\pi-4)\left(tan^{-1}n+\frac{1}{n}\right)-(\pi+4)(\frac{tan^{-1}n}{n}-1)}{(\pi-4)(\frac{(tan^{-1}n)^2}{n}-2tan^{-1}n+n}=0$$

**979.** $(a_n)_{n\geq1},(b_n)_{n\geq1},(c_n)_{n\geq1}$ **are sequences of real numbers such that:**

$$a_n=\sum_{k=1}^n\binom{n}{k}\cdot k^{\frac{1}{k}};\ b_n=\sum_{k=1}^n\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}};$$

$$c_n=\sum_{k=1}^n\frac{1}{2^k}\left(2\cos\frac{\pi}{2(k+1)}-\sin\frac{\pi(k+1)}{2(k+2)}\right)$$

**Find:**

$$\Omega=\lim_{n\to\infty}(a_n\cdot b_n\cdot c_n^{6n})$$

***Proposed by Florică Anastase-Romania***

***Solution1 by Adrian Popa-Romania***

$$c_n=\sum_{k=1}^n\frac{1}{2^k}\left(2cos\frac{\pi}{2(k+1)}-sin\frac{\pi(k+1)}{2(k+2)}\right)$$

$$=\sum_{k=1}^n\frac{1}{2^k}\left(2sin\left(\frac{\pi}{2}-\frac{\pi}{2(k+1)}\right)-sin\frac{\pi(k+1)}{2(k+2)}\right)$$

$$=\frac{1}{2}\cdot2sin\frac{\pi}{2\cdot2}-\frac{1}{2}sin\frac{2\pi}{2\cdot3}+\frac{1}{2^2}\cdot2sin\frac{2\pi}{2\cdot3}-\frac{1}{2^2}\cdot sin\frac{3\pi}{2\cdot4}+\frac{1}{2^3}\cdot2sin\frac{3\pi}{2\cdot4}+\cdots$$

$$+\frac{1}{2^n}\cdot2sin\frac{n\pi}{2(n+1)}-\frac{1}{2^n}\cdot sin\frac{(n+1)\pi}{2(n+2)}=\frac{\sqrt{2}}{2}-\frac{1}{2^n}\cdot sin\frac{(n+1)\pi}{2(n+2)}$$

$$a_n \cdot b_n \cdot c_n^{6n} = \sum_{k=1}^{n}\binom{n}{k}\cdot k^{\frac{1}{k}} \cdot \sum_{k=1}^{n}\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}}\cdot\left(\frac{\sqrt{2}}{2}-\frac{1}{2^n}\cdot sin\frac{(n+1)\pi}{2(n+2)}\right)^{6n}$$

$$\frac{(n+1)\pi}{2(n+2)} < \frac{\pi}{2} \Rightarrow 0 < \underbrace{sin\frac{(n+1)\pi}{2(n+2)}}_{\to 0} < 1$$

$$a_n \cdot b_n \cdot c_n^{6n} < \sum_{k=1}^{n}\binom{n}{k}\cdot k^{\frac{1}{k}} \cdot \sum_{k=1}^{n}\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}}\cdot\left(\frac{\sqrt{2}}{2}\right)^{6n} = \frac{\sum_{k=1}^{n}\binom{n}{k}\cdot k^{\frac{1}{k}}\cdot\sum_{k=1}^{n}\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}}}{2^{3n}}$$

$$a_n = \sum_{k=1}^{n}\binom{n}{k}\cdot k^{\frac{1}{k}} < \sqrt[e]{e}\sum_{k=1}^{n}\binom{n}{k} = \sqrt[e]{e}\cdot(2^n-1)$$

$$b_n = \sum_{k=1}^{n}\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}} < \sum_{k=1}^{n}\binom{n}{k} = 2^n-1$$

$$0 < a_n \cdot b_n \cdot c_n^{6n} < \frac{\sqrt[e]{e}\cdot(2^n-1)^2}{2^{3n}} \to 0$$

***So:***

$$\Omega = \lim_{n\to\infty}(a_n \cdot b_n \cdot c_n^{6n}) = 0$$

## *Solution 2 by Naren Bhandari-Bajura-Nepal*

***We note that:*** $sin\frac{\pi(k+1)}{2(k+2)} = cos\left(\frac{\pi}{2}\left(1-\frac{k+1}{k+2}\right)\right) = cos\frac{\pi}{2(k+2)}$ ***and hence we have***

$c_n = \sum_{k=1}^{n}\frac{1}{2^k}\left(2cos\frac{\pi}{2(k+1)} - cos\frac{\pi}{2(k+2)}\right)$ ***writing the*** $c_n$ ***is series from we encounter with telescoping series giving partial sum as*** $c_n = cos\frac{\pi}{4} - \frac{1}{2^n}cos\frac{\pi}{2(n+2)}$ ***(1)***

***For all*** $k \geq 1$; $k+1 \geq 2$ ***and hence*** $\frac{1}{k+1} \leq \frac{1}{2} < 1$ ***which follows***

$\sqrt[k+1]{k+1} \leq 1 + \frac{k+1}{k+1} = 2$ ***by Bernoulli inequality and hence***

$\sum_{k=1}^{n}\binom{n}{k} \leq a_n \leq 2\sum_{k=1}^{n}\binom{n}{k}$ ***which now is deduced to***

$2^n - 1 \leq a_n \leq 2(2^n-1)$ ***as*** $n \to \infty$ ***by squeeze theorem we deduced that:***

$$\lim_{n\to\infty} a_n = \infty$$

***Here*** $k \leq n \Rightarrow k^k \leq n^n$ ***also*** $k^k \geq k+1$ ***for all*** $k \geq 2$ ***and we have***

$$\frac{1}{n^n}\sum_{k=1}^{n}\binom{n}{k} < b_n < \sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1} = \sum_{k=1}^{n}\binom{n}{k}\int_0^1 x^k dx = \int_0^1 (1+x)^n dx = \int_1^2 t^n dt =$$

$\frac{1}{n+1}(2^{n+1}-1)$ ***show*** $\lim_{n\to\infty} b_n \le \infty$ ***and to evaluate*** $\lim_{n\to\infty}(a_n\cdot b_n\cdot c_n^{6n})$

***Here,*** $\lim_{n\to\infty}(c_n^{6n}) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}-\frac{1}{2^n}\cos\frac{\pi}{2(n+2)}\right)^{6n}$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}-\frac{1}{2^n}\cos\frac{\pi}{2(n+2)}\right)^{6n} \le \lim_{n\to\infty}(c_n^{6n}) \le \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}-\frac{1}{2^n}\right)^{6n} \Rightarrow$$

$$\lim_{n\to\infty}\frac{1}{2^{3n}} \le \lim_{n\to\infty}(c_n^{6n}) \le \lim_{n\to\infty}\frac{1}{2^{6n}}\left(1-\frac{1}{2^{n-1}}\right)^{6n}$$

$0 \le \lim_{n\to\infty}(c_n^{6n}) \le \lim_{n\to\infty}\frac{1}{2^{6n}\cdot\sqrt[2^{n-1}]{e^{6n}}} = 0$ ***and thus by squeeze theorem***

$$\lim_{n\to\infty}(c_n^{6n}) = 0$$

***So:*** $\Omega = \lim_{n\to\infty}(a_n\cdot b_n\cdot c_n^{6n}) = 0$

## *Solution 3 by proposer*

$\therefore$ *If* $x_k \in [a,b], \forall t_k \in \mathbb{R}_+, \forall k = \overline{1,n}$ *then*:

$$\left(\sum_{k=1}^{n} t_k x_k\right)\left(\sum_{k=1}^{n}\frac{t_k}{x_k}\right) \le \frac{(a+b)^2}{4ab}\left(\sum_{k=1}^{n} t_k\right)^2 \ldots (\text{Kantorovici inequality})$$

$$1; \sqrt{2} \le \sqrt[3]{3};\ 1 < \sqrt[k]{k} \overset{induction}{\Rightarrow} \sqrt[n]{n} \le \sqrt[3]{3}, \forall n \ge 3$$

$$a = 1; b = \sqrt[3]{3};\ x_k = \sqrt[k]{k};\ t_k = \binom{n}{k} \Rightarrow$$

$$0 \le a_n\cdot b_n \le 2^{2n}\frac{(1+\sqrt[3]{3})^2}{4\sqrt[3]{3}} \ldots (1)$$

$$c_n = \sum_{k=1}^{n}\frac{1}{2^k}\left(2\cos\frac{\pi}{2(k+1)} - \sin\frac{\pi(k+1)}{2(k+2)}\right) \overset{u_k=\frac{\pi}{2(k+1)}}{=} \sum_{k=1}^{n}\frac{1}{2^k}(2\cos(u_k) - \cos(u_{k+1}))$$

$$= \sum_{k=1}^{n}\left(\frac{1}{2^{k-1}}\cos(u_k) - \frac{1}{2^k}\cos(u_{k+1})\right) = \cos(u_1) - \frac{1}{2^n}\cos(u_{n+1})$$

$$= \cos\frac{\pi}{4} - \underbrace{\frac{1}{2^n}\cos(u_{n+1})}_{\to 0} \to \frac{1}{\sqrt{2}}$$

$$0 \le a_n \cdot b_n \cdot c_n^{6n} \le 2^{2n} \cdot \frac{(1+\sqrt[3]{3})^2}{4\sqrt[3]{3}} \cdot \left(\frac{1}{\sqrt{2}}\right)^{6n} \to 0$$

*So:*

$$\Omega = \lim_{n\to\infty}(a_n \cdot b_n \cdot c_n^{6n}) = 0$$

**980.**

$$\text{For } \omega_n = \left(\left(\prod_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1}\right)\left(\sum_{k=0}^{2n} \cot\left(x + \frac{k\pi}{2n+1}\right)\right)\right)^{-1}, \text{ find:}$$

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{4n+2}}\left(\frac{\tan x}{\tan\left(\frac{\pi}{4n+2}\right)}\right)^{\omega_n}\right)$$

*Proposed by Marian Ursărescu and Florică Anastase-Romania*

*Solution 1 by Naren Bhandari-Bajura-Nepal*

*Firstly we recall Euler's Formula* $e^{imx} = \cos mx + i\sin mx = (\cos x + i\sin x)^m$ *and further solving gives*

$$\frac{\cos mx + i\sin mx}{\sin^m x} = \sum_{r=0}^{m}\binom{m}{r}\cot^{m-r}x\, i^r$$

$$= \sum_{k=0}^{\left[\frac{m}{2}\right]+l}(-1)^r\binom{m}{2r}\cot^{m-2r}x + i\sum_{k=0}^{\left[\frac{m}{2}\right]}(-1)^r\binom{m}{2r-1}\cot^{m-2r+1}x$$

*where* $l = 1, 0$ *if* $m$ *is even odd respectively.We note that latter sum is the imaginary part and hence equating the imaginar parts we get*

$$\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m}{2r-1}\cot^{m-2r+1}x = \frac{\sin mx}{\sin^m x}$$

*on setting* $m = 2n+1, x = \frac{k\pi}{m}$ *and further expansion we get*

$$\binom{2n+1}{1}\cot^{2n}\frac{k\pi}{2n+1} - \binom{2n+1}{3}\cot^{2n-2}\frac{k\pi}{2n+1} + \dots + \binom{2n+1}{2n+1} = 0$$

*as the polynomial equation is of even degree and hence by Vieta's formula we have*

$$\prod_{k=1}^{n} cot^2\left(\frac{k\pi}{2n+1}\right) = (-1)^{2n}\frac{\binom{2n+1}{2n+1}}{\binom{2n+1}{1}} = \frac{1}{2n+1}$$

*Secondly we use the identity* $\prod_{k=0}^{n-1} sin\left(x+\frac{k\pi}{n}\right) = \frac{sin\, nx}{2^{n-1}}$, *taking log on both side and on differentiating with respect to x, we get that, ie*

$$\frac{d}{dx}log\left(\prod_{k=0}^{n-1} sin\left(x+\frac{k\pi}{n}\right)\right) = \frac{d}{dx}\left(\frac{sin\, nx}{2^{n-1}}\right)$$

$$\Rightarrow \sum_{k=0}^{n-1} cot\left(x+\frac{k\pi}{n}\right) = ncot(nx)$$

*Replacing* $n$ *by* $2n+1$ *we yield*

$$\sum_{k=0}^{2n} cot\left(x+\frac{k\pi}{2n+1}\right) = (2n+1)cot\big((2n+1)x\big) \Rightarrow$$

$$\omega_n = \left(\frac{(2n+1)cot\big((2n+1)x\big)}{2n+1}\right)^{-1} = tan\big((2n+1)x\big)$$

*Call* $\lim_{x\to\frac{\pi}{4n+2}}\left(\frac{tanx}{tan\beta}\right)^{\omega_n}$. *As we can observe that we have* $1^\infty$ *limit form.*

*Since as soon as* $x\to\frac{1}{4n+2}$, *function* $\omega_n\to\infty$ *and* $\frac{tanx}{tan\beta}\to 1$ *with* $\beta = \frac{\pi}{4n+2}$.

*So we can either make the direct use the formula for* $1^\infty$ *or without of too, ie*

$$\lim_{x\to\beta}\left(\frac{tanx}{tan\beta}\right)^{\omega_n} = \lim_{x\to\beta}\left(1+\frac{tanx}{tan\beta}-1\right)^{\omega_n} = exp\left(\lim_{x\to\beta}\left(\frac{tanx}{tan\beta}-1\right)tan\big((2n+1)x\big)\right)$$

$$= exp\left(\lim_{x\to\beta}\frac{tanx-tan\beta}{tan\beta cot\big((2n+1)x\big)}\right) \overset{L'H}{=} exp\left(\lim_{x\to\beta}\frac{sec^2x}{-(2n+1)csc^2\big((2n+1)x\big)tan\beta}\right)$$

$$= exp\left(\frac{sec\beta}{-sin\beta(2n+1)}\right)$$

*And therefore,*

$$\lim_{n\to\infty} L(n) = exp\left(-\lim_{n\to\infty}\frac{\frac{4\pi}{\pi(4n+2)}}{tan\frac{\pi}{4n+2}}\right) = exp\left(-\frac{4}{\pi}\lim_{n\to\infty}\frac{\frac{\pi}{(4n+2)}}{tan\frac{\pi}{4n+2}}\right) = e^{-\frac{4}{\pi}}$$

***Solution 2 by Sergio Esteban-Argentina and Oscar Emilio Quesada-Brazil***

$$S=\frac{1}{2n+1}\sum_{k=0}^{2n}cot\left(x+\frac{k\pi}{2n+1}\right)=\frac{1}{2n+1}\sum_{k=0}^{2n}cot\left(x+\frac{(2n-k)\pi}{2n+1}\right)$$

$$\Rightarrow 2S=\frac{1}{2n+1}\sum_{k=0}^{2n}\left[cot\left(x+\frac{k\pi}{2n+1}\right)+cot\left(x+\frac{k\pi}{2n+1}\right)\right]$$

*We use* $coty=-cot(\pi-y)$

$$2S=\frac{1}{2n+1}\sum_{k=0}^{2n}\left[cot\left(x+\frac{k\pi}{2n+1}\right)-cot\left(\frac{\pi}{2n+1}-x+\frac{k\pi}{2n+1}\right)\right]$$

*And as* $x\to\frac{\pi}{4n+2}$ , $x=\frac{\pi}{4n+2}+y, y\to 0$ *then*

$$S=\frac{1}{2}\cdot\frac{1}{2n+1}\sum_{k=0}^{2n}\left[cot\left(y+\frac{\pi}{4n+2}+\frac{k\pi}{2n+1}\right)-cot\left(\frac{\pi}{4n+2}-y+\frac{k\pi}{2n+1}\right)\right]$$

*Now, we will analyze*

$$f(y)=\ cot\left(y+\frac{\pi}{4n+2}+\frac{k\pi}{2n+1}\right)-cot\left(\frac{\pi}{4n+2}-y+\frac{k\pi}{2n+1}\right)$$

*By Taylor series*

$$f(0)=0, f'(0)=-2csc^2\left(\frac{\pi}{4n+2}+\frac{k\pi}{2n+1}\right)$$

$$f(y)=-2ycsc^2\left[\frac{(2k+1)\pi}{4n+2}\right]+O(y^2)$$

$$\Rightarrow S=\frac{1}{2}\cdot\frac{1}{2n+1}\sum_{k=0}^{2n}f(y)=-\frac{y}{2n+1}\sum_{k=0}^{2n}csc^2\left[\frac{(2k+1)\pi}{4n+2}\right]+O(y^2)$$

$$S=-y\alpha_n+O(y^2),\ \textit{where}\ \alpha_n=\sum_{k=0}^{2n}csc^2\left[\frac{(2k+1)\pi}{4n+2}\right]$$

$$\omega_n=\frac{1}{-y\alpha_n+O(y^2)}$$

*Now, we will analyze*

$$\lim_{x\to\frac{\pi}{4n+2}}\left[\frac{tanx}{tan\left(\frac{\pi}{4n+2}\right)}\right]^{\omega_n}=L_n\Rightarrow\lim_{y\to 0}\left[\frac{tan\left(y+\frac{\pi}{4n+2}\right)}{tan\left(\frac{\pi}{4n+2}\right)}\right]^{\omega_n}=L_n$$

$$log(L_n)=\lim_{y\to 0}\omega_n log\left(\frac{tan\left(y+\frac{\pi}{4n+2}\right)}{tan\left(\frac{\pi}{4n+2}\right)}\right)$$

***By Taylor, we will analyze***

$$f(y) = \ log\left(\frac{tan\left(y+\frac{\pi}{4n+2}\right)}{tan\left(\frac{\pi}{4n+2}\right)}\right), f(0) = 0, f'(0) = \frac{1}{sin\left(\frac{\pi}{4n+2}\right)cos\left(\frac{\pi}{4n+2}\right)}$$

***Then***

$$f(y) = \frac{2y}{sin\left(\frac{\pi}{2n+1}\right)} + O(y^2)$$

$$log(L_n) = \lim_{y\to 0}\left(\frac{1}{-y\alpha_n + O(y^2)}\right)\left(\frac{2y}{sin\left(\frac{\pi}{2n+1}\right)} + O(y^2)\right)$$

***As $O(y^2) = yO(y)$ and $O(y) \underset{y\to 0}{\longrightarrow} 0$ then***

$$log(L_n) = \left(\frac{1}{-\alpha_n + O(y)}\right)\left(\frac{2y}{sin\left(\frac{\pi}{2n+1}\right)} + O(y)\right) = \frac{-2}{\alpha_n sin\left(\frac{\pi}{2n+1}\right)}$$

***But $d_n(y) = y + O(y^3) = y\left(1 + O(y^2)\right) \Rightarrow sin\left(\frac{\pi}{2n+1}\right) = \frac{\pi}{2n+1}\left(1 + O\left(\frac{1}{n^2}\right)\right)$***

***Function $y \to csc^2y$ is not defined on $y = 0$, then we can't use taylor. In fact,***

$$csc^2y = \frac{1}{sin^2y} \text{ and } sin\, y \sim y \Rightarrow csc^2y \sim \frac{1}{y^2} \Rightarrow y^2csc^2y \sim 1$$

***Function $y \to y^2csc^2y$ is defined and by Taylor $f(y) = y^2csc^2y$***

$$\lim_{y\to 0} f(y) = 1, f'(y) = \frac{2y}{sin^2y} - \frac{2y^2cosy}{sin^3y}$$

$$\lim_{y\to 0} f'(y) = 0 \Rightarrow f(y) = 1 + O(y^2) \Rightarrow csc^2y = \frac{1}{y^2} + O(1)$$

$$csc^2\left(\frac{(2k+1)\pi}{4n+2}\right) = \frac{4(2n+1)^2}{\pi^2(2k+1)^2} + O(1)$$

$$\alpha_n = \frac{1}{2n+1}\sum_{k=0}^{2n}\left[\frac{4(2n+1)^2}{\pi^2(2k+1)^2} + O(1)\right] = \frac{4(2n+1)}{\pi^2}\sum_{k=0}^{2n}\frac{1}{(2k+1)^2} + \frac{2n}{2n+1}O(1)$$

***Then***

$$\alpha_n sin\left(\frac{\pi}{2n+1}\right) = \left(\frac{\pi}{2n+1} + O\left(\frac{1}{n^3}\right)\right)\alpha_n = \frac{4}{\pi}\sum_{k=0}^{2n}\frac{1}{(2k+1)^2} + O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\alpha_n sin\left(\frac{\pi}{2n+1}\right)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi}{2}$$

$$\lim_{n\to\infty}log(L_n)=\lim_{n\to\infty}\frac{-2}{\alpha_n sin\left(\frac{\pi}{2n+1}\right)}=-\frac{4}{\pi}$$

$$\Omega=e^{-\frac{4}{\pi}}$$

## Solution 3 by proposers

***Developing in two ways:***

$$(cosa+isina)^p=cos(ap)+isin(ap)=\sum_{k=0}^{p}\binom{p}{k}cos^{p-k}a\cdot(isina)^k$$

**We get:** $sin(ap)=\binom{p}{1}cos^{p-1}a\cdot sina-\binom{p}{3}cos^{p-3}a\cdot sin^3a+\cdots$

$$For: p=2n+1 \text{ și } a\in\left\{\frac{\pi}{2n+1},\frac{2\pi}{2n+1},\dots,\frac{n\pi}{2n+1}\right\}\to sin(2n+1)a=0,$$
$$sin^p a\neq 0, cos^p a\neq 0$$

**Equation:** $\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\cdots+(-1)^n\binom{2n+1}{2n+1}=0$ **admit the roots**

$$x_k=cot^2\frac{k\pi}{2n+1}, k=\overline{1,n}\ \overset{Vieta's}{\Rightarrow}\ \prod_{k=1}^{n}x_k=\frac{\binom{2n+1}{2n+1}}{\binom{2n+1}{1}}=\prod_{k=1}^{n}cot^2\frac{k\pi}{2n+1}=\frac{1}{2n+1}\quad(1)$$

***The roots of the equation***

$$z^{2n}-2z^n\cos nx+1=0\ are: z_k=cos\left(x+\frac{2k\pi}{n}\right)+isin\left(x+\frac{2k\pi}{n}\right) \text{ și}$$

$$\overline{z_k}=cos\left(x+\frac{2k\pi}{n}\right)-isin\left(x+\frac{2k\pi}{n}\right), k=\overline{0,n-1}$$

$$\prod_{k=0}^{n-1}[(z-z_k)(z-\overline{z_k})]=z^{2n}-2z^n\cos(nx)+1$$

***From*** $z=\pm1$, ***we have***

$$\prod_{k=0}^{n-1}\left[1-2cos\left(x+\frac{2k\pi}{n}\right)+1\right]=2(1-cosnx)$$

$$\prod_{k=0}^{n-1}\left[1+2cos\left(x+\frac{2k\pi}{n}\right)+1\right]=2(1+cosnx)$$

***With*** $\frac{x}{2} \to x$ *and* $n \to 2n+1$ , ***we obtain***

$$\prod_{k=0}^{2n} \sin\left(x + \frac{k\pi}{2n+1}\right) = \frac{\sin(2n+1)x}{2^{2n}}$$

$$\prod_{k=0}^{2n} \cos\left(x + \frac{k\pi}{2n+1}\right) = (-1)^n \frac{\cos(2n+1)x}{2^{2n}}$$

***Logarithmating the two relationship and differentiating them from x, we have:***

$$\sum_{k=0}^{2n} \frac{\cos\left(x + \frac{k\pi}{2n+1}\right)}{\sin\left(x + \frac{k\pi}{2n+1}\right)} = (2n+1)\cot(2n+1)x$$

$$\omega_n = (2n+1)\left(\sum_{k=0}^{2n} \tan\left(x + \frac{k\pi}{2n+1}\right)\right)^{-1} = \tan(2n+1)x$$

$$\lim_{x\to\frac{\pi}{4n+2}} \left(\frac{\tan x}{\tan\frac{\pi}{4n+2}}\right)^{\tan(2n+1)x} = e^{\lim\limits_{x\to\frac{\pi}{4n+2}} \frac{\tan x - \tan\left(\frac{\pi}{4n+2}\right)}{\tan\left(\frac{\pi}{4n+2}\right)} \cdot \tan(2n+1)x} = (*)$$

$$\lim_{x\to\frac{\pi}{4n+2}} \frac{\tan x - \tan\left(\frac{\pi}{4n+2}\right)}{\tan\left(\frac{\pi}{4n+2}\right)} \cdot \tan(2n+1)x =$$

$$= -\lim_{x\to\frac{\pi}{4n+2}} \left(\frac{\sin\left(x - \frac{\pi}{4n+2}\right)}{\cos\left(\frac{\pi}{4n+2}\right)\cos x \frac{\sin\left(\frac{\pi}{4n+2}\right)}{\cos\left(\frac{\pi}{4n+2}\right)}} \cdot \cot\left((2n+1)x - \frac{\pi}{2}\right)\right) =$$

$$= -\lim_{x\to\frac{\pi}{4n+2}} \left(\frac{\sin\left(x - \frac{\pi}{4n+2}\right)}{\cos x \sin\frac{\pi}{4n+2}} \cdot \frac{\cos\left(\frac{(4n+2)x - \pi}{2}\right)}{\sin\left(\frac{(4n+2)x - \pi}{2}\right)}\right) =$$

$$= -\lim_{x\to\frac{\pi}{4n+2}} \left(\frac{\cos\left(\frac{(4n+2)x - \pi}{2}\right)}{\cos x \cdot \sin\left(\frac{\pi}{4n+2}\right)} \cdot \frac{\sin\left(x - \frac{\pi}{4n+2}\right)}{\sin\left(\frac{(4n+2)x - \pi}{2}\right)}\right) =$$

$$= -\frac{2}{\sin\left(\frac{\pi}{4n+2}\right)}\lim_{x\to\frac{\pi}{4n+2}}\left(\frac{\sin\left(x-\frac{\pi}{4n+2}\right)}{x-\frac{\pi}{4n+2}}\cdot\frac{1}{2n+1}\cdot\frac{\frac{(4n+2)x-\pi}{2}}{\sin\left(\frac{(4n+2)x-\pi}{2}\right)}\right)$$

$$= -\frac{2}{(2n+1)\sin\left(\frac{\pi}{4n+2}\right)}$$

$$(*) = e^{-\frac{2}{(2n+1)\sin\left(\frac{\pi}{4n+2}\right)}}$$

$$\Omega = e^{-\lim\limits_{n\to\infty}\frac{2}{(2n+1)\sin\left(\frac{\pi}{4n+2}\right)}} = e^{-\lim\limits_{n\to\infty}\frac{4}{\pi\left(\frac{4n+2}{\pi}\right)\sin\left(\frac{\pi}{4n+2}\right)}} = e^{-\frac{4}{\pi}}$$

**981.** Let: $f_n(x) = \sqrt[m]{1+\frac{x}{f_{n-1}(x)}}$; $f_0(x) = \sqrt[m]{x+1}$; $m \ge 2$

**Prove that:**

$$\lim_{n\to\infty} f_n(x) = \sqrt[m+1]{x+\sqrt[m+1]{x+\sqrt[m+1]{x+\sqrt[m+1]{x+\cdots\infty}}}}$$

*Proposed by Mohammed Bouras-Morocco*

*Solution by Kamel Benaicha-Algiers-Algerie*

$$f_n(x) = \sqrt[m]{1+\frac{x}{f_{n-1}(x)}},\ f_0(x) = \sqrt[3]{1+x},\ m \ge 2 \text{ and } x > 0.$$

***If*** $(f_n)_{n\in\mathbb{N}}$ ***has a limit*** $l(x)$, ***then:***

$$l(x) = \sqrt[3]{1+\frac{x}{l(x)}},\ \text{so } l(x) \ge 1 \quad \textbf{(1)}$$

$$\therefore\ \left(l(x)\right)^{m+1} - l(x) - x = 0 \quad \textbf{(E)}$$

***Put*** $g(t) = t^{m+1} - t - x$, $g$ ***is a continuous function (polynomial function)*** **(2)**

$$g'(t) = (m+1)t^m - 1 = 0 \Leftrightarrow \begin{cases} t_{0,1} = \pm\sqrt[m]{\frac{1}{m+1}};\ if\ m \in 2\mathbb{N}; \\ t_0 = \sqrt[m]{\frac{1}{m+1}},\ if\ m \in 2\mathbb{N}+1 \end{cases}$$

*So: for all $m \geq 2$, if $t \geq \sqrt[m]{\frac{1}{m+1}}$ then $g'(t) \geq m > 0 \Rightarrow g$ increasing for*

$$t \in \left]\sqrt[m]{\frac{1}{m+1}};\ +\infty\right[\quad (3)$$

$$g(t_0) = \frac{1}{m+1}\sqrt[m]{\frac{1}{m+1}} - \sqrt[m]{\frac{1}{m+1}} - x = \sqrt[m]{\frac{1}{m+1}}\left(\frac{1}{m+1} - 1\right) - x < 0\quad (4)$$

$$\lim_{t\to+\infty} g(t) = +\infty\quad (5)$$

*Using the results: (2), (3), (4), (5) and applying the theory of intermediate values, we conclude that the equation (E) has a unique solution $l(x) > \sqrt[m]{\frac{1}{m+1}}$*

*By applying the fixed point method, we get the following solution:*

$$\text{(E)} \Leftrightarrow l(x) = \sqrt[m+1]{x + l(x)},\ l(x) \geq 1.$$

*Put:* $l_{n+1}(x) = \sqrt[m+1]{x + l_n(x)}$, *with* $l_0(x) = \sqrt[m+1]{1 + x}$.

*So:* $l(x) = \lim_{n\to+\infty} l_n(x) = \sqrt[m+1]{x + \sqrt[m+1]{x + \cdots + \sqrt[m+1]{x + \cdots}}}$

**982. Find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2n-1}}\sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Naren Bhandari-Bajura-Nepal***

$$Let\ b_n = \sqrt{2n-1}\ and\ a_n = \sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}$$

*We note that*

$$b_{n+1} - b_n = \frac{2}{\sqrt{2n+1} + \sqrt{2n-1}} > 0\ implies\ b_n\ is\ increasing\ sequences.$$

*Then we use Stolz-Cesaro theorem*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty}\frac{\sqrt{2n+1} + \sqrt{2n-1}}{2(2^{n+1}-1)(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{(2^{n+1}-1)(n+1)!}\left(\sqrt{2n+1}-\sqrt{2n-1}\right)} = 0$$

## Solution 2 by Ravi Prakash-New Delhi-India

$$\text{For } k \in \mathbb{N}, k \geq 1 \Rightarrow k! \geq 2^k - 1 > 2^{k-1} \Rightarrow$$

$$\left(2^{k-1}\right)^2 < \left(2^k-1\right)^2 \leq \left(2^k-1\right)k! \leq (k!)^2 \Rightarrow$$

$$\frac{1}{k!} \leq \frac{1}{\sqrt{(2^k-1)k!}} < \frac{1}{2^{k-1}} \Rightarrow$$

$$e - 1 < \sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{(2^k-1)k!}} < \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 2, \forall n \in \mathbb{N}$$

$$\frac{e-1}{\sqrt{2n-1}} < \frac{1}{\sqrt{2n-1}} \sum_{k=1}^{n} \frac{1}{\sqrt{(2^k-1)k!}} < \frac{2}{\sqrt{2n-1}}, \forall n \in \mathbb{N}$$

$$\text{Since } \lim_{n\to\infty} \frac{e-1}{\sqrt{2n-1}} = \lim_{n\to\infty} \frac{2}{\sqrt{2n-1}} = 0$$

**We get**

$$\Omega = \lim_{n\to\infty} \left( \frac{1}{\sqrt{2n-1}} \sum_{k=1}^{n} \frac{1}{\sqrt{(2^k-1)k!}} \right)$$

## Solution 3 by Remus Florin Stanca-Romania

$$\Omega \overset{L.C-S}{=} \lim_{n\to\infty} \frac{1}{\sqrt{(2^{n+1}-1)(n+1)!}} \cdot \frac{1}{\left(\sqrt{2n+1}-\sqrt{2n-1}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{(2^{n+1}-1)(n+1)!}} \cdot \frac{1}{\frac{2}{\sqrt{2n+1}+\sqrt{2n-1}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{(2^{n+1}-1)(n+1)!}} \cdot \frac{\sqrt{2n+1}+\sqrt{2n-1}}{2} = 0$$

## Solution 4 by Sergio Esteban-Argentina

**By** $Am \geq Gm \geq Hm$

$$\frac{2}{\frac{1}{k!}+\frac{1}{2^k-1}} \leq \sqrt{(2^k-1)k!} \leq \frac{1}{2}\left(k!+2^k-1\right) < k!$$

**But** $k! + 2^k - 1 < k! + 2^k < 2k!, \forall k \geq 4, k \in \mathbb{N}$

*Then* $\frac{\sum_{k=1}^{n}\frac{1}{k!}}{\sqrt{2n-1}} < \frac{\sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}}{\sqrt{2n-1}} \le \frac{\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k!}+\frac{1}{2^k-1}\right)}{\sqrt{2n-1}}$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k!}-1}{\sqrt{2n-1}} \le \Omega \le \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k!}-1}{2\sqrt{2n-1}} + \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{2^k-1}}{2\sqrt{2n-1}}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k!}-1}{2\sqrt{2n-1}} = \lim_{n\to\infty}\frac{e-1}{\sqrt{2n-1}} = 0$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{2^k-1}}{2\sqrt{2n-1}} = \lim_{n\to\infty}\frac{\int_1^n\frac{1}{2^k-1}dk}{2\sqrt{2n-1}} = \lim_{n\to\infty}\frac{log(2^n-1)}{4\sqrt{2n-1}} = 0$$

***So,***

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{2n-1}}\sum_{k=1}^{n}\frac{1}{\sqrt{(2^k-1)k!}}\right) = 0$$

**983. Find:**

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_0^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx - log\sqrt{n}\right)$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution by Kamel Benaicha-Algeirs-Algerie***

***Put:*** $I(k) = \int_0^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx = \int_0^{1}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx + \int_1^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx$

$$\int_1^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx \overset{t=\frac{1}{x}}{=} \int_0^{1}\frac{t^{k+1}}{(1+t^2)(1+t^k)^2}dt = \int_0^{1}\frac{t^{k-1}\cdot t^2}{(1+t^2)(1+t^k)^2}dt$$

$$= \int_0^{1}\frac{t^{k-1}(t^2-1+1)}{(1+t^2)(1+t^k)^2}dt = -\int_0^{1}\frac{t^{k-1}}{(1+t^2)(1+t^k)^2}dt + \int_0^{1}\frac{t^{k-1}}{(1+t^k)^2}dt$$

$$I(k) = \int_0^{1}\frac{x^{k-1}}{(1+x^k)^2}dx = -\frac{1}{k(1+x^k)}\Big|_0^1 = \frac{1}{2k}$$

$$\sum_{k=1}^{n}\int_0^{\infty}\frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} = \frac{H_n}{2} \Rightarrow$$

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_0^\infty \frac{x^{k-1}}{(1+x^2)(1+x^k)^2}dx - \log\sqrt{n}\right) = \frac{1}{2}\lim_{n\to\infty}(H_n - \log(n)) = \frac{\gamma}{2}$$

**984. Find:**

$$\Omega(a,b,c) = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)^n, a,b,c > 0$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution 1 by Sergio Esteban-Argentina*

$$\Omega(a,b,c) = \lim_{n\to\infty}\left(1+\frac{1}{\frac{c}{\sqrt[n]{a}-\sqrt[n]{b}}}\right)^{\frac{c}{\sqrt[n]{a}-\sqrt[n]{b}}\cdot\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\cdot n} = e^{\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)\cdot n} = e^{\frac{1}{c}\log\left(\frac{a}{b}\right)} = \sqrt[c]{\frac{a}{b}}$$

*where:*

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)\cdot n = \frac{1}{c}\cdot\lim_{n\to\infty}\sqrt[n]{b}\cdot\frac{\left(\sqrt[n]{\frac{a}{b}}-1\right)}{\frac{1}{n}} = \frac{1}{c}\log\left(\frac{a}{b}\right)$$

*Solution 2 by Kamel Benaicha-Algiers-Algerie*

$$\Omega(a,b,c) = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)^n = \lim_{n\to\infty} e^{n\log\left(1+\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)}$$

$$\log\left(1+\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)\underbrace{\sim}_{n\to\infty}\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}$$

**Put:** $t = \frac{1}{n}\overset{n\to\infty}{\Longrightarrow} t\to 0_+$

$$\Omega(a,b,c) = \lim_{t\to 0_+} e^{\frac{a^t-b^t}{ct}} = e^{\lim_{t\to 0_+}\frac{a^t-1-(b^t-1)}{ct}} = e^{\frac{\log(a)-\log(b)}{c}} = e^{\log\left(\frac{a}{b}\right)^{\frac{1}{c}}} = \sqrt[c]{\frac{a}{b}}$$

**985. Find:**

$$\Omega = \lim_{n\to\infty}\left(\sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}\sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\binom{n}{k}\right)$$

*Proposed by Daniel Sitaru-Romania*

*Solution 1 by Remus Florin Stanca-Romania*

$$\sqrt{\frac{k}{k+1}} \le 1 \Big| \cdot \binom{n}{k} \Rightarrow \sqrt{\frac{k}{k+1}} \cdot \binom{n}{k} \le \binom{n}{k} \Rightarrow$$

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \cdot \binom{n}{k} \le \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1 < 2^n \Rightarrow$$

$$\sqrt{\frac{2n+2}{n^3 \cdot 2^n(2^{n+1}-n)}} \sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{2n+2}{n^3 \cdot 2^n(2^{n+1}-n)}} \cdot 2^n \dots (1)$$

$$\lim_{n\to\infty} \sqrt{\frac{2n+2}{n^3 \cdot 2^n(2^{n+1}-n)}} \cdot 2^n = \lim_{n\to\infty} \sqrt{\frac{2^n}{2^n\left(2-\frac{n}{2^n}\right)}} \cdot \lim_{n\to\infty} \sqrt{\frac{2n+2}{n^3}} = 0 \overset{(1)}{\Rightarrow} \Omega = 0$$

*Solution 2 by Naren Bhandari-Bajura-Nepal*

***It's trivial to show that*** $\sqrt{\frac{k}{k+1}}, \forall k \in \mathbb{N}$***. Here***

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \cdot \binom{n}{k} < \prod_{k=1}^{n} \sqrt{\frac{k}{k+1}} \cdot \binom{n}{k}$$

$$= \frac{1}{\sqrt{n+1}} \prod_{k=1}^{n} \binom{n}{k} = \frac{1}{\sqrt{n+1}} \left(\frac{H^2(n)}{(n!)^{n+1}} - 1\right)$$

***Where H(n) is the hyperfactorial. Further***

$$\frac{H^2(n)}{(n!)^{n+1}} = \frac{1}{(n!)^{n+1}} \left(\prod_{k=1}^{n} k^k\right)^2 \sim \frac{1}{(n!)^{n+1}} \left(A_n^{\frac{6n^2+6n+1}{12}} e^{-\frac{n^2}{4}}\right)^2$$

***Here A is Glaisher-Kinklien and we use the Stirling approximation for***

$$n! \sim \sqrt{2n\pi} \left(\frac{n}{e}\right)^n$$

***and simplifying the result gives.***

$$\frac{H^2(n)}{(n!)^{n+1}} \sim \frac{A^2 e^{\frac{n^2+2n}{2}}}{(2\pi)^{\frac{n+1}{2}} n^{\frac{3n+2}{6}}}$$

*plugging the final obtained result to the inequality we have then*

$$0 < \Omega < \lim_{n\to\infty}\frac{\alpha}{\sqrt{n+1}}\left(\frac{A^2 e^{\frac{n^2+2n}{2}}}{(2\pi)^{\frac{n+1}{2}} n^{\frac{3n+2}{6}}} - 1\right)$$

*Where* $\alpha = \sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}$. *Evaluating the latter limit gives*

$$\lim_{n\to\infty}\sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}\cdot\frac{A^2 e^{\frac{n^2+2n}{2}}}{(2\pi)^{\frac{n+1}{2}} n^{\frac{3n+2}{6}}}$$

$$= \lim_{n\to\infty}\frac{\sqrt{2}A^2}{(\pi)^{\frac{n+1}{2}}\sqrt{2^{2n+2}(2^{n+1}-n)}}\cdot\lim_{n\to\infty}\frac{e^{\frac{n^2+2n}{2}}}{n^{\frac{3n+11}{6}}} = 0$$

*And hence by Squeeze theorem we have Ω=0*

***Solution 3 by Naren Bhandari-Bajura-Nepal***

*We note that* $\forall k \ge 1, \frac{k}{k+1} < 1 \Rightarrow \sqrt{\frac{k}{k+1}} < 1 \Rightarrow 0 < \sqrt{\frac{k}{k+1}} < 1$ *and hence we have*

$$0 < \sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\cdot\binom{n}{k} < \sum_{k=1}^{n}\binom{n}{k} = 2^n - 1$$

*and hence we have*

$$0 < \Omega < \lim_{n\to\infty}(2^n-1)\sqrt{\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}}$$

$$= \lim_{n\to\infty}\left(\frac{2n+2}{n^3\cdot 2^n(2^{n+1}-n)}\right)^{\frac{1}{2}} = \lim_{n\to\infty}\left(\frac{2(2^n-1)^2}{n^2\cdot 2^{2n+1}}\right)^{\frac{1}{2}} = 0$$

**986. Find:**

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n} sec\left(\frac{k}{n\sqrt{n}}\right)$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution by Naren Bhandari-Bajura-Nepal***

$$\lim_{n\to\infty}\prod_{k=1}^{n} sec\left(\frac{k}{n\sqrt{n}}\right) = exp\left(\lim_{n\to\infty}\sum_{k=1}^{n} log\left(sec\left(\frac{k}{n\sqrt{n}}\right)\right)\right)$$

$$= exp\left(\lim_{n\to\infty}\sum_{k=1}^{n} log\left(1+\frac{k^2}{2n^3}+\frac{5k^4}{24n^6}+\cdots\right)\right) = exp\left(\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2}{2n^3}+\frac{5k^4}{24n^6}+\cdots\right)\right)$$

$$= exp\left(\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2}{2n^3}\right)\right) = exp\left(\int_0^1 \frac{x^2}{2}dx\right) = \sqrt[6]{e}$$

**987. Find:**

$$\Omega = \lim_{n\to\infty}\int_a^b\left(\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n + x^n}\right)dx\,,0\le a<b$$

***Proposed by Vasile Mircea Popa-Romania***

***Solution by Kamel Benaicha-Algiers-Algerie***

***Put:***

$$f(x) = \frac{2x^2-2(a+b)x+b(a+b)}{b-a} - x = \frac{2x^2-(a+3b)x+b(a+b)}{b-a} = 0 \Leftrightarrow$$

$$2x^2-(a+3b)x+b(a+b) = 0 \Leftrightarrow x\in\left\{\frac{a+b}{2};b\right\}$$

$$\therefore \frac{2x^2-2(a+b)x+b(a+b)}{b-a} \ge x \text{ for } x\in\left[a;\frac{a+b}{2}\right]$$

$$\therefore \frac{2x^2-2(a+b)x+b(a+b)}{b-a} \le x \text{ for } x\in\left[\frac{a+b}{2};b\right]$$

$$\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n + x^n} \le \frac{2x^2-2(a+b)x+b(a+b)}{b-a} + x$$

***Put:*** $g(x) = 2x^2-(a+3b)x+b(a+b)$

$$g'(x) = 4x-(3a+b)$$

$$g'(x) = 0 \Leftrightarrow x = \frac{3a+b}{4}\in[a,b]$$

$$\therefore g'(x) \geq 0 \; for \; x \in \left[\frac{3a+b}{4}; b\right] \Rightarrow g(x) \leq g(b)$$

$$\therefore g'(x) \leq 0 \; for \; x \in \left[a; \frac{3a+b}{4}\right] \Rightarrow g(x) \leq g(a)$$

$$\therefore g(x) \leq max\{g(a), g(b)\}$$

$$g(a) = 2a^2 - 3a^2 - ab + ab + b^2 = b^2 - a^2$$

$$g(b) = 2b(b-a)$$

$$g(b) - g(a) = (b-a)(2b-a-b) = (b-a)^2 \Rightarrow g(a) \leq g(b)$$

$$\therefore \sqrt[n]{\left(\frac{2x^2 - 2(a+b)x + b(a+b)}{b-a}\right)^n + x^n} \leq 2b; \; \forall x \in [a; b]$$

$$\Omega = \int_a^b \left(\lim_{n\to\infty} \sqrt[n]{\left(\frac{2x^2 - 2(a+b)x + b(a+b)}{b-a}\right)^n + x^n}\right) dx$$

***We know that:***

$$\lim_{n\to\infty} (\alpha^n + \beta^n)^{\frac{1}{n}} = max\{\alpha; \beta\}$$

***Then:***

$$\Omega = \int_a^{\frac{a+b}{2}} \left(\frac{2x^2 - 2(a+b)x + b(a+b)}{b-a}\right) dx + \int_{\frac{a+b}{2}}^b x dx$$

$$= \frac{1}{b-a}\left(\frac{2x^3}{3} - (a+b)x^2 + b(a+b)x\right)\Bigg|_a^{\frac{a+b}{2}} + \frac{x^2}{2}\Bigg|_{\frac{a+b}{2}}^b$$

$$= \frac{1}{24}(b-a)(7a+17b)$$

***So:***

$$\Omega = \lim_{n\to\infty} \int_a^b \left(\sqrt[n]{\left(\frac{2x^2 - 2(a+b)x + b(a+b)}{b-a}\right)^n + x^n}\right) dx = \frac{1}{24}(b-a)(7a+17b)$$

**988.**

$$(x_n)_{n\ge1},(y_n)_{n\ge1};x_n=\sum_{k=1}^{n}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{n\pi}{4};$$

$$y_n=\sum_{k=1}^{n}\frac{1}{\cot^{-1}(2k+1)}.\ \text{Find: }\Omega=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}(H_1\cdot H_2\cdot\ldots\cdot H_k)^{\frac{1}{k}}}{\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi y_m^2}{4}\right)}$$

*Proposed by Florică Anastase-Romania*

***Solution by proposer:***

***From Hardy-Carleman's inequality, we have:***

$$\sum_{k=1}^{n}(H_1H_2\cdot\ldots\cdot H_k)^{\frac{1}{k}}<e\sum_{k=1}^{n}H_k\ \ldots(1)$$

$$\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{\pi}{4}=\cot^{-1}(2k+1)\Rightarrow$$

$$\sum_{k=1}^{m}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{m\pi}{4}=\sum_{k=1}^{m}\cot^{-1}(2k+1)\Leftrightarrow$$

$$\left(\sum_{k=1}^{m}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{m\pi}{4}\right)\left(\sum_{k=1}^{m}\frac{1}{\cot^{-1}(2k+1)}\right)=\left(\sum_{k=1}^{m}\cot^{-1}(2k+1)\right)\left(\sum_{k=1}^{m}\frac{1}{\cot^{-1}(2k+1)}\right)\overset{B.C.S.}{\geq}m^2$$

$$\Leftrightarrow x_my_m\ge m^2\ \ldots(2)$$

$$\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi y_m^2}{4}\right)\overset{Am-Gm}{\geq}\sum_{m=1}^{n}x_my_m\overset{(2)}{\geq}\sum_{m=1}^{n}m^2\ \ldots(3)$$

***From (1),(3) we have:***

$$0\le\Omega=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}(H_1\cdot H_2\cdot\ldots\cdot H_k)^{\frac{1}{k}}}{\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi y_m^2}{4}\right)}\le\lim_{n\to\infty}\frac{e\sum_{k=1}^{n}H_k}{\sum_{m=1}^{n}m^2}\overset{L.C-S}{=}e\lim_{n\to\infty}\frac{H_n}{n^2}\xrightarrow{L.C-S}0$$

**989. Find:**

$$\Omega=\lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right)$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Kamel Benaicha-Algiers-Algerie***

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right) = \lim_{n\to\infty}\frac{\log^2\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+2}-\frac{1}{2(n+2)^2}}$$

$$= \lim_{n\to\infty}\frac{2(n+2)^2}{(n+1)^2(n+3)} = 0$$

***Solution 2 by Igor Soposki-Skopje-Macedonia***

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right) = \lim_{n\to\infty}\frac{2\log\left(1+\frac{1}{n+1}\right)\cdot\frac{1}{1+\frac{1}{n+1}}\cdot\left(-\frac{1}{(n+1)^2}\right)}{\frac{1}{1+\frac{1}{n+2}}\cdot\left(-\frac{1}{(n+2)^2}\right)}$$

$$= \lim_{n\to\infty}\frac{2\log\left(1+\frac{1}{n+1}\right)\cdot\frac{n+1}{n+2}\cdot\frac{1}{(n+1)^2}}{\frac{n+2}{n+3}\cdot\frac{2}{(n+2)^2}}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{(n+1)(n+2)}\cdot\log\left(1+\frac{1}{n+1}\right)}{\frac{1}{(n+2)(n+3)}} = 2\lim_{n\to\infty}\frac{(n+3)\log\left(1+\frac{1}{n+1}\right)}{n+1}$$

$$= 2\lim_{n\to\infty}\frac{(n+1)\log\left(1+\frac{1}{n+1}\right)}{n+1} + 4\lim_{n\to\infty}\frac{\log\left(1+\frac{1}{n+1}\right)}{n+1} = 0$$

***Solution 3 by Izumi Ainsworth-Lima-Peru***

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right) = \lim_{n\to\infty}\frac{\left(\log\left(\overbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}^{\to e}\right)^{\frac{1}{n+1}}\right)^2}{\log\left(\overbrace{\left(1+\frac{1}{n+2}\right)^{n+2}}^{\to e}\right)^{\frac{1}{n+2}}}$$

$$= \lim_{n\to\infty}\frac{\left(\frac{loge}{n+1}\right)^2}{\frac{loge}{n+2}} = \lim_{n\to\infty}\frac{n+2}{(n+1)^2} = \lim_{n\to\infty}\frac{1}{n} = 0$$

***Solution 4 by Mohamed Arahman Jama-Somalia***

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(log\left(1+\frac{1}{n+1}\right)\right)^2}{log\left(1+\frac{1}{n+2}\right)}\right) \overset{n+1=x}{\cong} \lim_{x\to\infty}\left(\frac{\left(log\left(1+\frac{1}{x}\right)\right)^2}{log\left(1+\frac{1}{x+1}\right)}\right)$$

$$= \lim_{x\to\infty}\frac{2log\left(1+\frac{1}{x}\right)\cdot\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\left(-\frac{1}{x^2}\right)}{\left(\frac{1}{1+\frac{1}{x+1}}\right)\cdot\left(-\frac{1}{(1+x)^2}\right)} = \lim_{x\to\infty}\left(\frac{2log\left(1+\frac{1}{x}\right)}{x(x+1)}\cdot\frac{(1+x)^2(x+2)}{x+1}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2log\left(1+\frac{1}{x}\right)}{x(x+1)}\cdot(x+2)\right) = \lim_{x\to\infty}\left(\frac{2xlog\left(1+\frac{1}{x}\right)}{x} + \frac{4log\left(1+\frac{1}{x}\right)}{x}\right)$$

$$= \lim_{x\to\infty}\left(\underbrace{2log\left(1+\frac{1}{x}\right)}_{\to 0} + \frac{4log\left(1+\frac{1}{x}\right)}{x}\right) = \lim_{x\to\infty}\left(\frac{-4x}{(x+1)x^2}\right) = \lim_{x\to\infty}\left(\frac{-4}{(x+1)x}\right) = 0$$

***Solution 5 by Khaled Abd Imouti-Damascus-Syria***

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(log\left(1+\frac{1}{n+1}\right)\right)^2}{log\left(1+\frac{1}{n+2}\right)}\right)$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\cdot\frac{\log\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}}\right)^2}{\frac{1}{n+2}\cdot\frac{\log\left(1+\frac{1}{n+2}\right)}{\frac{1}{n+2}}} = \lim_{n\to\infty} \underbrace{\frac{n+2}{(n+1)^2}}_{\to 0}\cdot\underbrace{\frac{\left(\frac{\log\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}}\right)^2}{\frac{\log\left(1+\frac{1}{n+2}\right)}{\frac{1}{n+2}}}}_{\to 1} = 0$$

**Solution 6 by Rajeev Rastogi-India**

$$\Omega = \lim_{n\to\infty}\left(\frac{\left(\log\left(1+\frac{1}{n+1}\right)\right)^2}{\log\left(1+\frac{1}{n+2}\right)}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\frac{\log\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}}\right)^2\cdot\frac{\frac{1}{n+2}}{\log\left(1+\frac{1}{n+2}\right)}\cdot\frac{n+2}{(n+1)^2}\right) = 0$$

**990. Find:**

$$\Omega = \lim_{n\to\infty}\left(n - \sum_{k=1}^{n}\sqrt[4]{1+\frac{k^3}{n^4}}\right)$$

***Proposed by Vasile Mircea Popa-Romania***

**Solution 1 by Ali Jaffal-Lebanon**

*Let:* $\varphi(x) = \begin{cases} \frac{(1+x)^\alpha - \frac{\alpha(\alpha-1)}{2}x^2 - \alpha x - 1}{x^2}, x \in [0,1] \\ 0, x = 0 \end{cases}$; $\alpha = ct.$

*We know that:* $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3$

*So,* $\lim_{x\to 0}\frac{(1+x)^\alpha - \frac{\alpha(\alpha-1)}{2}x^2 - \alpha x - 1}{x^2} = 0$, *then* $\lim_{x\to 0}\varphi(x) = \varphi(0)$, $\varphi$ *–is continuous, then exist* $M > 0$ *such that* $|\varphi(x)| \le M, \forall x \in [0,1]$ $\quad (*)$

*Let:* $\Omega_n = n - \sum_{k=1}^{n}\sqrt[4]{1+\frac{k^3}{n^4}}$

$$\left(1+\frac{k^3}{n^4}\right)^{\frac{1}{n}}=1+\frac{1}{n}\cdot\frac{k^3}{n^4}-\frac{3}{32}\cdot\frac{k^6}{n^8}+\frac{k^6}{n^8}\varphi\left(\frac{k^3}{n^4}\right)$$

$$\Omega_n=-\frac{1}{4n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3+\frac{3}{32}\cdot\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^6+\sum_{k=1}^{n}\frac{k^6}{n^8}\varphi\left(\frac{k^3}{n^4}\right)$$

***We have:*** $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3=\int_0^1 x^3dx=\frac{1}{4}$

$$\lim_{n\to\infty}\frac{3}{32}\cdot\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^6=0\cdot\int_0^1 x^6dx=0$$

$$\left|\sum_{k=1}^{n}\frac{k^6}{n^8}\varphi\left(\frac{k^3}{n^4}\right)\right|\le\sum_{k=1}^{n}\frac{k^6}{n^8}\left|\varphi\left(\frac{k^3}{n^4}\right)\right|\le M\cdot\sum_{k=1}^{n}\frac{k^6}{n^8}\overset{(*)}{\Rightarrow}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^6}{n^8}=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^6=0$$

***So,*** $\lim_{n\to\infty}\Omega_n=-\frac{1}{16}$

## *Solution 2 by Sergio Esteban-Argentina*

$$\Omega=-\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt[4]{1+\frac{k^3}{n^4}}-1\right)$$

***We consider inequality:*** $\frac{x}{3x+4}<\sqrt[4]{1+x}-1<\frac{x}{4},x>-1$

***By Bernoulli's inequality:*** $\sqrt[4]{1+x}<1+\frac{x}{4},x>-1$ ***and by Gm-Hm:***

$$\sqrt[4]{(1+x)\cdot1\cdot1\cdot1}>\frac{4x+4}{3x+4},x>-1$$

***Then, for*** $x=\frac{k^3}{n^4}$ ***we have:***

$$\sum_{k=1}^{n}\frac{\frac{k^3}{n^4}}{3\cdot\frac{k^3}{n^4}+4}<\sum_{k=1}^{n}\left(\sqrt[4]{1+\frac{k^3}{n^4}}-1\right)<\sum_{k=1}^{n}\frac{k^3}{4n^4}$$

***i)*** $\lim_{n\to\infty}\frac{1}{4}\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{k^3}{4n^4}=\frac{1}{4}\int_0^1 x^3dx=\frac{1}{16}$

***ii)*** $0<\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{4n^4}-\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{k^3}{n^4}}{3\cdot\frac{k^3}{n^4}+4}=\lim_{n\to\infty}\frac{3k^6}{16n^8+3k^3\cdot4n^4}$

***but*** $\lim_{n\to\infty}\frac{3k^6}{16n^8+3k^3\cdot 4n^4} < \lim_{n\to\infty}\frac{3k^6}{16n^8} = \lim_{n\to\infty}\frac{3}{16}\cdot\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{k^6}{n^6}$

$$= \lim_{n\to\infty}\frac{1}{16}\cdot\frac{1}{n}\int_0^1 x^6dx = 0$$

***Which means*** $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{k^3}{n^4}}{3\cdot\frac{k^3}{n^4}+4} = \frac{1}{16}$

***By squeeze theorem:***

$$\Omega = -\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt[4]{1+\frac{k^3}{n^4}}-1\right) = -\frac{1}{16}$$

***Solution 3 by Rajeev Rastogi-India***

$$\lim_{n\to\infty}\left\{n-\left[\left(1+\frac{1^3}{n^4}\right)^{\frac{1}{4}}+\left(1+\frac{2^3}{n^4}\right)^{\frac{1}{4}}+\left(1+\frac{3^3}{n^4}\right)^{\frac{1}{4}}+\cdots+\left(1+\frac{n^3}{n^4}\right)^{\frac{1}{4}}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{n-\left[\left(1+\frac{1}{4}\cdot\frac{1^3}{n^4}+\cdots\right)+\left(1+\frac{1}{4}\cdot\frac{2^3}{n^4}+\cdots\right)+\cdots+\left(1+\frac{1}{4}\cdot\frac{n^3}{n^4}+\cdots\right)\right]\right\}$$

$$= \lim_{n\to\infty}\left\{n-n-\frac{1}{4}\left(\frac{1^3+2^3+3^3+\cdots+n^3}{n^4}\right)\right\}$$

$$= -\frac{1}{4}\lim_{n\to\infty}\frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} = -\frac{1}{16}$$

**991. Find:**

$$\Omega = \lim_{x\to 0}\left(\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x(\sin(\cos^{-1}x)-1)}\right)$$

***Proposed by Qusay Yousef-Amman-Jordan***

***Solution 1 by Igor Soposki-Skopje-Macedonia***

$$\Omega = \lim_{x\to 0}\left(\frac{\sin(\sin 2x - \sin x) - \sin(\tan 2x - \tan x)}{x(\sin(\cos^{-1}x) - 1)}\right)$$

$$= \lim_{x\to 0}\frac{2\cos\frac{\sin 2x - \sin x + \tan 2x - \tan x}{2}\cdot \sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{x(\sin(\cos^{-1}x) - 1)}$$

$$= 2\lim_{x\to 0}\frac{\sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}\cdot\frac{1}{2}\cdot\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x(\sqrt{1 - x^2} - 1)}$$

$$= \lim_{x\to 0}\frac{\sin 2x\left(1 - \frac{1}{\cos 2x}\right) - \sin x\left(1 - \frac{1}{\cos x}\right)}{x(\sqrt{1 - x^2} - 1)}\cdot\frac{\sqrt{1 - x^2} - 1}{\sqrt{1 - x^2} - 1}$$

$$= 2\lim_{x\to 0}\frac{\sin 2x\left(\frac{\cos 2x - 1}{\cos 2x}\right) - \sin x\left(\frac{\cos x - 1}{\cos x}\right)}{x(1 - x^2 - 1)}$$

$$= -2\lim_{x\to 0}\frac{\sin 2x\left(-\frac{2\sin^2 x}{\cos 2x}\right) - \sin x\left(-\frac{2\sin^2\frac{x}{2}}{\cos x}\right)}{x^3}$$

$$= 4\lim_{x\to 0}\frac{2\sin x\cos x\sin^3 x - 2\sin\frac{x}{2}\cos\frac{x}{2}\frac{\sin^2\frac{x}{2}}{\cos x}}{x^3}$$

$$= 8\left(\lim_{x\to 0}\frac{\cos x\sin^3 x}{x^3} - \lim_{x\to 0}\frac{\cos^3\frac{x}{2}}{\cos x}\cdot\frac{\sin^3\frac{x}{2}}{\left(\frac{x}{2}\right)^3\cdot 8}\right) = 7$$

## Solution 2 by Rajeev Rastogi-India

$$\Omega = \lim_{x\to 0}\left(\frac{\sin(\sin 2x - \sin x) - \sin(\tan 2x - \tan x)}{x(\sin(\cos^{-1}x) - 1)}\right)$$

$$=\lim_{x\to 0}\frac{2\cos\left(\frac{\sin 2x-\sin x+\tan 2x-\tan x}{2}\right)\cdot\sin\left(\frac{\sin 2x-\tan 2x+\tan x-\sin x}{2}\right)}{x\left(\sqrt{1-x^2}-1\right)}$$

$$=2\lim_{x\to 0}\frac{\left(\frac{\sin 2x-\tan 2x+\tan x-\sin x}{2}\right)\left(\sqrt{1-x^2}-1\right)}{x(1-x^2-1)}$$

$$=-2\lim_{x\to 0}\left(\frac{\sin 2x-\tan 2x}{x^3}-\frac{\tan x-\sin x}{x^3}\right)$$

$$=-2\lim_{x\to 0}\left(-\frac{\sin 2x}{x}\left(\frac{1-\cos 2x}{x^2}\right)\frac{1}{\cos 2x}+\frac{\sin x}{x}\left(\frac{1-\cos 2x}{x^2}\right)\frac{1}{\cos x}\right)=7$$

***Solution 3 by Yen Tung Chung-Taichung-Taiwan***

$$\Omega=\lim_{x\to 0}\left(\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x(\sin(\cos^{-1}x)-1)}\right)$$

$$=\lim_{x\to 0}\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x\left(\sqrt{1-x^2}-1\right)}$$

$$=\lim_{x\to 0}\frac{\sin\left(\left(2x-\frac{(2x)^3}{3!}+O(x^5)\right)-\left(x-\frac{x^3}{3!}+O(x^5)\right)\right)}{x\left(1-\frac{x^2}{2}+O(x^4)-1\right)}-$$

$$-\lim_{x\to 0}\frac{\sin\left(\left(2x+\frac{(2x)^3}{3!}+O(x^5)\right)-\left(x+\frac{x^3}{3!}+O(x^5)\right)\right)}{x\left(1-\frac{x^2}{2}+O(x^4)-1\right)}$$

$$=\lim_{x\to 0}\frac{\sin\left(x-\frac{7}{6}x^3+O(x^5)\right)-\sin\left(x+\frac{7}{3}x^3+O(x^5)\right)}{-\frac{1}{2}x^3+O(x^5)}$$

$$=\lim_{x\to 0}\frac{\left\{\left(x-\frac{7}{6}x^3+O(x^5)\right)-\frac{1}{3!}\left(x-\frac{7}{6}x^3+O(x^5)\right)^3+\cdots\right\}}{-\frac{1}{2}x^3+O(x^5)}-$$

$$-\lim_{x\to 0}\frac{\left\{\left(x+\frac{7}{3}x^3+O(x^5)\right)-\frac{1}{3!}\left(x+\frac{7}{3}x^3+O(x^5)\right)^3+\cdots\right\}}{-\frac{1}{2}x^3+O(x^5)}$$

$$=\lim_{x\to 0}\frac{\left(-\frac{7}{6}-\frac{1}{6}-\frac{7}{3}+\frac{1}{6}\right)x^3+O(x^5)}{-\frac{1}{2}x^3+O(x^5)}=7$$

**992. Find:**

$$\Omega(a,b)=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^n,a,b>0$$

***Proposed by Jalil Hajimir-Toronto-Canada***

***Solution 1 by Samir HajAli-Damascus-Syria***

*Let put* $\frac{\sqrt[n]{a}-1}{b}=m$ *then* $\sqrt[n]{a}=bm+1$ *thus* $\frac{1}{n}log(a)=log(bm+1)$

$$n=\frac{log(a)}{log(bm+1)}$$

*Now:* $\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^n=\lim_{m\to 0}(1+m)^{\frac{log(a)}{log(bm+1)}}=\lim_{m\to 0}(1+m)^{\frac{1}{m}\cdot\frac{mlog(a)}{log(bm+1)}}$

$$=exp\left(\lim_{m\to 0}\frac{bm}{log(bm+1)}\cdot\frac{log(a)}{b}\right)=exp\left(\frac{log(a)}{b}\right)=\sqrt[b]{a}$$

***Solution 2 by Adrian Popa-Romania***

$$\Omega(a,b)=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^n=\lim_{n\to\infty}\left(\underbrace{\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^{\frac{b}{\sqrt[n]{a}-1}}}_{\to e}\right)^{\frac{n(\sqrt[n]{a}-1)}{b}}=e^{\frac{log(a)}{b}}=\sqrt[b]{a}$$

*where*

$$\lim_{n\to\infty}\frac{n(\sqrt[n]{a}-1)}{b}=\lim_{n\to\infty}\frac{n(a^{\frac{1}{n}}-1)}{b}=\frac{1}{b}\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}=\frac{log(a)}{b}$$

***Solution 3 by Adil Abdullayev-Baku-Azerbaijan***

*Lema 1:* $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

***Lema 2:*** $\lim_{x\to 0}\frac{a^x-1}{x} = log(a)$

$$\Omega(a,b) = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^{\frac{b}{\sqrt[n]{a}-1}}\right)^{\frac{n(\sqrt[n]{a}-1)}{b}} =$$

$$= e^{\lim_{n\to\infty}\frac{1}{b}\cdot\frac{(\sqrt[n]{a}-1)}{\frac{1}{n}}} = e^{\frac{1}{b}log(a)} = a^{\frac{1}{b}}$$

**993. Prove that:**

$$\lim_{n\to\infty}\frac{(n+1)^{n^2}n!}{n^{n(n+1)}\sqrt{n}} = \sqrt{\frac{2\pi}{e}}$$

***Proposed by Srinivasa Raghava-AIRMC-India***

***Solution 1 by Naren Bhandari-Bajura-Nepal***

***We use Stirling approximation for*** $n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$

$$\lim_{n\to\infty}\frac{(n+1)^{n^2}n!}{n^{n(n+1)}\sqrt{n}} = \lim_{n\to\infty}\frac{(n+1)^{n^2}}{n^{n^2}\sqrt{n}}\cdot\frac{\sqrt{2\pi n}n^n}{n^n e^n} = \lim_{n\to\infty}\frac{(n+1)^{n^2}}{n^{n^2}}\cdot\frac{\sqrt{2\pi}}{e^n}$$

***Further using the property:*** $e^{log(x)} = x, x > 0$ ***we have:***

$$\lim_{n\to\infty} exp\left(n^2 log\left(1+\frac{1}{n}\right) - log\left(\frac{1}{e^n}\right)\right) = exp\left(\lim_{n\to\infty} n^2\left(\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)-n\right)\right) = \frac{1}{\sqrt{e}}$$

***And on combining above two result we have limit:***

$$\lim_{n\to\infty}\frac{(n+1)^{n^2}n!}{n^{n(n+1)}\sqrt{n}} = \sqrt{\frac{2\pi}{e}}$$

***Solution 2 by Kamel Benaicha-Algiers-Algerie***

$$\Omega = \lim_{n\to\infty}\frac{(n+1)^{n^2}n!}{n^{n(n+1)}\sqrt{n}} = \lim_{x\to\infty}\left(\frac{n+1}{n}\right)^{n^2}\frac{n^{n+\frac{1}{2}}\sqrt{2\pi}e^{-n}}{n^{n+\frac{1}{2}}} = \sqrt{2\pi}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}e^{-n}$$

$$= \sqrt{2\pi}\lim_{n\to\infty} e^{n^2 log\left(1+\frac{1}{n}\right)-n} = \sqrt{2\pi}\lim_{n\to\infty} e^{n^2\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)-n} = \sqrt{\frac{2\pi}{e}}$$

**994.** $f:(0,\infty)\to\mathbb{R}, f$ **−derivable,** $f(1) = 2,$

$(1 + 2x^2 log2)f(x) + xf'(x) = 1, \forall x > 0$. **Find:**

$$\Omega = \lim_{n\to\infty}(nf(n))$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Ali Jaffal-Lebanon***

***Let (E):*** $(1 + (2log2)x^2)y + xy' = 1$ ***and (H):*** $(1 + (2log2)x^2)y + xy' = 0$

***We will solve the equation (H):***

***We have:*** $xy' = -(1 + (2log2)x^2)y \Leftrightarrow \frac{y'}{y} = -\frac{1}{x} - (2lo2)x$

$$log|x| = -logx - (log2)x^2 + C = log\left(\frac{2^{-x^2}}{x}\right) + C$$

***Then*** $y = \frac{2^{-x^2}}{x} \cdot k$ ***is the general solution of (H) were k is constant.***

***So, we can take*** $y = \psi(x) \cdot u(x)$ ***as solution of (E) so that*** $u(x) = \frac{2^{-x^2}}{x}$***then***

$y' = \psi'(x) \cdot u(x) + \psi(x)u'(x)$ ***but*** $(1 + (2log2)x^2)y + xy' = 1$ ***then***

$$(1 + (2log2)x^2)\,\psi(x) \cdot u(x) + x\psi'(x) \cdot u(x) + x\psi(x) \cdot u'(x) = 1$$

***We know*** $(1 + (2log2)x^2)u(x) + xu'(x) = 0$

***So,*** $x\psi'(x)u(x) = 1$ ***then*** $\psi'(x) = 2^{x^2} \Rightarrow y = \frac{2^{-x^2}}{x}\left(\int 2^{x^2}dx + C\right)$ ***where C is constant.***

***But*** $f(x)$ ***is a solution of (E) and*** $f(1) = 2$ ***so,*** $f(x) = \frac{2^{-x^2}}{x}\left(\int 2^{x^2}dx + 4\right)$

$$\lim_{x\to\infty}\frac{\int_1^x 2^{t^2}dt}{2^{x^2}} \overset{L'H}{=} \lim_{x\to\infty}\frac{2^{x^2}}{(2log2)\cdot x\cdot 2^{x^2}} = 0$$

***So,*** $\lim_{x\to\infty} xf(x) = 0$ ***then*** $\lim_{n\to\infty} nf(n) = 0$

***Solution 2 by Tran Hong-Dong Thap-Vietnam***

$$g(x) = e^{\varphi(x)}f(x) \Rightarrow g'(x) = e^{\varphi(x)}[\varphi'(x)f(x) + f'(x)]$$

***So,*** $(1 + 2x^2log2)f(x) + xf'(x) = 1 \Leftrightarrow \frac{1+2x^2log2}{x} \cdot f(x) + f'(x) = \frac{1}{x}$

$$\Rightarrow \varphi'(x) = \frac{1}{x} + 2xlog2 \Rightarrow \varphi(x) = logx + x^2log2$$

$$\Rightarrow g'(x) = \left(f(x)\cdot e^{logx+x^2log2}\right)' = \frac{1}{x} = (logx)';\ x > 0$$

$$\Rightarrow f(x)\cdot e^{logx+x^2} = logx + C; C = const.$$

$$f(1) = 2 \Rightarrow C = 4$$

$$\Rightarrow f(x) = \frac{logx+4}{e^{logx+x^2log2}} = \frac{logx+4}{x\cdot 2^{x^2}} \Rightarrow f(n) = \frac{logn+4}{n\cdot 2^{n^2}}$$

$$\Rightarrow \lim_{n\to\infty} nf(n) = \lim_{n\to\infty}\frac{logn+4}{n\cdot 2^{n^2}} \overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{2n\cdot 2^{n^2}log2} = \lim_{n\to\infty}\frac{1}{n^2\cdot 2^{n^2+1}log2} = 0$$

## Solution 3 by Kamel Benaicha-Algiers-Algerie

**Let (E):** $(1+(2log2)x^2)f(x)+xf'(x)=1$ **and (H):** $(1+(2log2)x^2)f(x)+xf'(x)=0$

$$(H) \Leftrightarrow \frac{f'(x)}{f(x)} = -\frac{1+(2log2)x^2}{x} \Leftrightarrow log|f(x)| = -log|x| - (log2)x^2 + C$$

$$f(x) = A(x)\frac{e^{-(log2)x^2}}{x} = A(x)\frac{2^{-x^2}}{x}$$

$$f'(x) = A(x)\frac{-(2log2)x^2 2^{-x^2} - 2^{-x^2}}{x^2} + A'(x)\frac{2^{-x^2}}{x}$$

$$= -\frac{1}{x}f(x)(1+(2log2)x^2) + A'(x)\frac{2^{-x^2}}{x}$$

$$A'(x) = 2^{x^2} \Rightarrow A(x) = \frac{1}{2}\sqrt{\frac{\pi}{log2}}erfi(x\sqrt{log2}) + K$$

**So:** $f(x) = \frac{2^{-x^2}}{2x}\sqrt{\frac{\pi}{log2}}erfi(x\sqrt{log2}) + K\frac{2^{-x^2}}{2x}$

$$f(1) = 1 \Rightarrow K = 4 - \frac{1}{2}\sqrt{\frac{\pi}{log2}}erfi(x\sqrt{log2})$$

$$\lim_{n\to\infty} nf(n) = \sqrt{\frac{\pi}{log2}}\lim_{n\to\infty}\left(\frac{erfi(n\sqrt{log2})}{2^{n^2+1}} + \frac{K}{2^{n^2}}\right) = \lim_{n\to\infty}\frac{1}{(2log2)n} = 0$$

**995. Find:**

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\left(\frac{1}{\sqrt{1+x^2}}log\left(\frac{cos^3\left(tan^{-1}(x+\varepsilon)\right)}{sin^3\left(tan^{-1}(x+\varepsilon)\right)}\right)\right)dx\right)$$

***Proposed by Abdul Mukhtar-Nigeria***

***Solution1 by Kamel Benaicha-Algeirs-Algerie***

$$\Omega = \lim_{\varepsilon\to 0_+}\left(\int_{\varepsilon}^{1}\left(\frac{1}{\sqrt{1+x^2}}log\left(\frac{cos^3\left(tan^{-1}(x+\varepsilon)\right)}{sin^3\left(tan^{-1}(x+\varepsilon)\right)}\right)\right)dx\right)$$

$$= -\lim_{\varepsilon\to 0_+}\left(\int_{\varepsilon}^{1}\left(\frac{1}{\sqrt{1+x^2}}log\left(cot^3\left(tan^{-1}(x+\varepsilon)\right)\right)\right)dx\right)$$

$$= -3\lim_{\varepsilon\to 0_+}\left(\int_{\varepsilon}^{1}\left(\frac{1}{\sqrt{1+x^2}}log(x+\varepsilon)\right)dx\right)$$

$$= -3\lim_{\varepsilon\to 0_+}\left(\int_{0}^{\varepsilon}\left(\frac{log(x+\varepsilon)}{\sqrt{1+x^2}}\right)dx\right) - 3\lim_{\varepsilon\to 0_+}\left(\int_{0}^{1}\left(\frac{log(x+\varepsilon)}{\sqrt{1+x^2}}\right)dx\right)$$

$$\frac{log(x+\varepsilon)}{\sqrt{1+x^2}} \le \frac{log(1+x)}{\sqrt{1+x^2}} \qquad (1)$$

$f(x) = \frac{log(1+x)}{\sqrt{1+x^2}}$ ***is continuous function in*** $[0;\varepsilon]$

***Put*** $F(x)$ ***the primitive function of f.***

**So,** $\lim_{\varepsilon\to 0_+}\left(\int_0^{\varepsilon}\left(\frac{log(x+\varepsilon)}{\sqrt{1+x^2}}\right)dx\right) \le \lim_{\varepsilon\to 0_+}\left(F(\varepsilon)-F(0)\right) = 0$

$$\Omega = -3\int_0^1\left(\frac{log(x)}{\sqrt{1+x^2}}\right)dx = 3 - 3\int_0^1\left(\frac{(xlog(x)-x)x}{(1+x^2)\sqrt{1+x^2}}\right)dx = 3\int_0^1\frac{log\left(x+\sqrt{1+x^2}\right)}{x}dx$$

***Put:*** $t = log\left(x+\sqrt{1+x^2}\right) \Rightarrow x = sinh(t), dx = cosh(t)dt$

$$\therefore \Omega = 3\int_0^{sin^{-1}h(1)} tcot\left(h(t)\right)dt \overset{IBP}{=} -3\int_0^{sin^{-1}h(1)} log\left(sinh(t)\right)dt$$

$$= -3\int_0^{\sin^{-1}h(1)} \log\left(\frac{1-e^{-2t}}{2e^{-t}}\right)dt$$

$$= 3\log 2\cdot \sin^{-1}h(1) - 3\int_0^{\sin^{-1}h(1)} t\,dt + 3\sum_{n=1}^{+\infty}\frac{1}{n}\cdot\int_0^{\sin^{-1}h(1)} e^{-2nt}dt$$

$$= 3\log 2\cdot \sin^{-1}h(1) - \frac{3}{2}\sin^{-1}h^2(1) - \frac{3}{2}\cdot\sum_{n=1}^{+\infty}\frac{e^{-2\sin^{-1}h(1)}-1}{n^2}$$

$$= 3\log 2\cdot\log(1+\sqrt{2}) - \frac{3}{2}\log^2(1+\sqrt{2}) - \frac{3}{2}Li_2\left(e^{-2\log(1+\sqrt{2})}\right) + \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4} + 3\log 2\cdot\log(1+\sqrt{2}) - \frac{3}{2}\log^2(1+\sqrt{2}) - \frac{3}{2}Li_2\left(\frac{1}{3+2\sqrt{2}}\right)$$

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_\varepsilon^1\left(\frac{1}{\sqrt{1+x^2}}\log\left(\frac{\cos^3\left(\tan^{-1}(x+\varepsilon)\right)}{\sin^3\left(\tan^{-1}(x+\varepsilon)\right)}\right)\right)dx\right)$$

$$= \frac{\pi^2}{4} - \frac{3}{2}\left(Li_2\left(\frac{1}{3+2\sqrt{2}}\right) - \log 2\cdot\log(3+2\sqrt{2}) + \log^2(1+\sqrt{2})\right)$$

***Solution 2 by Precious Itsuokor-Nigeria***

$$\Omega = \int_\varepsilon^1\left(\frac{1}{\sqrt{1+x^2}}\log\left(\cot^3\left(\tan^{-1}(x+\varepsilon)\right)\right)\right)dx$$

$$= -3\int_0^1\frac{\log(x)}{\sqrt{1+x^2}}dx \overset{x=\sinh(y)}{=} -3\int_0^{\log(1+\sqrt{2})}\log(\sinh(y))dy$$

$$= -3\int_0^{\log(1+\sqrt{2})}\log\left(\frac{e^{2x}-1}{2e^x}\right)dx = -3\int_0^{\log(1+\sqrt{2})}\log\left(\frac{1-e^{-2x}}{2e^{-x}}\right)dx$$

$$= -3\int_0^{\log(1+\sqrt{2})}\log(1-e^{-2x})dx + 3\log 2\log(1+\sqrt{2}) - \frac{3\log^2(1+\sqrt{2})}{2}$$

$$= 3\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\log(1+\sqrt{2})} e^{-2nx}dx + 3\log 2\log(1+\sqrt{2}) - \frac{3\log^2(1+\sqrt{2})}{2}$$

$$= -3\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{2n}\left(\frac{1}{(1+\sqrt{2})^{2n}} - 1\right)\right] + 3\log 2\log(1+\sqrt{2}) - \frac{3\log^2(1+\sqrt{2})}{2}$$

$$= -\frac{3}{2}Li_2\left(\frac{1}{3+2\sqrt{2}}\right) + \frac{3}{2}\zeta(2) + 3\log 2\log(1+\sqrt{2}) - \frac{3\log^2(1+\sqrt{2})}{2}$$

**996. Find:**

$$\Omega = \lim_{n\to\infty}\left(\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)\right)^{\frac{1}{\sin(\gamma - H_n + \log n)}}$$

*Proposed by Daniel Sitaru-Romania*

*Solution 1 by Adrian Popa-Romania*

*We know that:* $\lim_{n\to\infty}(H_n - \log n) = \gamma$

$$\Omega = \lim_{n\to\infty}\left(\left(1 + \tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right) - 1\right)^{\frac{1}{\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right) - 1}}\right)^{\frac{\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right) - 1}{\sin(\gamma - H_n + \log n)}}$$

$$\therefore \tan\left(\frac{\pi}{4} + \alpha\right) = \frac{\tan\frac{\pi}{4} + \tan\alpha}{1 - \tan\frac{\pi}{4}\cdot\tan\alpha} = \frac{1 + \tan\alpha}{1 - \tan\alpha}$$

$$\therefore \frac{1 + \tan\alpha}{1 - \tan\alpha} - 1 = \frac{2\tan\alpha}{1 - \tan\alpha}$$

*Let:* $\gamma - H_n + \log n = \alpha$

$$\lim_{\alpha\to 0}\frac{2\tan\alpha}{(1 - \tan\alpha)\sin\alpha} = \lim_{\alpha\to 0}\frac{2}{(1 - \tan\alpha)\cos\alpha} = 2 \Rightarrow \Omega = e^2$$

*Solution 2 by Khaled Abd Imouti-Damascus-Syria*

$$H_n = \log n + \gamma + \frac{1}{2n} + \frac{\varepsilon_n}{n^2}$$

$(\varepsilon_n)_{n\geq 1}$ – is a bounded sequence

So: $\gamma - H_n + \log n = -\frac{1}{2n} - \frac{\varepsilon_n}{n^2}$

$$u_n = \left(\tan\left(-\frac{1}{2n} - \frac{\varepsilon_n}{n^2} + \frac{\pi}{4}\right)\right)^{\frac{1}{\sin\left(-\frac{1}{2n}-\frac{\varepsilon_n}{n^2}\right)}} = \left(\tan\left(-\frac{1}{2n} - \frac{\varepsilon_n}{n^2} + \frac{\pi}{4}\right)\right)^{\frac{-1}{\sin\left(\frac{1}{2n}+\frac{\varepsilon_n}{n^2}\right)}}$$

$$= \left(\left[1 + \tan\left(-\frac{1}{2n} - \frac{\varepsilon_n}{n^2} + \frac{\pi}{4}\right) - 1\right]^{\frac{1}{\tan\left(-\frac{1}{2n}-\frac{\varepsilon_n}{n^2}+\frac{\pi}{4}\right)-1}}\right)^{\frac{-\tan\left(-\frac{1}{2n}-\frac{\varepsilon_n}{n^2}+\frac{\pi}{4}\right)-1}{\sin\left(\frac{1}{2n}+\frac{\varepsilon_n}{n^2}\right)}}$$

$$v_n = \frac{\tan\left(\frac{1}{2n} + \frac{\varepsilon_n}{n^2} + \frac{\pi}{4}\right) - 1}{\sin\left(\frac{1}{2n} + \frac{\varepsilon_n}{n^2}\right)} = \frac{\tan\left(t_n + \frac{\pi}{4}\right) - 1}{\sin t_n}$$

$$= \frac{\frac{\tan t_n}{1 - \tan t_n} - 1}{\sin t_n} = \frac{2\tan t_n}{\sin t_n(1 - \tan t_n)} = 2 \cdot \frac{\frac{\tan t_n}{t_n}}{\frac{\sin t_n}{t_n}} \cdot \frac{1}{1 - \tan t_n}$$

$$\lim_{n\to\infty} v_n = 2 \Rightarrow \lim_{n\to\infty} u_n = e^2$$

## Solution 3 by Sergio Esteban-Argentina

$$\Omega = \lim_{n\to\infty}\left(\tan\left(\alpha + \frac{\pi}{4}\right)\right)^{\frac{1}{\sin\alpha}}, \text{ where } \alpha = \gamma - H_n + \log n$$

$$\Omega = \lim_{n\to\infty}\left(\tan\left(\alpha + \frac{\pi}{4}\right)\right)^{\frac{1}{\sin\alpha}} = \lim_{\alpha\to 0}\left(1 + \tan\left(\alpha + \frac{\pi}{4}\right) - 1\right)^{\frac{1}{\sin\alpha}}$$

$$= \lim_{\alpha\to 0}\left(1 + \frac{1}{\frac{1}{\tan\left(\alpha + \frac{\pi}{4}\right) - 1}}\right)^{\frac{1}{\tan\left(\alpha+\frac{\pi}{4}\right)-1}\cdot\frac{\tan\left(\alpha+\frac{\pi}{4}\right)-1}{\sin\alpha}}$$

$$= e^{\lim_{\alpha\to 0}\frac{\tan\left(\alpha+\frac{\pi}{4}\right)-1}{\sin\alpha}} \overset{L'H}{=} e^{\lim_{\alpha\to\infty}\frac{\sec^2\left(\alpha+\frac{\pi}{4}\right)}{\cos\alpha}} = e^2$$

$$\Omega = e^2$$

**Solution 4 Surjeet Singhania-India**

$$\text{Let: } f(n) = \left(\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)\right)^{\frac{1}{\sin(\gamma - H_n + \log n)}}$$

$$\log f(n) = \frac{\log\left(\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)\right)}{\sin(\gamma - H_n + \log n)}$$

$$\frac{d}{dn}H_n = \frac{d}{dn}(\psi(n+1) + \gamma) = \psi'(n+1)$$

$$\psi(n+1) = \frac{1}{n} + \psi(n)$$

$$\psi'(n+1) = -\frac{1}{n^2} + \psi'(n) = -\frac{1}{n^2} - \frac{1}{(n-1)^2} - \frac{1}{(n-2)^2} - \cdots - \frac{1}{1^2} + \psi'(1)$$

$$= -H_n^{(2)} + \psi'(1) = -H_n^{(2)} - \int_0^1 \frac{\log(x)}{1-x}dx = -H_n^{(2)} + \zeta(2)$$

$$\frac{d}{dn}\log\left(\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)\right) =$$

$$= \frac{\sec^2\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)}{\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)} \cdot \left(\frac{1}{n} + H_n^{(2)} - \zeta(2)\right)$$

*Also*

$$\frac{d}{dn}\sin(\gamma - H_n + \log n) = \cos(\gamma - H_n + \log n) \cdot \left(\frac{1}{n} + H_n^{(2)} - \zeta(2)\right)$$

*Hence*

$$\lim_{n\to\infty} \frac{\sec^2\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)}{\tan\left(\gamma - H_n + \frac{\pi}{4} + \log n\right)} \cdot \frac{\left(\frac{1}{n} + H_n^{(2)} - \zeta(2)\right)}{\cos(\gamma - H_n + \log n) \cdot \left(\frac{1}{n} + H_n^{(2)} - \zeta(2)\right)}$$

$$= \frac{\sec^2\frac{\pi}{4}}{\tan\frac{\pi}{4}} = 2$$

$$\Omega = e^2$$

**997. Find:**

$$\Omega = \lim_{x\to\infty}\left(xe^{x^2}\left(\int_0^x e^{-t^2}dt - \int_0^\infty e^{-t^2}\,dt\right)\right)$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Daniel Sitaru-Romania*

$$f:[0,\infty)\to\mathbb{R}, f(t) = e^{-t^2}, F:[0,\infty)\to\mathbb{R}, F'(t) = f(t)$$

$$\lim_{x\to\infty} xe^{x^2}\left(\int_0^x e^{-t^2}dt - \int_0^\infty e^{-t^2}dt\right) = \lim_{x\to\infty}\frac{\int_0^x e^{-t^2}dt - \int_0^\infty e^{-t^2}dt}{\frac{1}{xe^{x^2}}} =$$

$$= \lim_{x\to\infty}\frac{F(x) - F(0) - \frac{\sqrt{\pi}}{2}}{\frac{1}{xe^{x^2}}} = \lim_{x\to\infty}\frac{F'(x)}{\frac{-e^{x^2} - 2x^2e^{x^2}}{x^2e^{2x^2}}} =$$

$$= \lim_{x\to\infty}\frac{f(x)}{\frac{-1-2x^2}{x^2e^{x^2}}} = \lim_{x\to\infty}\frac{e^{-x^2}\cdot x^2e^{x^2}}{-1-2x^2} = \lim_{x\to\infty}\frac{x^2}{-1-2x^2} = -\frac{1}{2}$$

**998. Determine the values of $a, n$ if:**

$$\lim_{x\to 0}\frac{cos\left(1-\frac{sinx}{x}\right) + cos\left(2-\frac{sin2x}{x}\right) + \cdots + cos\left(k-\frac{sinkx}{x}\right) - k}{ax^n} = 1$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Yen Tung-Taichung-Taiwan*

Since: $cos\left(m-\frac{\sin mx}{x}\right) = cos\left(m - \frac{mx-\frac{1}{3!}(mx)^3+O(x^5)}{x}\right) = cos\left(\frac{m^3}{3!}x^2 + O(x^4)\right)$

$$= 1 - \frac{\left(\frac{m^3}{3!}x^2 + O(x^4)\right)^2}{2!} + \cdots = 1 - \frac{m^6}{(3!)^2 2!}x^4 + O(x^6)$$

$$\lim_{x\to 0}\frac{cos\left(1-\frac{sinx}{x}\right)+cos\left(2-\frac{sin2x}{x}\right)+\cdots+cos\left(k-\frac{sinkx}{x}\right)-k}{ax^n}$$

$$=\lim_{x\to 0}\frac{\sum_{m=1}^{n}\left(1-\frac{m^6}{(3!)^2 2!}x^4+O(x^6)\right)-k}{ax^n}=\lim_{x\to 0}\frac{-\frac{1}{72}\sum_{m=1}^{k}m^6x^4+O(x^6)}{ax^n}=1$$

***We have*** $n=4$ ***and*** $-\frac{1}{72a}\sum_{m=1}^{k}m^6=1\Rightarrow a=-\frac{72}{\sum_{m=1}^{k}m^6}$

**999.**

$$\Omega(a)=\int_0^a\frac{\text{sinht}\cdot\text{cost}}{(\text{sint}+\text{cosht})(\text{sinht}+\text{cost})}\text{dt}\,,\,a>0$$

**Find:**

$$\Omega=\lim_{x\to 0}(2\Omega(x))^x$$

***Proposed by Daniel Sitaru-Romania***

***Solution 1 by Ali Jaffal-Lebanon***

$$\Omega(x)=\int_0^x\frac{sinht\cdot cost}{(sint+cosht)(sinht+cost)}dt$$

***So:*** $lim_{x\to 0}\Omega(x)=0$

$$\lim_{x\to 0}xlog(2\Omega(x))=\lim_{x\to 0}\left(xlog2+xlog\Omega(x)\right)=0+\lim_{x\to 0}\left(xlog\Omega(x)\right)$$

***But*** $lim_{x\to 0}\left(xlog\Omega(x)\right)=lim_{x\to 0}\frac{log\Omega(x)}{\frac{1}{x}}\overset{L'H}{=}lim_{x\to 0}\frac{-x^2}{\Omega(x)}\cdot\Omega'(x)$

***We know that:*** $\Omega'(x)=\frac{sinhx\cdot cosx}{(sinx+coshx)(sinhx+cosx)}$ ***then*** $lim_{x\to 0}\Omega'(x)=0$ ***and***

$$\lim_{x\to 0}\frac{-x^2}{\Omega(x)}=\lim_{x\to 0}\frac{-2x}{\Omega'(x)}=\lim_{x\to 0}\frac{-2}{\Omega''(x)}=-2$$

***Then:*** $lim_{x\to 0}\left(xlog\Omega(x)\right)=0$ ***therefore*** $\Omega=lim_{x\to 0}(2\Omega(x))^x=1$

***Solution 2 by Igor Soposki-Skopje-Macedonia***

$$\Omega(a)=\int_0^a \frac{sinht\cdot cost}{(sint+cosht)(sinht+cost)}dt$$

$$=\frac{1}{2}\int_0^a \frac{2sinht\cdot cost}{(sint+cosht)(sinht+cost)}dt$$

$$=\frac{1}{2}\int_0^a \frac{sin^2t+cos^2t-(cos^2ht-sin^2ht)+2sinht\cdot cost}{(sint+cosht)(sinht+cost)}dt$$

$$=\frac{1}{2}\int_0^a \frac{sin^2ht+2sinht\cdot cost+cos^2t+(sin^2t-cos^2ht)}{(sint+cosht)(sinht+cost)}dt$$

$$=\frac{1}{2}\int_0^a \frac{(cost+sinht)^2+(sint-cosht)(sint+cosht)}{(sint+cosht)(sinht+cost)}dt$$

$$=\frac{1}{2}\int_0^a \left(\underbrace{\frac{cost+sinht}{sint+cosht}}_{x=sint+cosht}+\underbrace{\frac{sint-cosht}{sinht+cost}}_{x=sinht+cost}\right)dt$$

$$=\frac{1}{2}\left(log(sint+cosht)-log(sinht+cost)\right)\Big|_0^a$$

$$=\frac{1}{2}\left(log\left(\frac{sina+cosha}{cosa+sinha}\right)-\underbrace{log\left(\frac{sin0+cosh0}{cos0+sinh0}\right)}_{=0}\right)$$

$$=\frac{1}{2}\,log\left(\frac{sina+cosha}{cosa+sinha}\right)\Rightarrow\Omega(x)=\frac{1}{2}\,log\left(\frac{sinx+coshx}{cosx+sinhx=}\right)$$

$$\Omega=\lim_{x\to 0}\big(2\Omega(x)\big)^x=\lim_{x\to 0}\left(log\left(\frac{sinx+coshx}{cosx+sinhx}\right)\right)^x=e^{\lim_{x\to 0}xlog\left(log\left(\frac{sinx+coshx}{cosx+sinhx}\right)\right)}=e^{L_1}$$

$$L_1=\lim_{x\to 0}xlog\left(log\left(\frac{sinx+coshx}{cosx+sinhx}\right)\right)=\lim_{x\to 0}\frac{log\left(log\left(\frac{sinx+coshx}{cosx+sinhx}\right)\right)}{\frac{1}{x}}$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{1}{log\left(\frac{sinx+coshx}{cosx+sinhx}\right)}\cdot\frac{1}{\frac{sinx+coshx}{cosx+sinhx}}\cdot\frac{2cosxsinhx}{(cosx+sinhx)^2}}{-\frac{1}{x^2}}$$

$$= -\lim_{x\to 0}\frac{2x^2cosxsinhx}{(sinx+coshx)(cosx+sinhx)log\left(\frac{sinx+coshx}{cosx+sinhx}\right)} \overset{L'H}{=}$$

$$-2\lim_{x\to 0}\frac{2xcosxsinhx+x^2coshxcosx-x^2sinxsinhx}{((cosx+sinhx)^2+(cos^2hx-sin^2x))log\left(\frac{sinx+coshx}{cosx+sinhx}\right)+2sinhxcosx}$$

$$\overset{L'H}{=} -2\lim_{x\to 0}\frac{cosxsinhx}{cosxcoshx} = 0$$

$$\Omega = \lim_{x\to 0}(2\Omega(x))^x = e^{L_1} = 1$$

***Solution 3 by Tran Hong-Dong Thap-Vietnam***

$$\Omega(a) = \int_0^a \frac{sinht\cdot cost}{(sint+cosht)(sinht+cost)}dt, a > 0$$

$$\Omega'(a) = \frac{sinha\cdot cosa}{(sina+cosha)(sinha+cosa)} \sim \frac{a}{(1+a)^2} = \frac{1}{1+a} - \frac{1}{(1+a)^2}$$

$$= (1-a) - (1-a)(1-a) + o(a), a \to 0$$

$$\therefore \frac{a}{(1+a)^2} = a + o(a), \Omega(0) = 0$$

***So:*** $\Omega(a) = \frac{1}{2}a^2 + o(a^2)$

***Then:*** $\Omega = lim_{x\to 0}(2\Omega(x))^x = e^{lim_{x\to 0}xlog(2\Omega(x))} = e^{lim_{x\to 0}2xlog|x|} = 1$

**1000.**

$$\Omega_k(m) = 2\lim_{x\to 0}\left(\frac{1-(\cos kx)^{\frac{1}{k^{m+2}}}}{x^2}\right), k, m \in \mathbb{N}^*$$

**Find a closed form for:**

$$\Omega = \left(\sum_{k=1}^{\infty}\Omega_k(2)\right)\left(\sum_{k=1}^{\infty}\Omega_k(3)\right)$$

*Proposed by Daniel Sitaru-Romania*

*Solution 1 by Adrian Popa-Romania*

$$\Omega_k(m) = 2\lim_{x\to 0}\frac{-\frac{1}{k^{m+2}}\cdot(\cos kx)^{\frac{1}{k^{m+2}}-1}\cdot(-k)\cdot\sin kx}{2x} = \frac{k^2}{k^{m+2}} = \frac{1}{k^m}$$

$$\Omega = \left(\sum_{k=1}^{\infty}\Omega_k(2)\right)\left(\sum_{k=1}^{\infty}\Omega_k(3)\right) = \sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(2)\zeta(3)$$

*Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo*

$$\Omega_k(m) = 2\lim_{x\to 0}\frac{1-\cos kx}{x^2\cdot\sum_{n=0}^{k^{m+2}}\left((\cos nx)^{\frac{1}{k^{m+2}}}\right)^n} = 2k^2\cdot\underbrace{\lim_{x\to 0}\frac{1-\cos kx}{(kx)^2}}_{\frac{1}{2}}\cdot\frac{1}{\underbrace{1+1+\cdots+1}_{k^{m+2}}}$$

$$= 2k^2\cdot\frac{1}{2k^{m+2}} = \frac{1}{k^m}$$

$$\sum_{k=1}^{\infty}\Omega_k(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{\infty}\Omega_k(3) = \sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{\pi^2}{6}\cdot\zeta(3)$$

*Solution 3 by Kamel Benaicha-Algeirs-Algerie*

$$\Omega_k(m) = 2\lim_{x\to 0}\left(\frac{1-(\cos kx)^{\frac{1}{k^{m+2}}}}{x^2}\right)$$

$$= \frac{1}{k^{m+1}}\lim_{x\to 0}\left(\frac{\sin(kx)}{x}\cdot\left(\cos(kx)\right)^{\frac{1}{k^{m+2}}-1}\right) = \frac{1}{k^m}$$

***So:***

$$\Omega = \left(\sum_{k=1}^{\infty}\Omega_k(2)\right)\left(\sum_{k=1}^{\infty}\Omega_k(3)\right) = \sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(2)\zeta(3) = \frac{\pi^2\zeta(3)}{6}$$

***It's nice to be important but more important it's to be nice.***

***At this paper works a TEAM.***

***This is RMM TEAM.***

***To be continued!***

***Daniel Sitaru***