Generating inequalities using Schweitzer's theorem

Daniel Sitaru and Claudia Nănuţi

In 1914 P. Schweitzer published a theorem (see [1]) that later featured in the 1978 Russian Olympiad. Romanian mathematician Daniel Culea has since proposed several applications of Schweitzer's theorem [2]. In this article, we will present other applications of this theorem.

Theorem. (Kantorovic) If $p_k \in (0,\infty); k \in 1\ldots n; x_k \in \mathbb{R}; 0 < m \le x_k \le M$ then

$$\Big(\sum_{k=1}^{n} p_k x_k\Big)\Big(\sum_{k=1}^{n} \frac{p_k}{x_k}\Big) \le \frac{(m+M)^2}{4mM}\Big(\sum_{k=1}^{n} p_k\Big)^2 - \frac{(m-M)^2}{4mM} \cdot \min_A \Big(\sum_{i\in A} p_i - \sum_{j\in B p_j} p_j\Big)^2,$$

where $A \cup B = \{1,2,\cdots,n\}; A\cap B = \emptyset$

Proof. From $(x_k - m)(x_k - M) \le 0$ we obtain successively:

$$x_k^2 - (m+M)x_k + mM \le 0,$$
$$x_k + \frac{mM}{x_k} \le m + M,$$
$$\frac{mM}{x_k} \le m + M - x_k,$$
$$\frac{1}{x_k} \le \frac{m+M-x_k}{mM},$$
$$\frac{p_k}{x_k} \le \frac{(m+M)p_k - p_k x_k}{mM}, \quad k \in 1\ldots n,$$
$$\sum_{k=1}^{n} \frac{p_k}{x_k} \le \frac{1}{mM}\sum_{k=1}^{n}\Big((m+M)p_k - p_k x_k\Big),$$

$$\Big(\sum_{k=1}^{n} p_k x_k\Big)\Big(\sum_{k=1}^{n} \frac{p_k}{x_k}\Big) \le \Big(\sum_{k=1}^{n} p_k x_k\Big)\frac{1}{mM}\Big((m+M)\sum_{k=1}^{n} p_k - \sum_{k=1}^{n} p_k x_k\Big). \qquad (1)$$

We compute the maximum value of the right-hand side from (1). Let $x_i = m, i \in A, x_j = M, j \in B, A\cap B = \emptyset$, $A \cap B = \{1,2,...,n\}$ and $\alpha = \sum_{i\in A} p_i; \beta = \sum_{j \in B} p_j$. Then

$$\frac{1}{mM}\Big(m\sum_{i\in B} p_i\Big)\Big[(m+M)\Big(\sum_{i\in A} p_i + \sum_{j\in B} p_j\Big) - m\sum_{i\in A} p_i - M\sum_{j\in B} p_j\Big]$$
$$= \Big(m\sum_{i\in A} p_i + M\sum_{j\in B} p_j\Big)\Big(\frac{\sum_{i\in A} p_i}{m} + \frac{\sum_{j\in B} p_j}{M}\Big)$$
$$= (m\alpha + M\beta)\Big(\frac{\alpha}{m} + \frac{\beta}{M}\Big) = \frac{(2m\beta + 2M\alpha)(2m\alpha + 2M\beta)}{4mM} =$$

$$= \frac{(m\alpha+m\beta+M\alpha+M\beta-m\alpha+m\beta+M\alpha-M\beta)(m\alpha+m\beta+M\alpha+M\beta+m\alpha-m\beta-M\alpha+M\beta)}{4mM}$$
$$= \frac{[(m+M)(\alpha+\beta)-(m-M)(\alpha-\beta)][(m+m)(\alpha+\beta)+(m-M)(\alpha-\beta)]}{4mM}$$
$$= \frac{(m+M)^2(\alpha+\beta)^2-(m-M)^2(\alpha-\beta)^2}{4mM}$$
$$= \frac{(m+M)^2(\alpha+\beta)^2}{4mM} - \frac{(m-M)^2(\alpha-\beta)^2}{4mM}.$$

The maximum value of the right-hand side from (1) is obtained when $(\alpha-\beta)^2$ is minimum, namely when

$$\min_A\Big(\sum_{i\in A} p_i - \sum_{j\in B} p_j\Big), \quad A\cup B = \{1,2,...,n\}, \quad A\cap B = \emptyset.$$

□

Theorem. (Schweitzer) If $x_k \in \mathbb{R}; k \in 1\dots n$ and $0 < m \le x_k \le M$ then

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2 n^2}{4mM} - \frac{(m-M)^2[1+(-1)^{n+1}]}{8mM}.$$

Proof. In the Kantorovic theorem, let the weights be $p_k = 1, k \in 1\dots n, x = |A|, n-x = |B|$. It follows that:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2 n^2}{4mM} - \frac{(m-M)^2}{4mM}\cdot \min[4(x^2-nx)+n^2]$$

and

$$\min(4x^2-4nx) = -\frac{16n^2}{16} = -n^2.$$

The minimum value is reached for $x = \frac{n}{2}$ if n is even when $\min(4x^2-4nx+n^2) = 0$.

If n is odd, the minimum is reached when $x = \frac{n-1}{2} < \frac{n}{2}$.

$$\begin{aligned}\min[4(x^2-nx)+n^2] &= 4\Big[\Big(\frac{n-1}{2}\Big)^2 - n\frac{n-1}{2}\Big] + n^2\\ &= 4\Big(\frac{n^2-2n+1-2n^2+2n}{4}\Big) + n^2\\ &= -n^2+1+n^2 = 1.\end{aligned}$$

For n even:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2 n^2}{4mM}.$$

For n odd:

$$\Big(\sum_{k=1}^n x_k\Big)\Big(\sum_{k=1}^n \frac{1}{x_k}\Big) \le \frac{(m+M)^2 n^2}{4mM} - \frac{(m-M)^2}{4mM}.$$

For $n = 2, 3, 4$ the inequality becomes, respectively,

$$(x_1 + x_2)\Big(\frac{1}{x_1} + \frac{1}{x_2}\Big) \leq \frac{(m+M)^2}{mM},$$

$$(x_1 + x_2 + x_3)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\Big) \leq \frac{(m+M)^2 \cdot 9 - (m-M)^2}{4mM} = 5 + 2\Big(\frac{m}{M} + \frac{M}{m}\Big),$$

$$(x_1 + x_2 + x_3 + x_4)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\Big) \leq \frac{(m+M)^2 \cdot 16}{4mM} = 8 + 4\Big(\frac{m}{M} + \frac{M}{m}\Big).$$

Let $0 < a \leq b, m = a, M = b$ and $x_1, x_2, x_3, x_4 \in [a, b]$. The inequality is

$$(x_1 + x_2)\Big(\frac{1}{x_1} + \frac{1}{x_2}\Big) \leq \frac{(a+b)^2}{ab}, \tag{2}$$

$$(x_1 + x_2 + x_3)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\Big) \leq 5 + 2\Big(\frac{a}{b} + \frac{b}{a}\Big), \tag{3}$$

$$(x_1 + x_2 + x_3 + x_4)\Big(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\Big) \leq 8 + 4\Big(\frac{a}{b} + \frac{b}{a}\Big). \tag{4}$$

The following inequality is well known:

$$0 < a \leq \sqrt{\frac{2a^2b^2}{a^2+b^2}} \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq b \tag{5}$$

□

Problem. Prove that if $x, y, z, t \in [a, b], 0 < a \leq b$ then:

$$\frac{x+y+z+t}{\sqrt{xy} + \sqrt{yz} + \sqrt{zt} + \sqrt{tx}} \leq \frac{a+b}{2\sqrt{ab}}.$$

Proof. From (2) for $m = a, M = b, x_1 = x, x_2 = y$ it follows that

$$(x+y)\Big(\frac{1}{x} + \frac{1}{y}\Big) \leq \frac{(a+b)^2}{ab},$$
$$\frac{(x+y)^2}{xy} \leq \frac{(a+b)^2}{ab}$$
$$ab(x+y)^2 \leq xy(a+b)^2,$$
$$(x+y)\sqrt{ab} \leq \sqrt{xy}(a+b).$$

Analogously, $(y+z)\sqrt{ab} \leq \sqrt{yz}(a+b)$, $(z+t)\sqrt{ab} \leq \sqrt{zt}(a+b)$ and $(t+x)\sqrt{ab} \leq \sqrt{tx}(a+b)$ and by adding

$$2(x+y+z+t)\sqrt{ab} \leq (a+b)(\sqrt{xy} + \sqrt{yz} + \sqrt{zt} + \sqrt{tx}),$$

and we obtain the result. □

Problem. In triangle ABC, let $M, N, P \in [BC]$. Prove that

$$\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\Big) \le \frac{5}{3}+\frac{2}{3}\Big(\frac{AB}{AC}+\frac{AC}{AB}\Big).$$

Proof. WLOG we assume that $AB < AC$. In (3) we take $m = AB, M = AC$ and then $AM, AN, AP \in [m, M]$. Let be $x_1 = AM, x_2 = AN, x_3 = AP$. Then

$$(AM+AN+AP)\Big(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\Big) \le 5+2\Big(\frac{AB}{AC}+\frac{AC}{AB}\Big)$$

From AM-GM inequality, we obtain:

$$AM+AN+AP \ge 3\sqrt[3]{AM \cdot AN \cdot AP}.$$

It follows that

$$3\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\Big) \le 5+2\Big(\frac{AB}{AC}+\frac{AC}{AB}\Big),$$
$$\sqrt[3]{AM \cdot AN \cdot AP}\Big(\frac{1}{AM}+\frac{1}{AN}+\frac{1}{AP}\Big) \le \frac{5}{3}+\frac{2}{3}\Big(\frac{AB}{AC}+\frac{AC}{AB}\Big).$$

□

Problem. Prove that if $0 < a \le b$ then

$$(a+\sqrt{ab}+\frac{a+b}{2}+b)\Big(\frac{1}{a}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}+\frac{1}{b}\Big) \le 8+4\Big(\frac{a}{b}+\frac{b}{a}\Big), \tag{6}$$

$$\Big(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}+\sqrt{\frac{2}{a^2+b^2}}\Big) \le 8+4\Big(\frac{a}{b}+\frac{b}{a}\Big). \tag{7}$$

Proof. For (6), in (4) take $m = a, M = b, x_1 = a, x_2 = \sqrt{ab}, x_3 = \frac{a+b}{2}, x_4 = b$.

For (7), in (4) take $m = a, M = b, x_1 = \frac{2ab}{a+b}, x_2 = \sqrt{ab}, x_3 = \frac{a+b}{2}, x_4 = \sqrt{\frac{a^2+b^2}{2}}$.

References.

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