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**In $\Delta ABC$ the following relationship holds:**

$$1+\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\le\frac{2R(h_a+h_b+h_c)}{r(r_a+r_b+r_c)}$$

*Proposed by Bogdan Fuştei -Romania*

*Solution 1 by Rahim Shahbazov-Baku-Azerbaijan; Solution 2 by Soumava Chakraborty-Kolkata-India*

*Solution 1 by Rahim Shahbazov-Baku-Azerbaijan*

$$r_a+r_b+r_c=4R+r \text{ and } h_a+h_b+h_c=\frac{s^2+r^2+4Rr}{2R} \quad (1)$$

$$\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\le\frac{s^2}{4Rr+r^2} \quad (2)$$

$$l_a\ge s_a\Rightarrow\frac{m_a}{l_a}\le\frac{b^2+c^2}{2bc}$$

$$\overset{(2)}{\Rightarrow}\frac{2s^2}{4Rr+r^2}\ge\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$$

*Let: $a=x+y; b=y+z; c=x+z$ inequality becomes:*

$$\frac{2(x+y+z)^2}{xy+yz+zx} \geq 2(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)-3$$

$$\frac{1}{2}+\frac{x^2+y^2+z^2}{xy+yz+zx} \geq \frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} \text{ true from } \frac{1}{x}+\frac{1}{y} \geq \frac{4}{x+y}$$

*Solution 2 by Soumava Chakraborty-Kolkata-India*

$$By\ Tsintsifas, LHS \leq 1+\frac{1}{2}\sum\frac{b^2+c^2}{bc} = 1+\frac{\sum ab(2s-c)}{8Rrs} = 1+\frac{s^2-2Rr+r^2}{4Rr}$$

$$= \frac{s^2+2Rr+r^2}{4Rr} \Rightarrow LHS \overset{(1)}{\leq} \frac{s^2+2Rr+r^2}{4Rr}$$

$$Now, RHS = \frac{2R\left(\sum\frac{bc}{2R}\right)}{r(4R+r)} \overset{(2)}{=} \frac{s^2+4Rr+r^2}{r(4R+r)} \therefore (1),(2) \Rightarrow it\ suffices\ to\ prove$$

$$: \frac{s^2+2Rr+r^2}{4Rr} \leq \frac{s^2+4Rr+r^2}{r(4R+r)}$$

$$\Leftrightarrow r(4R+r)s^2+r(4R+r)(2Rr+r^2) \leq 4Rrs^2+4Rr(4Rr+r^2) \Leftrightarrow r^2s^2$$

$$\leq (4Rr+r^2)(2Rr-r^2) \Leftrightarrow s^2 \overset{(3)}{\leq} 8R^2-2Rr-r^2$$

$$Now, s^2 \overset{Gerretsen}{\leq} 4R^2+4Rr+3r^2 \overset{?}{\leq} 8R^2-2Rr-r^2 \Leftrightarrow 2R^2-3Rr-2r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2R+r) \overset{?}{\geq} 0 \rightarrow true \because R \overset{Euler}{\geq} 2r$$

$$\Rightarrow (3)\ is\ true \therefore, 1+\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c} \leq \frac{2R(h_a+h_b+h_c)}{r(r_a+r_b+r_c)}\ (Proved)$$

**Note by editor:**

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