ABOUT FIBONACCI - LUCAS - KANTOROVICH - SURÁNYI - COLLABORATION INEQUALITIES

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Abstract. In this paper we present some certain results on Fibonacci - Lucas inequalities using Kantorovich and Surányi inequalities.

Theorem 1. Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the Fibonacci and the Lucas sequence, respectively. Then:

$$F_n^m F_{n+1}^m \sum_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}} \geq n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}, \forall n \in \mathbb{N}, \forall m \in \mathbb{R}_+$$

Proof. 1. We use AM-GM inequality:

$$\sum_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}} \overset{\text{AM-GM}}{\geq} n \cdot \sqrt[n]{\prod_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}}} = \frac{n \cdot \sqrt[n]{\prod_{k=1}^{n} L_k^{m+1}}}{\left(\sqrt[n]{\prod_{k=1}^{n} F_k^2} \right)^m} \overset{\text{AM-GM}}{\geq} \frac{n \cdot \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}}{\left(\frac{\sum_{k=1}^{n} F_k^2}{n} \right)^m} =$$

$$= \frac{n^{m+1} \cdot \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}}{\left(\sum_{k=1}^{n} F_k^2 \right)^m} = \frac{n^{m+1} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{m}}}{F_n^m F_{n+1}^m}$$

which is equivalent with the given inequality, and we are done.

$$\text{Above we used the well-known } \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

□

Proof 2. We use the inequality of J. Radon and AM-GM inequality.

$$\sum_{k=1}^{n} \frac{L_k^{m+1}}{F_k^{2m}} = \sum_{k=1}^{n} \frac{L_k^{m+1}}{(F_k^2)^m} \overset{\text{Radon}}{\geq} \frac{\left(\sum_{k=1}^{n} L_k \right)^{m+1}}{\left(\sum_{k=1}^{n} F_k^2 \right)^m} = \frac{1}{F_n^m F_{n+1}^n} \left(\sum_{k=1}^{n} L_k \right)^{m+1} \overset{\text{AM-GM}}{\geq}$$

$$\overset{\text{AM-GM}}{\geq} \frac{1}{F_n^m F_{n+1}^m} \left(n \cdot \sqrt[n]{\prod_{k=1}^{n} L_k} \right)^{m+1} = \frac{n^{m+1}}{F_n^m F_{n+1}^m} \left(\prod_{k=1}^{n} L_k \right)^{\frac{m+1}{n}}$$

$$\text{We used the well-known } \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

□

Theorem 2. If $a, b, c > 0$, then:

$$2\left(\left(\frac{a}{F_n b + F_{n+1} c} \right)^3 + \left(\frac{b}{F_n c + F_{n+1} a} \right)^3 + \left(\frac{c}{F_n a + F_{n+1} b} \right)^3 \right) +$$

$$+3\cdot\frac{abc}{(F_na+F_{n+1}b)(F_nb+F_{n+1}c)(F_nc+F_{n+1}a)}\geq\frac{9}{F_{n+2}^3}\text{ for any positive integer }n.$$

Proof. By János Surányi's inequality we have that if $x_k>0(k=1,2,\ldots,n)$ then the following inequality holds:

$$\text{(S)}\qquad (n-1)\sum_{k=1}^n x_k^n+n\prod_{k=1}^n x_k\geq\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^{n-1}\right)$$

The equality holds if and only if $x_k=x(k=1,2,\ldots,n)$.
If $n=3$, then (S) becomes:

$$\text{(1)}\quad 2\sum_{k=1}^3 x_k+3x_1x_2x_3\geq(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)\overset{\text{Bergström}}{\geq}\frac{1}{3}(x_1+x_2+x_3)^3$$

If in (1) we take $x_1=\frac{a}{F_nb+F_{n+1}c},x_2=\frac{b}{F_nc+F_{n+1}a},x_3=\frac{c}{F_na+F_{n+1}b}$, then we obtain

$$\text{(2)}\quad A=2\sum_{cyc}\left(\frac{a}{F_nb+F_{n+1}c}\right)^3+\frac{3abc}{\prod_{cyc}(aF_n+bF_{n+1})}\geq\frac{1}{3}\left(\sum_{cyc}\frac{a}{F_nb+F_{n+1}c}\right)^3$$

$$\text{But, } B=\sum_{cyc}\frac{a}{F_nb+F_{n+1}c}=\sum_{cyc}\frac{a^2}{F_nab+F_{n+1}ac}\overset{\text{Bergström}}{\geq}\frac{(a+b+c)^2}{\sum_{cyc}(F_nab+F_{n+1}ac)}=$$

(3)

$$=\frac{(a+b+c)^2}{(F_n+F_{n+1})(ab+bc+ca)}=\frac{(a+b+c)^2}{F_{n+2}(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{F_{n+2}(ab+bc+ca)}=\frac{3}{F_{n+2}}$$

From (2) and (3) we obtain that: $A\geq\frac{1}{3}\cdot\frac{27}{F_{n+2}^3}=\frac{9}{F_{n+2}^9}$.
The equality holds if and only if $a=b=c$. □

Theorem 3. If $a,b,c>0$, then prove that:

$$2\left(\left(\frac{a}{L_nb+L_{n+1}c}\right)^3+\left(\frac{b}{L_nc+L_{n+1}a}\right)^3+\left(\frac{c}{L_na+L_{n+1}b}\right)^3\right)+$$

$$+3\cdot\frac{abc}{(L_na+L_{n+1}b)(L_nb+L_{n+1}c)(L_nc+L_{n+1}a)}\geq\frac{9}{L_{n+2}^3},\text{ for any positive integer }n.$$

Proof. By János Surányi's inequality we have that if $x_k>0(k=1,2,\ldots,n)$ then the following inequality holds:

$$\text{(S)}\qquad (n-1)\sum_{k=1}^n x_k^n+n\prod_{k=1}^n x_k\geq\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n x_k^{n-1}\right)$$

The equality holds if and only if $x_k=x(k=1,2,\ldots,n)$.
If $n=3$, then (S) becomes:

$$\text{(1)}\quad 2\sum_{k=1}^3 x_k+3x_1x_2x_3\geq(x_1+x_2+x_3)(x_1^2+x_2^2+x_3)^2\overset{\text{Bergström}}{\geq}\frac{1}{3}(x_1+x_2+x_3)^3$$

If in (1) we take $x_1=\frac{a}{L_n+L_{n+1}c},x_2=\frac{b}{L_nc+L_{n+1}a},x_3=\frac{c}{L_na+L_{n+1}b}$, then we obtain: □

(2) $A = 2\sum_{cyc}\left(\frac{a}{L_n b + L_{n+1}c}\right)^3 + \frac{3abc}{\prod_{cyc}(aL_n + bL_{n+1})} \geq \frac{1}{3}\left(\sum_{cyc}\frac{a}{L_n b + L_{n+1}c}\right)^3$

But, $B = \sum_{cyc}\frac{a}{L_n b + L_{n+1}c} = \sum_{cyc}\frac{a^2}{L_n ab + L_{n+1}ac} \overset{\text{Bergström}}{\geq} \frac{(a+b+c)^2}{\sum_{cyc}(L_n ab + L_{n+1}ac)} =$

(3)

$$= \frac{(a+b+c)^2}{(L_n + L_{n+1})(ab+bc+ca)} = \frac{(a+b+c)^2}{L_{n+2}(ab+bc+ca)} \geq \frac{3(ab+bc+ca)}{L_{n+2}(ab+bc+ca)} = \frac{3}{L_{n+2}}$$

From (2) and (3) we obtain that $A \geq \frac{1}{3}\cdot\frac{27}{L_{n+2}^3} = \frac{9}{L_{n+2}^3}$

The equality holds if and only if $a = b = c$.

Theorem 4. If $a, b, c > 0$, then:

$$2\left(\left(\frac{a}{F_n^2 b + F_{n+1}^2 c}\right)^3 + \left(\frac{b}{F_n^2 c + F_{n+1}^2 a}\right) + \left(\frac{c}{F_n^2 a + F_{n+1}^2 b}\right)^3\right) +$$

$$+3\cdot\frac{abc}{(F_n^2 a + F_{n+1}^2 b)(F_n^2 b + F_{n+1}^2 c)(F_n^2 c + F_{n+1}^2 a)} > \frac{9}{F_{2n+1}^3},$$ for any positie integer n.

Proof. By János Surányi's inequality we have that if $x_k > 0 (k = 1, 2, \ldots, n)$ then the following inequality holds:

(S) $$(n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right)$$

The equality holds if and only if $x_k = x (k = 1, 2, \ldots, n)$.

If $n = 3$, then (S) becomes:

(1) $2\sum_{k=1}^{3} x_k + 3x_1x_2x_3 \geq (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^3) \overset{\text{Bergström}}{\geq} \frac{1}{3}(x_1 + x_2 + x_3)^3$

If in (1) we take $x_1 = \frac{a}{F_n^2 b + F_{n+1}^2 c}, x_2 = \frac{b}{F_n^2 c + F_{n+1}^2 a}, x_3 = \frac{c}{F_n^2 a + F_{n+1}^2 b}$, then we obtain:

(2) $A = 2\sum_{cyc}\left(\frac{a}{F_n^2 b + F_{n+1}^2 c}\right)^3 + \frac{3abc}{\prod_{cyc}(aF_n^2 + bF_{n+1}^2)} \geq \frac{1}{3}\left(\sum_{cyc}\frac{a}{F_n^2 b + F_{n+1}^2 c}\right)^3$

But, $B = \sum_{cyc}\frac{a}{F_n^2 b + F_{n+1}^2 c} = \sum_{cyc}\frac{a^2}{F_n^2 ab + F_{n+1}^2 ac} \overset{\text{Bergström}}{\geq} \frac{(a+b+c)^2}{\sum_{cyc}(F_n^2 ab + F_{n+1}^2 ac)} =$

(3)

$$= \frac{(a+b+c)^2}{(F_n^2 + F_{n+1}^2)(ab+bc+ca)} = \frac{(a+b+c)^2}{F_{2n+1}(ab+bc+ca)} \geq \frac{3(ab+bc+ca)}{F_{2n+1}(ab+bc+ca)} = \frac{3}{F_{2n+1}}$$

From (2) and (3) we obtain that:

$$A \geq \frac{1}{3}\cdot\frac{27}{F_{2n+1}^3} = \frac{9}{F_{2n+1}^3}$$

□

The inequality is strictly because $F_n^2 \neq F_{n+1}^2$.

Theorem 5. If $e_n = (1+\frac{1}{n})^n$, then:

$$\left(\sum_{k=1}^{n} e_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{e_k}\right) \leq \frac{(e+2)^2}{8e} F_n^2 F_{n+1}^2$$

for any positive integer n.

Proof. From Kantorivich's inequality we have:
If $0 < m \leq x_k \leq M, y_k > 0, (k = 1, 2, \ldots, n)$ then:

$$\text{(K)} \quad \left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{k=1}^{n} \frac{y_k}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}\left(\sum_{k=1}^{n} y_k\right)^2 \text{ for, any positive integer } n$$

Since, $e_1 = 2$ and $\lim_{n\to\infty} e_n = e$ we have that: if $x_k = e_k$, then $x_k \in [2, e]$, for any positive integer k. So, in this case we have $m = 2, M = e$. If we take $y_k = F_k^2$, then by (K) we obtain that:

$$\text{(1)} \quad \left(\sum_{k=1}^{n} e_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{e_k}\right) \leq \frac{(e+2)^2}{8e}\left(\sum_{k=1}^{n} F_k^2\right)^2$$

If we are taking into account that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, then by (1) we obtain the desired inequality. □

Theorem 6.

$$\text{If } e_n = (1+\frac{1}{n})^n \text{ then } \left(\sum_{k=1}^{n} e_k F_{2k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2k-1}}{e_k}\right) \leq \frac{(e+2)^2}{8e} F_{2n}^2$$

for any positive integer n.

Proof. From Kantorivich's inequality we have:
If $0 < m \leq x_k \leq M, y_k > 0, (k = 1, 2, \ldots, n)$, then:

$$\text{(K)} \quad \left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{k=1}^{n} \frac{y_k}{x_k}\right) \leq \frac{(M+n)^2}{4Mm}\left(\sum_{k=1}^{n} y_k\right)^2 \text{ for any positive integer } n.$$

Since, $e_1 = 2$ and $\lim_{n\to\infty} = e$ we have that: if $x_k = e_k$, then $x_k \in [2, e]$, for any positive integer k. So, in this case we have: $m = 2, M = e$. If we take $y_k = F_{2k-1}$, then by (K) we obtain that:

$$\text{(1)} \quad \left(\sum_{k=1}^{n} e_k F_{2k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2k-1}}{e_k}\right) \leq \frac{(e+2)^2}{8e}\left(\sum_{k=1}^{n} F_{2k-1}\right)^2$$

If we take into account that $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$, then by (1) we obtain the desired inequality. □

Theorem 7.

$$\text{If } x_n = \sum_{k=1}^{n} \frac{1}{k^2} \text{ then } \left(\sum_{k=1}^{n} x_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{x_k}\right) \leq \frac{(\pi^2+6)^2}{24\pi^2} \cdot F_n^2 F_{n+1}^2$$

for any positive integer n.

Proof. From Kantorivich's inequality we have:
If $0 < m \leq x_k \leq M, y_k > 0, (k = 1, 2, \dots, n)$ then:

$$\text{(K)} \quad \left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{k=1}^{n} \frac{y_k}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}\left(\sum_{k=1}^{n} y_k\right)^2 \text{ for any positie integer } n.$$

Since, $x_1 = 1$ and $x_n = \frac{\pi^2}{6}$ we have that: if $x_k = \sum_{i=1}^{k} \frac{1}{i^2}$, then $x_k \in \left[1, \frac{\pi^2}{6}\right]$, for any positive integer k. So, in this case we have $m = 1, M = \frac{\pi^2}{6}$. If we take $y_k = F_k^2$, then by (K) we obtain that:

$$\text{(1)} \quad \left(\sum_{k=1}^{n} x_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{x_k}\right) \leq \frac{(\pi^2+6)^2}{24\pi^2}\left(\sum_{k=1}^{n} F_k^2\right)^2$$

If we are taking into account that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ then by (1) we obtain the desired inequality. □

Theorem 8. If $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$, then:

$$\left(\sum_{k=1}^{n} x_k F_{2k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2k-1}}{x_k}\right) \leq \frac{(\pi^2+6)^2}{24\pi^2} F_{2n}^2$$

for any positive integer n.

Proof. From Kantorivich's inequality we have:
If $0 < m \leq x_k \leq M, y_k > 0, (k = 1, 2, \dots, n)$, then:

$$\text{(K)} \quad \left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{k=1}^{n} \frac{y_k}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}\left(\sum_{k=1}^{n} y_k\right)^2, \text{ for any positive intefer } n$$

Since, $x_1 = 1$ and $x_n = \frac{\pi^2}{6}$ we have that: if $x_k = \sum_{i=1}^{k} \frac{1}{i^2}$, then $x_k \in [1, \frac{\pi^2}{6}]$, for any positive integer k. So, in this case we have $m = 1, M = \frac{\pi^2}{6}$. If we make $y_k = F_{2k-1}$, then (K) we obtain that:

$$\text{(1)} \quad \left(\sum_{k=1}^{n} x_k F_{2k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2k-1}}{x_k}\right) \leq \frac{(\pi^2+6)^2}{24\pi^2}\left(\sum_{k=1}^{n} F_{2k-1}\right)^2$$

If we take into account that $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$, then by (1) we obtain the desired inequality. □

Theorem 9. If $x_n = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$, then:

$$\left(\sum_{k=1}^{n} x_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{x_k}\right) \leq \frac{(\pi^2+8)^2}{32\pi^2} F_n^2 F_{n+1}^2$$

for any positive integer n.

Proof. From Kantorivich's inequality we have:
If $0 < m \leq x_k \leq M, y_k > 0, (k = 1, 2, \dots, n)$, then:

$$\text{(K)} \quad \left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{k=1}^{n} \frac{y_k}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}\left(\sum_{k=1}^{n} y_k\right)^2, \text{ for any positive integer } n$$

Since $x_1 = 1$ and $\lim_{n\to\infty} = \frac{\pi^2}{8}$ we have that: if $x_k = \sum_{i=1}^{k} \frac{1}{(2i-i)^2}$, then $x_k \in [1, \frac{\pi^2}{8}]$, for any positive integer k. So, in this case we have $m = 1, M = \frac{\pi^2}{8}$. If we take $y_k = F_k^2$, then by (K) we obtain that:

$$(1) \qquad \left(\sum_{k=1}^{n} x_k F_k^2\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{x_k}\right) \leq \frac{(\pi^2+8)^2}{32\pi^2}\left(\sum_{k=1}^{n} F_k^2\right)^2$$

If we take into account that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, then by (1) we obtain the desired inequality. □

Theorem 10.

$$n^{n-2}(n-1)\sum_{k=1}^{n} F_k^n + n^{n-1}\prod_{k=1}^{n} F_k > (F_{n+2}-1)^n, \text{ for any positive integer } n \geq 2.$$

Proof. From János Surányi's inequality we have that if $x_k > 0 (k = 1, 2, \ldots, n)$ then the following inequality holds:

$$(S) \qquad (n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right)$$

The equality holds if and only if $x_k = x (k = 1, 2, \ldots, n)$.
If we take in (1) $x_k = F_k (k = 1, 2, \ldots, n)$ we obtain that:

$$(1) \qquad (n-1)\sum_{k=1}^{n} F_k^n + n\prod_{k=1}^{n} F_k \geq \left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} F_k^{n-1}\right)$$

From J. Radon's inequality we have that:

$$(2) \qquad \sum_{k=1}^{n} F_k^{n-1} \geq \frac{\left(\sum_{k=1}^{n} F_k\right)^{n-1}}{\left(\underbrace{1+1+\ldots+1}_{n}\right)^{n-2}} = \frac{1}{n^{n+2}}\left(\sum_{k=1}^{n} F_k\right)^{n-1}$$

From (1) and (2) we deduce that:
(3)

$$(n-1)\sum_{k=1}^{n} F_k^n + n\prod_{k=1}^{n} F_k \geq \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n} F_k\right)^n \Leftrightarrow n^{n-2}(n-1)\sum_{k=1}^{n} F_k^n + n^{n-1}\prod_{k=1}^{n} F_k > \left(\sum_{k=1}^{n} F_k\right)^n$$

If we take into account $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ then (3) becomes:

$$n^{n-2}(n-1)\sum_{k=1}^{n} F_k^n + n^{n-1}\prod_{k=1}^{n} F_k > (F_{n+2}-1)^n$$

The inequality is strictly since $F_k \neq F_{k+1}$, for any $k \geq 2$.

□

Theorem 11.

$$n^{n-2}(n-1)\sum_{k=1}^{n} L_k^n + n^{n-1}\prod_{k=1}^{n} L_k > (L_{n+2}-3)^n, \text{ for any positie integer } n \geq 2.$$

Proof. From János Surányi's inequality we have:
If $x_k > 0 (k = 1, 2, \ldots, n)$ then the following inequality holds:

$$(n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right) \tag{S}$$

The equality holds if and only if $x_k = x (k = 1, 2, \ldots, n)$
If we take in (S) $x_k = L_k (k = 1, 2, \ldots, n)$ we obtain that:

$$(n-1)\sum_{k=1}^{n} L_k^n + n\prod_{k=1}^{n} L_k \geq \left(\sum_{k=1}^{n} L_k\right)\left(\sum_{k=1}^{n} L_k^{n-1}\right) \tag{1}$$

From J. Radon's inequality we have that:

$$\sum_{k=1}^{n} L_k^{n-1} \geq \frac{\left(\sum_{k=1}^{n} L_k\right)^{n-1}}{(\underbrace{1+1+\ldots+1}_{n})} = \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n} L_k\right)^{n-1} \tag{2}$$

From (1) and (2) we deduce that:
(3)

$$(n-1)\sum_{k=1}^{n} L_k^n + n\prod_{k=1}^{n} L_k \geq \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n} L_k\right)^n \Leftrightarrow n^{n-2}(n-1)\sum_{k=1}^{n} L_k^n + n^{n-1}\prod_{k=1}^{n} L_k > \left(\sum_{k=1}^{n} L_k\right)^n$$

If we take into account that $\sum_{k=1}^{n} L_k = L_{n+2} - 3$ then by (3) yields the desired inequality. The inequality is strictly since $L_k \neq L_{k+1}$, for any $k \in \mathbb{N}$ □

Theorem 12.

$$n^{n-2}(n-1)\sum_{k=1}^{n} F_k^{2n} + n^{n-1}\prod_{k=1}^{n} F_k^2 > F_n^n F_{n+1}^n, \text{ for any positive integer } n \geq 2.$$

Proof. From János Surányi's inequality we have that:
If $x_k > 0, (k = 1, 2, \ldots, n)$ then the following inequality holds:

$$(n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right) \tag{S}$$

The equality holds if and only if $x_k = x, (k = 1, 2, \ldots, n)$. If we take in (S) $x_k = F_k^2 (k = 1, 2, \ldots, n)$ we obtain that:

$$(n-1)\sum_{k=1}^{n} F_k^{2n} + n\prod_{k=1}^{n} F_k^2 \geq \left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n} F_k^{2(n-1)}\right) \tag{1}$$

From J. Radon's inequality we have that:

$$\sum_{k=1}^{n} (F_k^2)^{n-1} \geq \frac{\left(\sum_{k=1}^{n} F_k^2\right)^{n-1}}{(\underbrace{1+1+\ldots+1}_{n})^{n-2}} = \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n} F_k^2\right)^{n-1} \tag{2}$$

From (1) and (2) we deduce that:
(3)

$$(n-1)\sum_{k=1}^{n} F_k^{2n} + n\prod_{k=1}^{n} F_k^2 \geq \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n} F_k^2\right)^n \Leftrightarrow n^{n-2}(n-1)\sum_{k=1}^{n} F_k^{2n} + n^{n-1}\prod_{k=1}^{n} F_k^2 > \left(\sum_{k=1}^{n} F_k^2\right)^n$$

If we take into account that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, then (3) becomes:

$$n^{n-2}(n-1)\sum_{k=1}^{n} F_k^{2n} + n^{n-1}\prod_{k=1}^{n} F_k^2 > F_n^n F_{n+1}^n$$

The inequality is strictly since $F_k \neq F_{k+1}$, for any $k \geq 2$. □

Theorem 13.

$n^{n-2}(n-1)\sum_{k=1}^{n}\left(\binom{n}{k}F_k\right)^n + n^{n-1}\prod_{k=1}^{n}\binom{n}{k}F_k > F_{2n}^n$, for any positive integer $n \geq 2$.

Proof. From János Surányi's inequality we have that:
If $x_k > 0; (k = 1, 2, \dots, n)$ then the following inequality holds:

$$\text{(S)} \qquad (n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right)$$

The equality holds if and only if $x_k = x; (k = 1, 2, \dots, n)$. If we take in (S) $x_k = \binom{n}{k}F_k (k = 1, 2, \dots, n)$ we obtain that:

$$(1) \quad (n-1)\sum_{k=1}^{n}\left(\binom{n}{k}F_k\right)^n + n\prod_{k=1}^{n}\binom{n}{k}F_k \geq \left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)\left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^{n-1}$$

From J. Radon's inequality we have that:

$$(2) \quad \left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^{n-1} \geq \frac{\left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^{n-1}}{(\underbrace{1+1+\ldots+1}_{n})^{n-2}} = \frac{1}{n^{n-2}}\left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^{n-1}$$

From (2) and (2) we deduce that:

$$(3) \qquad n^{n-2}(n-1)\sum_{k=1}^{n}\left(\binom{n}{k}F_k\right)^n + n^{n-1}\prod_{k=1}^{n}\binom{n}{k}F_k > \left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^n$$

If we take into account that $\sum_{k=1}^{n}\binom{n}{k}F_k = F_{2n}$ then from (3) we obtain the desired inequality. The inequality is strictly since $F_k \neq F_{k+1}$, for any $k \geq 2$. □

Theorem 14.

$$n(n+1)^{n+1}\sum_{k=0}^{n}\left(\binom{n}{k}L_k\right)^{n+1} + (n+1)^n\prod_{k=0}^{n}\binom{n}{k}L_k > L_{2n}^{n+1}$$

for any positive integer n.

Proof. From János Surányi's inequality we have that:
If $x_k > 0 (k = 1, 2, \dots, n)$ then the following inequality holds:

$$\text{(S)} \qquad (n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right)$$

If instead of n we take $n+1$ the inequality (S) becomes:

$$\text{(S')} \qquad n\sum_{k=1}^{n+1} x_k^{n+1} + (n+1)\prod_{k=1}^{n+1} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n+1} x_k^n\right)$$

The equality holds if and only if $x_k = x (k = 1, 2, \ldots, n+1)$. If we take in(S') $x_k = \binom{n}{k-1} L_{k-1} (k = 1, 2, \ldots, n+1)$ we obtain that:

$$(1) \quad n\sum_{k=0}^{n}\left(\binom{n}{k} L_k\right)^{n+1} + (n+1)\prod_{k=0}^{n}\binom{n}{k} L_k \geq \left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)\left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)^n$$

From J. Radon's inequality we have that:

$$(2) \qquad \sum_{k=0}^{n}\left(\binom{n}{k} L_k\right)^n \geq \frac{\left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)^n}{\left(\underbrace{1+1+\ldots+1}_{n+1}\right)^{n-1}} = \frac{1}{(n+1)^{n-1}}\left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)^n$$

From (1) and (2) we deduce that:

$$(3) \quad (n+1)^{n-1} n\left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)^{n+1} + (n+1)^n\prod_{k=0}^{n}\binom{n}{k} L_k > \left(\sum_{k=0}^{n}\binom{n}{k} L_k\right)^{n+1}$$

If we take into account that $\sum_{k=0}^{n}\binom{n}{k} L_k = L_{2n}$, then from (2) we obtain the desired inequality. The inequality is strictly since $L_k \neq L_{k+1}$, for any $k \in \mathbb{N}$. □

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