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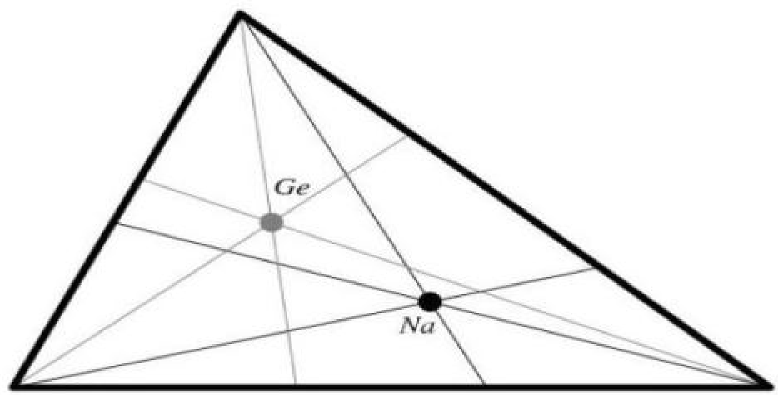
ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(IV)

By Bogdan Fuştei-Romania



Note by Editor: *The article is written as a story of discovery triangle inequalities. The author give us a detailed mind process of these discoveries. I consider it an innovative and outstanding method to show results to readers.*

In any ΔABC we have: $m_a^2 = \frac{2(b^2+c^2)-a^2}{4}$ (and analogs)

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \text{ (and analogs)}$$

It's easy to prove that:

$$m_a^2 = s(s-a) + \frac{1}{4}(b-c)^2 \text{ (and analogs)}$$

$$w_a^2 = s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2 \text{ (and analogs)}$$

So, we have the following relationship:

$$m_a^2 w_a^2 = \left(s(s-a) + \frac{1}{4}(b-c)^2\right)\left(s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2\right) =$$

$$= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4} - \frac{s^2(s-a)^2(b-c)^2}{(b+c)^2} - \frac{s(s-a)(b-c)^4}{4(b+c)^2} =$$

$$= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}[(b+c)^2 - 4s(s-a) - (b-c)^2] =$$

$$\left(\therefore s(s-a) = \frac{a+b+c}{2}\cdot\left(\frac{a+b+c}{2} - a\right) = \frac{(a+b+c)(b+c-a)}{4}\right)$$

$$m_a^2 w_a^2 = s^2(s-a)^2 +$$

$$+ \frac{s(s-a)(b-c)^2}{4(b+c)^2}[b^2 + c^2 + 2bc - (a+b+c)(b+c-a) - (b-c)^2] =$$

$$= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}(2bc + a^2 - b^2 - c^2) =$$

$$= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}(a^2 - (b-c)^2) \overset{a^2-(b-c)^2=4rr_a}{\cong}$$

$$= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}\cdot 4rr_a \overset{a^2-(b-c)^2=4(s-b)(s-c)}{\cong}$$

$$= s^2(s-a)^2 + \frac{s(s-a)(s-b)(s-c)(b-c)^2}{(b+c)^2} \xRightarrow{Heron}$$

$$m_a^2 w_a^2 = s^2(s-a)^2 + \frac{S^2(b-c)^2}{(b+c)^2} \text{ (and analogs)}$$

We show it that:

$$n_a^2 = s(s-a) + \frac{(b-c)^2 s}{a} \text{ (and analogs)}$$

$$g_a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a} \text{ (and analogs)}$$

$$n_a^2 g_a^2 = s(s-a)\left(s - \frac{(b-c)^2}{a}\right)\left(s - a + \frac{(b-c)^2}{a}\right) =$$

$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$

$$n_a^2 g_a^2 = s(s-a)\left(s(s-a) + (b-c)^2 \cdot \frac{a^2-(b-c)^2}{a^2}\right) \text{(and analogs)}$$

$$n_a^2 g_a^2 = s^2(s-a)^2 + \frac{4s(s-a)(s-b)(s-c)}{a^2}(b-c)^2 =$$

$$= s^2(s-a)^2 + \frac{4S^2}{a^2}(b-c)^2$$

We show that: $n_a g_a \geq m_a w_a$ (and analogs)

$$s^2(s-a)^2 + \frac{4S^2}{a^2}(b-c)^2 \geq s^2(s-a)^2 + \frac{S^2(b-c)^2}{(b+c)^2}$$

$$\frac{4S^2}{a^2}(b-c)^2 \geq \frac{S^2(b-c)^2}{(b+c)^2}$$

If $b = c$ we have equality.

If $b \neq c$ we have: $\frac{4S^2}{a^2} > \frac{S^2}{(b+c)^2} \Rightarrow \frac{4}{a^2} > \frac{1}{(b+c)^2} \Rightarrow 4(b+c)^2 > a^2 \Rightarrow 2(b+c) > a$ true.

So, we have: $n_a g_a \geq m_a w_a$ (and analogs)

$$n_a^2 g_a^2 = s^2(s-a)^2 + \frac{4S^2}{a^2}(b-c)^2 - \frac{S^2(b-c)^2}{(b+c)^2} + \frac{S^2(b-c)^2}{(b+c)^2} =$$

$$= m_a^2 w_a^2 + S^2(b-c)^2\left[\frac{4}{a^2} - \frac{1}{(b+c)^2}\right] \overset{S^2=\frac{a^2h_a^2}{4}}{\cong} m_a^2 w_a^2 + \frac{a^2 h_a^2}{4}(b-c)^2 \frac{4(b-c)^2-a^2}{a^2(b+c)^2}$$

$$n_a^2 g_a^2 = m_a^2 w_a^2 + h_a^2(b-c)^2 \frac{4(b-c)^2-a^2}{4(b+c)^2}$$

Finally, we get a new inequality:

$$n_a^2 g_a^2 = m_a^2 w_a^2 + h_a^2(b-c)^2\left[1 - \frac{a^2}{4(b+c)^2}\right] \text{(and analogs)}$$

Using means inequality for AQ-AM, we have: $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \Rightarrow \sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y)$

For $x^2 = m_a^2 w_a^2 \Rightarrow x = m_a w_a$

$$y^2 = h_a^2(b-c)^2\left[1-\frac{a^2}{4(b+c)^2}\right] \Rightarrow y = h_a|b-c|\sqrt{1-\frac{a^2}{4(b+c)^2}}$$

Finally, we get a new inequality:

$$n_a g_a \geq \frac{1}{\sqrt{2}}\left(m_a w_a + h_a|b-c|\sqrt{1-\frac{a^2}{4(b+c)^2}}\right) \text{ (and analogs)}$$

$$n_a^2 g_a^2 = s^2(s-a)^2 + \frac{4S^2}{a^2}(b-c)^2 \overset{4S^2=a^2h_a^2}{=} s^2(s-a)^2 + h_a^2(b-c)^2$$

Finally, we get a new equality:

$$n_a^2 g_a^2 = s^2(s-a)^2 + h_a^2(b-c)^2$$

From: $n_a g_a \geq m_a w_a$ (and analogs) $\Rightarrow n_a n_b n_c g_a g_b g_c \geq m_a m_b m_c w_a w_b w_c \Rightarrow$

$$\frac{n_a n_b n_c}{m_a m_b m_c} \geq \frac{w_a w_b w_c}{g_a g_b g_c}$$

From: $\frac{n_a}{m_a} \geq \frac{w_a}{g_a}$ (and analogs) $\Rightarrow \frac{n_a}{m_a} + \frac{n_b}{m_b} + \frac{n_c}{m_c} \geq \frac{w_a}{g_a} + \frac{w_b}{g_b} + \frac{w_c}{g_c}$

From: $\frac{n_a}{w_a} \geq \frac{m_a}{g_a}$ (and analogs) $\Rightarrow \frac{n_a}{w_a} + \frac{n_b}{w_b} + \frac{n_c}{w_c} \geq \frac{m_a}{g_a} + \frac{m_b}{g_b} + \frac{m_c}{g_c}$

From: $m_a w_a \geq r_b r_c$ (Panaitopol inequality) $\overset{r_b r_c = s(s-a)}{\Rightarrow}$

$$m_a w_a \geq s(s-a) \text{(and analogs)} \Rightarrow n_a g_a \geq m_a w_a \geq r_b r_c \text{ (and analogs)}$$

We know that: $s_a = \frac{2bc}{b^2+c^2} \cdot m_a$ (and analogs)

$$w_a s_a = \frac{2bc}{b^2+c^2} \cdot m_a w_a \leq \frac{2bc}{b^2+c^2} \cdot n_a g_a$$

$$w_a s_a \leq \frac{2bc}{b^2+c^2} \cdot n_a g_a \Rightarrow \frac{b^2+c^2}{2bc} \leq \frac{n_a g_a}{w_a s_a} \text{ (and analogs)} \Rightarrow \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right) \leq \frac{n_a g_a}{w_a s_a}$$

Finally, we get a new inequality:

$$\sum_{cyc}\frac{n_a g_a}{w_a s_a} \geq \frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

But: $\frac{b+c}{a} = \frac{r_a+r}{r_a-r}$ (and analogs) so, we have the following relationship:

$$\sum_{cyc}\frac{n_a g_a}{w_a s_a} \geq \frac{1}{2}\sum_{cyc}\frac{r_a+r}{r_a-r}$$

But: $\sum_{cyc}\frac{b+c}{a} = \sum_{cyc}\frac{h_a+h_b}{h_c}$ so, we have the following relationship:

$$\sum_{cyc}\frac{n_a g_a}{w_a s_a} \geq \frac{1}{2}\sum_{cyc}\frac{h_a+h_b}{h_c}$$

$$h_a = \left(1+\frac{b+c}{a}\right)r \text{ (and analogs)} \Rightarrow \frac{h_a-r}{r} = \frac{b+c}{a} \text{ (and analogs)}$$

$$\frac{1}{2}\sum_{cyc}\frac{b+c}{a} = \frac{h_a+h_b+h_c-3r}{2r}$$

So, we have the following relationship:

$$\sum_{cyc}\frac{n_a g_a}{w_a s_a} \geq \frac{h_a+h_b+h_c-3r}{2r}$$

But: $\frac{b+c}{a} = \frac{r_a+h_a}{r_a} = 1+\frac{h_a}{r_a}$ (and analogs)

So, we have the following relationship:

$$\sum_{cyc}\frac{b+c}{a} = 3+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}$$

Finally, we get a new inequality:

$$3+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c} \leq 2\sum_{cyc}\frac{n_a g_a}{w_a s_a}$$

We show it that:

$$4m_a^2 = n_a^2+g_a^2+2r_b r_c \text{ (and analogs)}$$

$$m_a w_a \geq r_b r_c \text{ (and analogs)} \Rightarrow n_a^2 + g_a^2 + 2m_a w_a \geq 4m_a^2$$

$$n_a g_a \geq m_a w_a \text{ (and analogs)} \Rightarrow n_a^2 + g_a^2 + 2n_a g_a = (n_a + g_a)^2 \geq n_a^2 + g_a^2 + 2m_a w_a$$

Finally, we get a new inequality:

$$n_a + g_a \geq \sqrt{n_a^2 + g_a^2 + 2m_a w_a} \geq 2m_a \text{ (and analogs)} \Rightarrow$$

$$\sum_{cyc}(n_a + g_a) \geq \sum_{cyc}\sqrt{n_a^2 + g_a^2 + 2m_a w_a} \geq 2(m_a + m_b + m_c)$$

We show it that:

$$b^2 + c^2 = n_a^2 + g_a^2 + 2rr_a \text{ (and analogs) and } bc = r_b r_c + rr_a \text{ (and analogs)}$$

$$(b - c)^2 = n_a^2 + g_a^2 - 2rr_a = n_a^2 + g_a^2 - n_a g_a - 2rr_a + 2n_a g_a =$$

$$= (n_a - g_a)^2 + 2(n_a g_a - r_b r_c)$$

Finally, we get a new inequality:

$$(b - c)^2 = (n_a - g_a)^2 + 2(n_a g_a - r_b r_c) \text{(and analogs)}$$

$$\sqrt{\frac{(n_a - g_a)^2 + 2(n_a g_a - r_b r_c)}{2}} \overset{AQM}{\geq} \frac{n_a - g_a + \sqrt{2(n_a g_a - r_b r_c)}}{2} \Rightarrow$$

$$|b - c| \geq \frac{1}{\sqrt{2}}\left(n_a - g_a + \sqrt{2(n_a g_a - r_b r_c)}\right) \text{ (and analogs)}$$

From: $n_a g_a \geq m_a w_a$ (and analogs) we get:

$$(b - c)^2 \geq (n_a - g_a)^2 + 2(m_a w_a - r_b r_c) \text{(and analogs)}$$

From: $(b - c)^2 = (n_a - g_a)^2 + 2(n_a g_a - r_b r_c)$(and analogs) and

$n_a g_a \geq m_a w_a \geq r_b r_c$ (and analogs) we get: $|b - c| \geq n_a - g_a$ (and analogs)

$$n_a^2 + g_a^2 - 2m_a w_a \geq n_a^2 + g_a^2 - 2n_a g_a = (n_a - g_a)^2 \Rightarrow$$

$$\sqrt{n_a^2 + g_a^2 - 2m_a w_a} \geq n_a - g_a \text{ (and analogs)} \Rightarrow$$

$$\sum_{cyc}\sqrt{n_a^2 + g_a^2 - 2m_a w_a} \geq \sum_{cyc} n_a - \sum_{cyc} g_a$$

From: $n_a g_a \geq m_a w_a \Rightarrow n_a \geq \frac{m_a w_a}{g_a}$ (and analogs)

But: $g_a \leq AI + 2 \leq w_a$ (and analogs) $\Rightarrow n_a \geq \frac{m_a w_a}{g_a} \geq \frac{m_a(AI+r)}{g_a} \geq m_a$ (and analogs)

$$\Rightarrow n_a + n_b + n_c \geq \sum_{cyc} \frac{m_a w_a}{g_a} \geq \sum_{cyc} \frac{m_a(AI + r)}{g_a} \geq m_a + m_b + m_c$$

References:

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