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In ΔABC the following relationship holds:

$$\frac{r_a+r}{r_a-r}+\frac{r_b+r}{r_b-r}+\frac{r_c+r}{r_c-r}\le\sqrt{\frac{2R}{r}}\sum\frac{h_a}{s_a}\sqrt{\frac{h_a}{r_a}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_a+r}{r_a-r}=\sum\frac{\frac{rs}{s-a}+\frac{rs}{s}}{\frac{rs}{s-a}-\frac{rs}{s}}=\sum\frac{\frac{b+c}{s(s-a)}}{\frac{a}{s(s-a)}}\overset{(1)}{=}\sum\frac{b+c}{a}$$

$$\text{Tsintsifas}\Rightarrow m_a\le\frac{b^2+c^2}{2bc}w_a\Rightarrow\frac{2bc}{b^2+c^2}m_a\le w_a\Rightarrow s_a\le w_a\text{ and analogs}$$

$$\Rightarrow\sqrt{\frac{2R}{r}}\sum\frac{h_a}{s_a}\sqrt{\frac{h_a}{r_a}}$$

$$\ge\sqrt{\frac{2R}{r}}\sum\left(\frac{2rs}{aw_a}\sqrt{\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}.\,s\tan\frac{A}{2}}}\right)=\sum\frac{2rs(b+c)}{a.\,2bc\sin\frac{A}{2}\cos\frac{A}{2}}=\sum\frac{2rs(b+c)}{4Rrs\sin A}$$

$$=\sum\frac{2R.\,2rs(b+c)}{4Rrsa}=\sum\frac{b+c}{a}\overset{\text{by }(1)}{=}\sum\frac{r_a+r}{r_a-r}$$

$$\therefore\ \frac{r_a+r}{r_a-r}+\frac{r_b+r}{r_b-r}+\frac{r_c+r}{r_c-r}\le\sqrt{\frac{2R}{r}}\sum\frac{h_a}{s_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(Proved)}$$