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**In $\Delta ABC, m(\sphericalangle BAC) = 90°, AD \perp, D \in (BC), I, I_1, I_2$- incenters in $\Delta ABC, \Delta ABD$ respectively $\Delta ACD$. Prove that:**

$$[AI_1I_2] = \frac{rh_a}{2}$$

*Proposed by Mehmet Şahin-Ankara-Turkey*

*Solution 1 by Tran Hong-Dong Thap-Vietnam, Solution 2 by Thanasis Gakopoulos-Larisa-Greece*

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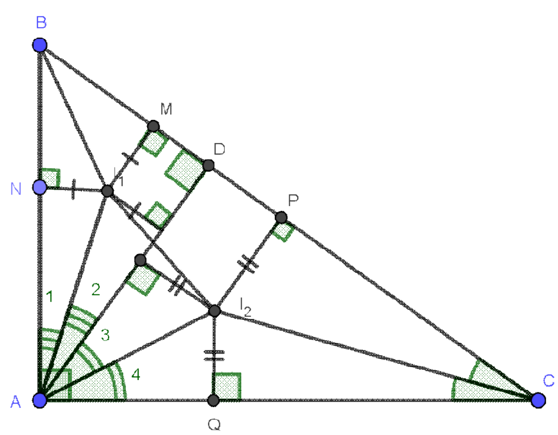



$$[AI_1I_2] = [AI_1DI_2] - [DI_1I_2]$$

$$DI_1 \perp DI_2 \Rightarrow DI_1 = r_1\sqrt{2}, DI_2 = r_2\sqrt{2}$$

$$\Rightarrow [DI_1I_2]=\frac{1}{2}\cdot DI_1\cdot DI_2=r_1r_2=$$

$$=\frac{1}{4}[AD^2+BD\cdot CD-(AB+AC-BC)AD+AB\cdot AC-(BD\cdot AC+AB\cdot CD)]$$

$$=\frac{1}{4}\left[2h_a^2-(b+c-a)h_a+ah_a-\frac{bc}{a}(b+c)\right]$$

$$=\frac{1}{4}[h_a^2-2rh_a-h_a(b+c-a)]=\frac{1}{4}[2h_a^2-4rh_a]$$

$$[AI_1DI_2]=[AI_1D]+[AI_2D]$$

$$=\frac{1}{2}r_a\cdot AD+\frac{1}{2}\cdot r_2\cdot AD=\frac{1}{2}(r_1+r_2)\cdot AD=\frac{1}{2}(r_1+r_2)h_a$$

$$=\frac{1}{2}\cdot\frac{1}{2}(2AD+BC-AB-AC)h_a$$

$$=\frac{1}{4}\big(2h_a-(b+c-a)\big)h_a=\frac{1}{4}(2h_a-2r)h_a$$

$$=\frac{1}{2}(h_a-r)h_a=\frac{1}{2}(h_a^2-rh_a)$$

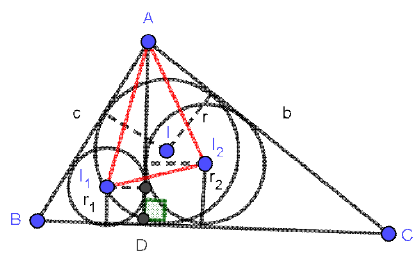
**($\because$ Because:** $r_1=\frac{1}{2}(AD+BD-AB);r_2=\frac{1}{2}(AD+CD-AC);r=\frac{1}{2}(AB+AC-BC)$**)**

**Hence,**

$$[AI_1I_2]=\frac{1}{2}(h_a^2-rh_a)-\frac{1}{4}(2h_a^2-4rh_a)=\frac{rh_a}{2}$$

**Proved.**

## Solution 2 by Thanasis Gakopoulos-Larisa-Greece




$$AB=c,AC=b,BC=\sqrt{b^2+c^2},BD=\frac{c^2}{\sqrt{b^2+c^2}},CD=\frac{b^2}{\sqrt{b^2+c^2}}$$

$$AD=\frac{bc}{\sqrt{b^2+c^2}}=d,r_1=\frac{BD\cdot AD}{BD+AD+AB}\to r_1=\frac{bc^2}{\left(b+c+\sqrt{b^2+c^2}\right)\cdot\sqrt{b^2+c^2}}$$

$$r_2=\frac{b^2c}{\left(b+c+\sqrt{b^2+c^2}\right)\sqrt{b^2+c^2}},\ r=\frac{AB\cdot AC}{AB+AC+BC}\to r=\frac{bc}{b+c+\sqrt{b^2+c^2}}$$

$$[AI_1I_2]=\frac{1}{2}\left\|\begin{vmatrix}1 & 1 & 1\\ 0 & -r_1 & r_2\\ d & r_1 & r_2\end{vmatrix}\right\|=\frac{1}{2}[2r_1r_2-d(r_1+r_2)]\ \text{(“PLAGIOGONAL” system)}$$

$$[AI_1I_2]-\frac{r\cdot h_a}{2}=\frac{1}{2}\left[2\frac{b^3c^3}{(b^2+c^2)\left(b+c+\sqrt{b^2+c^2}\right)}-\frac{bc}{\sqrt{b^2+c^2}}\left(\frac{bc^2+b^2c}{\sqrt{b^2+c^2}\cdot\left(b+c+\sqrt{b^2+c^2}\right)}+\frac{bc}{b+c+\sqrt{b^2+c^2}}\right)\right]=0\to$$

$$\to[AI_1I_2]=\frac{rh_a}{q}$$

$$[AI_1I_2]-\frac{rh_a}{2}=\frac{1}{2}[2r_1r_2-h_a(r_1+r_2+r)]=0\to[AI_1I_2]=\frac{rh_a}{2}$$