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In ΔABC the following relationship holds:

$$\left(3+\sum\frac{h_b+h_c}{h_a}\right)\sum AI\ \geq \frac{2}{3}\left(\sum(m_a+8w_a)\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\{(b+c)w_a\}=\sum\left\{(b+c)\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\right\}=\sum\frac{2abc\cos\frac{A}{2}}{a}=\sum\frac{8Rrs\cos\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=2s\sum\frac{r}{\sin\frac{A}{2}}=2s\sum AI$$

$$\Rightarrow\sum AI\overset{(1)}{=}\frac{1}{2s}\sum\{(b+c)w_a\}$$

WLOG, we may assume $a\geq b\geq c\ \therefore\ w_a\leq w_b\leq w_c$ and $b+c\leq c+a\leq a+b$

$\therefore$ using Chebyshev, we get

$$\frac{1}{2s}\sum\{(b+c)w_a\}\geq\frac{1}{6s}\left\{\sum(b+c)\right\}\left(\sum w_a\right)=\frac{4s}{6s}\sum w_a$$

$$=\frac{2}{3}\sum w_a\overset{\text{by }(1)}{\Rightarrow}\sum AI\overset{(2)}{\geq}\frac{2}{3}\sum w_a$$

Now, $3+\sum\frac{h_b+h_c}{h_a}=3+\sum\frac{ca+ab}{bc}=3+\sum\left(\frac{a}{b}+\frac{a}{c}\right)=3+\sum\left(\frac{b}{c}+\frac{c}{b}\right)$

$$=3+\sum\frac{b^2+c^2}{bc}\overset{\text{Tsintsifas}}{\geq}3+\sum\frac{2m_a}{w_a}$$

$$\therefore 3+\sum\frac{h_b+h_c}{h_a}\overset{(3)}{\geq}3+\sum\frac{2m_a}{w_a}\ \therefore (2),(3)$$

$$\Rightarrow \boxed{LHS\geq\frac{2}{3}\left(3+\sum\frac{2m_a}{w_a}\right)\sum w_a}\overset{?}{\geq}\frac{2}{3}\left(\sum(m_a+8w_a)\right)$$

$$\Leftrightarrow 3\sum w_a+\sum w_a\sum\frac{2m_a}{w_a}\overset{?}{\geq}\sum m_a+8\sum w_a$$

$$\Leftrightarrow \sum w_a\sum\frac{2m_a}{w_a}-6\sum w_a\overset{?}{\geq}\sum m_a-\sum w_a$$

$$\Leftrightarrow \left(\sum w_a\right)\sum\left(\frac{2m_a}{w_a}-2\right)\overset{?}{\geq}\sum(m_a-w_a)$$

$$\Leftrightarrow \left(\sum w_a\right)\left\{\frac{2(m_a-w_a)}{w_a}+\frac{2(m_b-w_b)}{w_b}+\frac{2(m_c-w_c)}{w_c}\right\}\overset{?}{\geq}(m_a-w_a)+(m_b-w_b)+(m_c-w_c)$$

$$\Leftrightarrow (m_a-w_a)\left(\frac{2\sum w_a}{w_a}-1\right)+(m_b-w_b)\left(\frac{2\sum w_a}{w_b}-1\right)+(m_c-w_c)\left(\frac{2\sum w_a}{w_c}-1\right)\overset{?}{\geq}0$$

$$\Leftrightarrow (m_a-w_a)\left(\frac{w_a+2w_b+2w_c}{w_a}\right)+(m_b-w_b)\left(\frac{w_b+2w_c+2w_a}{w_b}\right)+(m_c-w_c)\left(\frac{w_c+2w_a+2w_b}{w_c}\right)\overset{?}{\geq}0\to\text{true}$$

$$\therefore \left(3+\sum\frac{h_b+h_c}{h_a}\right)\sum AI\ \geq\frac{2}{3}\left(\sum(m_a+8w_a)\right)\ \text{(Proved)}$$