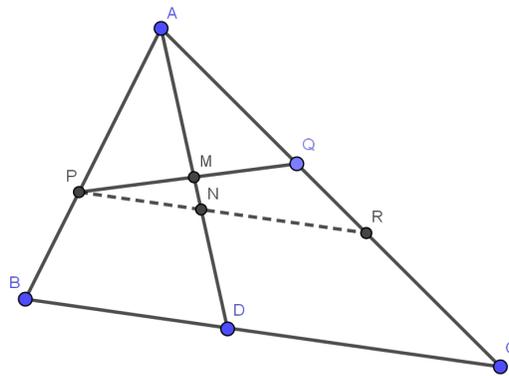


GAKOPOULOS' LEMMA (II)

THANASIS GAKOPOULOS - LARISA - GREECE

(A) LEMMA (I)

$$(I) \quad \frac{\frac{MP}{MQ}}{\frac{AP}{AQ}} = \frac{\frac{DB}{DC}}{\frac{AB}{AC}}$$



Proof.

Let $PR \parallel BC$.

$$(1) \quad \text{Is: } \frac{NP}{NR} = \frac{DB}{DC}$$

$$(2) \quad \frac{AP}{AR} = \frac{AB}{AC} \quad (\text{Thales's theorem})$$

Is: (Menelaus's theorem $\Delta PQR, \overline{AMN}$):

$$\frac{NP}{NR} \cdot \frac{MQ}{MP} \cdot \frac{AR}{AQ} = 1 \xrightarrow{(1)} \frac{DB}{DC} \cdot \frac{MQ}{MP} \cdot \frac{AR}{AQ} = 1 \xrightarrow{(2)} \frac{DB}{DC} \cdot \frac{MQ}{MP} \cdot \frac{1}{AQ} \cdot \frac{AC}{AB} \cdot AP = 1 \rightarrow$$

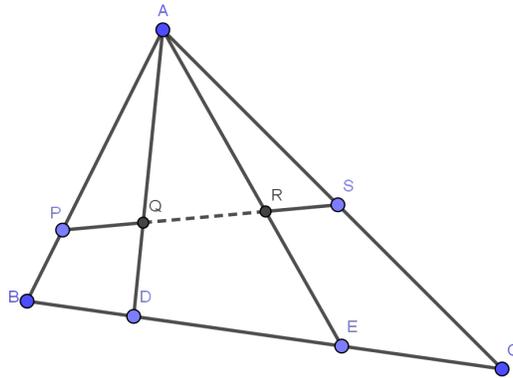
$$\rightarrow \frac{\frac{MP}{MQ}}{\frac{AP}{AQ}} = \frac{\frac{DB}{DC}}{\frac{AB}{AC}}$$

Note: the lemma (I) has been proved (with PLAGIOGONAL system) and so is known in my previous paper, which has been published in RMM group. \square

(B) LEMMA (II)

Let: $\frac{PQ}{BD} = a, \frac{RS}{EC} = b, \frac{AP}{AB} = p, \frac{AQ}{AD} = q, \frac{AR}{AE} = r, \frac{AS}{AC} = s.$

$$(II) \quad \frac{a}{b} = \frac{p \cdot q}{r \cdot s}$$



Proof.

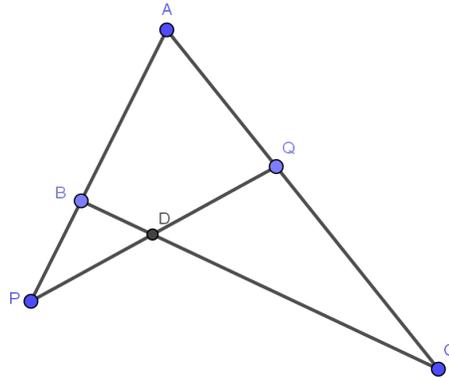
$$\left. \begin{array}{l} \Delta ABE \text{ (lemma I): } \frac{PQ}{AP} = \frac{BD}{AB} \\ \Delta ADC \text{ (lemma I): } \frac{QR}{AS} = \frac{DE}{AC} \end{array} \right\} \begin{array}{l} (\times) \frac{PQ}{AP} \cdot \frac{QR}{AS} = \frac{BD}{AB} \cdot \frac{DE}{AC} \rightarrow \frac{PQ}{BD} = \frac{AP}{AB} \cdot \frac{AQ}{AD} \rightarrow \\ \rightarrow \frac{a}{b} = \frac{p \cdot q}{r \cdot s} \end{array}$$

□

(C) APPLICATIONS. Relationships that hold to any COMPLETE QUADRILATERAL.

(C1)

$$(C1) \quad \frac{DB \cdot DQ}{AB \cdot AQ} = \frac{DP \cdot DC}{AP \cdot AC}$$



Proof.

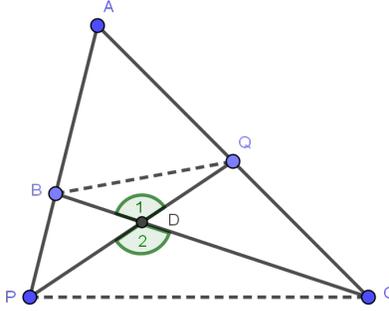
From lemma (I), if $M \equiv D$ then we have:

$$\frac{DB}{DC} = \frac{DP}{DQ} \rightarrow \frac{DB \cdot DQ}{AB \cdot AQ} = \frac{DP \cdot DC}{AP \cdot AC}$$

□

(C2)

$$(C2) \quad \frac{[DBQ]}{[ABQ]} = \frac{[DPC]}{[APC]}$$



There is a harmonious relationship between the triangle's areas.

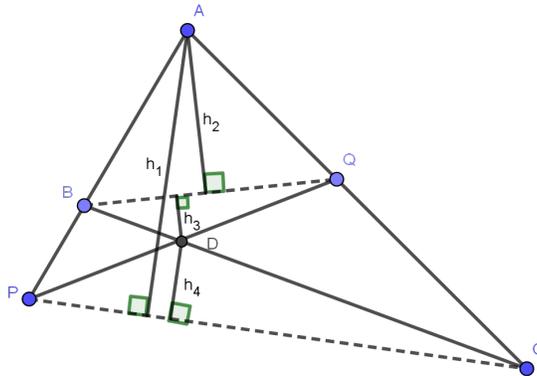
Proof.

$$\text{From (C1): } \frac{DB \cdot DQ}{AB \cdot AQ} = \frac{DP \cdot DC}{AP \cdot AC} \rightarrow \frac{DB \cdot DQ \cdot \sin D_1}{AB \cdot AQ \cdot \sin A} = \frac{DP \cdot DC \cdot \sin D_2}{AP \cdot AC \cdot \sin A} \rightarrow \frac{[DBQ]}{[ABQ]} = \frac{[DPC]}{[APC]} \quad \square$$

(C3)

$$(C3) \quad h_1 \cdot h_3 = h_2 \cdot h_4$$

There is a harmonious relationship between the segments' lengths.

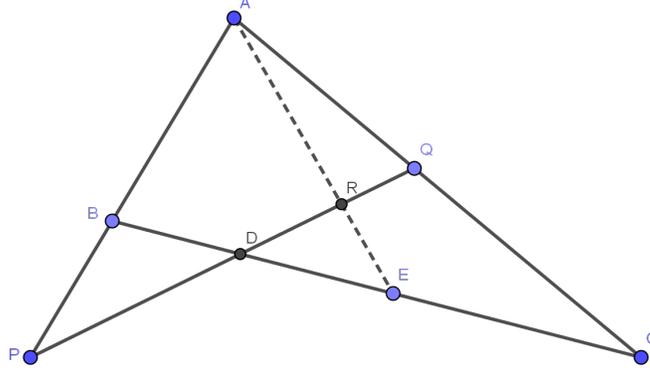


Proof.

$$\text{From (C2): } \frac{[DBQ]}{[ABQ]} = \frac{[DPC]}{[APC]} \rightarrow \frac{BQ \cdot h_3}{BQ \cdot h_2} = \frac{PC \cdot h_4}{PC \cdot h_1} \rightarrow h_1 \cdot h_3 = h_2 \cdot h_4 \quad \square$$

(C4)

$$(C4) \quad \frac{BD \cdot RQ}{PD \cdot EC} = \frac{AB \cdot AR \cdot AQ}{AP \cdot AE \cdot AC}$$



Proof.

From lemma (II), we have: $\left(\frac{a}{b} = \frac{p \cdot q}{r \cdot s}\right) \rightarrow \frac{\frac{PD}{BD}}{\frac{RQ}{EC}} = \frac{\left(\frac{AP}{AB}\right) \cdot \left(\frac{AD}{AQ}\right)}{\left(\frac{AR}{AE}\right) \cdot \left(\frac{AC}{AC}\right)} \rightarrow \frac{BD \cdot RQ}{PD \cdot EC} = \frac{AB \cdot AR \cdot AQ}{AP \cdot AE \cdot AC}$ \square

(C5)

$$(C5.1) \quad \frac{\frac{DR}{DE}}{\frac{QR}{CE}} \cdot \frac{AQ}{AC} = \frac{\frac{DF}{DS}}{\frac{BF}{PS}} \cdot \frac{AB}{AP} = 1$$

$$(C5.2) \quad \frac{BF \cdot QR}{PS \cdot CE} = \frac{AB \cdot AF \cdot AQ \cdot AR}{AP \cdot AS \cdot AC \cdot AE}$$

Proof.

From lemma (I): $\Delta ADC : \frac{\frac{DR}{QR}}{\frac{AD}{AQ}} = \frac{\frac{DE}{CE}}{\frac{AD}{AC}} \rightarrow \frac{DR}{QR} \cdot \frac{AQ}{AC} = 1$
 From lemma (I): $\Delta ADP : \frac{\frac{DF}{BF}}{\frac{AD}{AB}} = \frac{\frac{PS}{AS}}{\frac{AD}{AP}} \rightarrow \frac{DF}{BF} \cdot \frac{AB}{AP} = 1$ $\left. \vphantom{\begin{matrix} \text{From lemma (I):} \\ \text{From lemma (I):} \end{matrix}} \right\} \rightarrow \frac{DR}{QR} \cdot \frac{AQ}{AC} = \frac{DF}{BF} \cdot \frac{AB}{AP} = 1$ \square

Proof.

From lemma (II): $\left(\frac{a}{b} = \frac{p \cdot q}{r \cdot s}\right) \rightarrow \frac{\frac{PS}{BF}}{\frac{RQ}{EC}} = \frac{\left(\frac{AP}{AB}\right) \cdot \left(\frac{AS}{AF}\right)}{\left(\frac{AR}{AE}\right) \cdot \left(\frac{AQ}{AC}\right)} \rightarrow \frac{BF \cdot QR}{PS \cdot CE} = \frac{AB \cdot AF \cdot AQ \cdot AR}{AP \cdot AS \cdot AC \cdot AE}$ \square

(D) Complete Quadrilateral $OAF - OBE$
 Cyclic Quadrilateral $ABCD$

$$(D1) \quad \frac{AG \cdot BH}{DE \cdot CF} = \frac{OG \cdot OH}{OE \cdot OF}$$

$$(D2) \quad \frac{AR \cdot CQ}{DP \cdot BS} = \frac{OR \cdot OQ}{OP \cdot OS}$$

