A “probabilistic” method for proving inequalities

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In this paper we solve a class of inequalities using an identity familiar from probability theory and classical mechanics.

In the year 2000, Fuhua Wei and Shan - He Wu from the Department of Mathematics and Computer Science, Longyan University, Longyan, Fujian 364012, P.R. China published the article: “Several proofs and generalisations of a fractional inequality with constraints.” In this article, they give ten different proofs for the 2nd problem of the 36th IMO, held at Toronto (Canada) in 1995.

In proof 5, the authors used a method based on a key random variable to prove that if a, b, c are positive real numbers with $abc = 1$ then:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Proof. We make the substitutions $x := bc$, $y := ca$, $z := ab$, and $s := x + y + z$.

Then:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = \frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}.$$

We consider the random variable ξ defined as follows:

$$\xi = \begin{cases} \frac{x}{s-x} : & (p = \frac{s-x}{2s}), \\ \frac{y}{s-y} : & (p = \frac{s-y}{2s}), \\ \frac{z}{s-z} : & (p = \frac{s-z}{2s}). \end{cases}$$

It follows that

$$E(\xi) = \frac{x}{s-x} \cdot \frac{s-x}{2s} + \frac{y}{s-y} \cdot \frac{s-y}{2s} + \frac{z}{s-z} \cdot \frac{s-z}{2s} = \frac{x+y+z}{2s} = \frac{1}{2}$$

and also

$$\begin{aligned} E(\xi^2) =& \Big(\frac{x}{s-x}\Big)^2 \cdot \frac{s-x}{2s} + \Big(\frac{y}{s-y}\Big)^2 \cdot \frac{s-y}{2s} + \Big(\frac{z}{s-z}\Big)^2 \cdot \frac{s-z}{2s} \\ =& \frac{1}{2s}\Big(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\Big). \end{aligned}$$

Now, the variance of ξ is given by $V(\xi) = E(\xi^2) - (E(\xi))^2$. This is always nonnegative, and positive unless ξ can take only one value (in which case $x = y = z$ and $a = b = c$.) We thus have

$$\frac{1}{2s}\Big(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\Big) \geq \frac{1}{4}$$

and so

$$\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z} \geq \frac{1}{2}s = \frac{1}{2}(x+y+z) \overbrace{\geq}^{AM-GM} \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}.$$

Hence

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

and this is strict unless $a=b=c$. □

The method of proof used here is based on the positivity of variance:

$$E(\xi^2)-(E(\xi))^2 = V(\xi) = E((\xi-E(\xi))^2) \geq 0\ ,$$

whence

$$E(\xi^2) \geq (E(\xi))^2\ .$$

It can be applied to other problems as well. The technique is to construct a random variable such that its variance is the quantity, or difference, that we wish to show positive. (Readers familiar with classical mechanics may prefer to consider this in terms of the parallel axis theorem for moments of inertia - a "mechanical" method of proof?)

Example 1. Prove that if $x,y,z>0$ then:

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}} \leq \sqrt{6\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)}$$

Solution. Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{x}{y}}: & (p=\frac{1}{6}),\\ \sqrt{\frac{y}{z}}: & (p=\frac{2}{6}),\\ \sqrt{\frac{z}{x}}: & (p=\frac{3}{6}), \end{cases} \qquad \text{then} \quad \xi^2 = \begin{cases} \frac{x}{y}: & (p=\frac{1}{6}),\\ \frac{y}{z}: & (p=\frac{2}{6}),\\ \frac{z}{x}: & (p=\frac{3}{6}). \end{cases}$$

It follows that

$$E(\xi) = \frac{1}{6}\sqrt{\frac{x}{y}}+\frac{2}{6}\sqrt{\frac{y}{z}}+\frac{3}{6}\sqrt{\frac{z}{x}} \quad \text{and} \quad E(\xi^2) = \frac{1}{6}\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)\ .$$

As

$$E(\xi^2) \geq (E(\xi))^2,$$

we have

$$\frac{1}{6}\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big) \geq \Big[\frac{1}{6}\Big(\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\Big)\Big]^2,$$

$$\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x} \geq \frac{1}{6}\Big(\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\Big)^2,$$

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}} \leq \sqrt{6\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)}$$

and, again, equality holds only for $x = y = z$. $\square$

Example 2. Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big)}$$

Solution. Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{a}{b+c}} : & (p = \frac{1}{7}), \\ \sqrt{\frac{b}{c+a}} : & (p = \frac{2}{7}), \\ \sqrt{\frac{c}{a+b}} : & (p = \frac{4}{7}). \end{cases}$$

As before we get

$$E(\xi) = \frac{1}{7}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big) \quad \text{and} \quad E(\xi^2) = \frac{1}{7}\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big),$$

and the inequality

$$\frac{1}{7}\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big) \geq \frac{1}{49}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big)^2.$$

Therefore

$$\frac{1}{\sqrt{7}} \cdot \sqrt{\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}} \geq \frac{1}{7}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big),$$

and

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big)},$$

with equality only for $a = b = c$. $\square$

Application 3. Prove that in any triangle ABC the following relationship holds for the medians m_a, m_b, m_c and altitudes h_a, h_b, h_c:

$$3\sqrt{\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}}$$

Solution. Let be the probability distribution sequence of random variable ξ below: Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{m_a}{m_b}} : & (p = \frac{1}{9}) \\ \sqrt{\frac{m_b}{m_c}} : & (p = \frac{2}{9}) \\ \sqrt{\frac{m_c}{m_a}} : & (p = \frac{6}{9}). \end{cases}$$

It follows that

$$E(\xi)=\frac{1}{9}\Big(\sqrt{\frac{m_a}{m_b}}+2\sqrt{\frac{m_b}{m_c}}+6\sqrt{\frac{m_c}{m_a}}\Big)\quad\text{and}\quad E(\xi^2)=\frac{1}{9}\Big(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\Big)\ ,$$

and, $m_a\geq h_a$, $m_b\geq h_b$, and $m_c\geq h_c$, we have

$$\begin{aligned}\frac{1}{9}\Big(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\Big)&\geq\frac{1}{81}\Big(\sqrt{\frac{m_a}{m_b}}+2\sqrt{\frac{m_b}{m_c}}+6\sqrt{\frac{m_c}{m_a}}\Big)^2\\&\geq\frac{1}{81}\Big(\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}}\Big)^2\ ,\end{aligned}$$

whence

$$9\Big(\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}\Big)\geq\Big(\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}}\Big)^2$$

and

$$3\sqrt{\frac{m_a}{m_b}+\frac{2m_b}{m_c}+\frac{6m_c}{m_a}}\geq\sqrt{\frac{h_a}{m_b}}+2\sqrt{\frac{h_b}{m_c}}+6\sqrt{\frac{h_c}{m_a}},$$

which completes the solution. □

Of course, applying this process in reverse is an intriguing way to invent new inequalities!

References

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