120 YEARS OF LALESCU SEQUENCES

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Abstract. In this paper we present new methods to calculate certain limits from math journals from all over the world .

1. Results – Lalescu type limits
Problems from math journals from all over the world

The Stolz-Cesàro criterion (C-S). Let the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ such that $(y_n)_{n\geq 0}$ is strictly monotonous and boundless. If there exists the limit $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = l$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = l$.
The Cauchy - D'Alembert criterion (C-D'A). Let the sequence $(x_n)_{n\geq 0}$ with $x_n > 0$. If there exists the limit $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$, then $\lim_{n\to\infty} \sqrt[n]{x_n} = l$.
We present new solutions of certain problems of Lalescu type limits from math journals from all over the world.
Next we use the following abbreviations: Crux Mathematicorum (CM); Gazeta Matematică Seria B (GMB); La Gaceta de la RSME (LG); Math Problems (MP); Pi Mu Epsilon Journal (PME); Mathematics Magazine from Timişoara (RMT); Mathematical Recreation from Iaşi (RM); Revista Escolar de la Olimpiada Iberoamericana de Matematica (REOIM); Romanian Mathematical Magazine (RMM); School Science and Mathematics (SSM); The Spark of the Mind (SM); The American Mathematical Monthly (AMM); The College Mathematics Journal (CMJ); The Pentagon (P); The Fibonacci Quarterly (FQ).

Problem 1. Traian Lalescu's limit, GMB, Vol. VI, 1900-1901, problem 579, p. 148.

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right) \overset{k!\cong\left(\frac{k}{e}\right)^k \Rightarrow \sqrt[k]{k!}\cong\frac{k}{e}}{=} \lim_{n\to\infty}\left(\frac{n+1}{e}-\frac{n}{e}\right)=\frac{1}{e}$$

Problem 2. D. M. Bătineţu-Giurgiu's limit, GMB, Vol. XCIV, 1989, problem C:890, p. 139.

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right) \overset{k!\cong\left(\frac{k}{e}\right)^k \Rightarrow \sqrt[k]{k!}\cong\frac{k}{e}}{=} \lim_{n\to\infty}\left(\frac{(n+1)^2}{\frac{(n+1)}{e}}-\frac{n^2}{\frac{n}{e}}\right)=$$

$$=\lim_{n\to\infty}((n+1)e-ne)=e.$$

Problem 3. D. M. Bătineţu-Giurgiu, Neculai Stanciu, GMB 5/2012.
If $(I_n(t))_{n\geq 2}$ is defined by $I_n(t) = n^{1-t}((n+1)^t(\sqrt[n+1]{n+1})^t - n^t(\sqrt[n]{n})^t), \forall t \in \mathbb{R}^*$, then compute $\lim_{n\to\infty} I_n(t)$.

Key words and phrases. Limits of Lalescu type sequences; Limits of Lalescu type functions; Problem Solving.

Solution.
$I_n(t) = n \cdot \sqrt[n]{n^t} \cdot (u_n - 1) = (\sqrt[n]{n})^t \cdot \frac{u_n-1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2$, where
$u_n = \left(\frac{n+1}{n}\right)^t \cdot \left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right)^t, \forall n \geq 2; \lim_{n\to\infty} u_n = 1$ and then $\lim_{n\to\infty} \frac{u_n-1}{\ln u_n} = 1$.

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} e_n^t \cdot \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right)^{nt} = e^t \cdot \lim_{n\to\infty}\left(\frac{n+1}{n} \cdot \frac{1}{\sqrt[n+1]{n+1}}\right)^t = e^t \cdot 1 = e^t$$

$$\text{So } \lim_{n\to\infty} I_n(t) = 1 \cdot 1 \cdot \ln\left(\lim_{n\to\infty} u_n^n\right) = \ln e^t = t.$$

□

Observation. For $t = 1$ we deduce $\lim_{n\to\infty} I_n = \lim_{n\to\infty} I_n(1) = 1$, i.e. the limit of Romeo T. Ianculescu.

Problem 4. D. M. Bătineţu-Giurgiu, Neculai Stanciu, CM 7/2013.
Let $(a_n)_{n\geq}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{n} = a \in \mathbb{R}_+^*$. We define $a_n!$ by $a_{n+1} = a_n! \cdot a_{n+1}, \forall n \in \mathbb{N}^*$. Compute:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n}\right)$$

Solution.

$$\lim_{n\to\infty} \frac{a_n}{n^2} \overset{\text{C-S}}{=} \lim_{n\to\infty} \frac{a_{n+1}-a_n}{(n+1)^2-n^2} = \lim_{n\to\infty}\left(\frac{a_{n+1}-a_n}{n} \cdot \frac{n}{2n+1}\right) = \frac{a}{2}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n!}}{n^2} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n!}{n^{2n}}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{a_{n+1}!}{(n+1)^{2(n+1)}} \cdot \frac{n^{2n}}{a_n!} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^2} \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n} = \frac{a}{2} \cdot \frac{1}{e^2} = \frac{a}{2e^2}$$

$$\text{So, } L = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n}\right) = \lim_{n\to\infty} \frac{\sqrt[n]{a_n!}}{n}(u_n - 1) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n!}}{n^2} \cdot \frac{u_n-1}{\ln u_n} \cdot \ln u_n^n\right)$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} \cdot \frac{n}{\sqrt[n]{a_n!}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{(n+1)^2} \cdot \frac{n^2}{\sqrt[n]{a_n!}} \cdot \frac{n+1}{n}\right) = \frac{a}{2e} \cdot \frac{2e}{a} \cdot 1 = 1,$$

$$\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{\sqrt[n]{a_n!}} \cdot \frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{a_{n+1}!}{a_n!} \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{\sqrt[n+1]{a_{n+1}!}}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^2} \cdot \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}!}}\right) \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{a}{2} \cdot \frac{2e^2}{a} \cdot \frac{1}{e} = e.$$

$$\text{We obtain } L = \frac{a}{2e^2} \cdot 1 \cdot \ln e = \frac{a}{2e^2}$$

□

Problem 5. D. M. Bătineţu-Giurgiu, Neculai Stanciu, RM 2/2012.

If $u_n = \frac{(n+2)^{n+1}}{(n+1)^n}, \forall n \in \mathbb{N}^*$, then compute $\lim_{n\to\infty}\left(\sqrt[n+1]{u_1 u_2 \ldots u_n u_{n+1}} - \sqrt[n]{u_1 u_2 \ldots u_n}\right)$.

Solution.

$$B_n = \sqrt[n+1]{u_1u_2\dots u_nu_{n+1}} - \sqrt[n]{u_1u_2\dots u_n} = \sqrt[n]{u_1u_2\dots u_n}(v_n-1) =$$

$$= \sqrt[n]{u_1u_2\dots u_n}\cdot\frac{v_n-1}{\ln v_n}\cdot\ln v_n = \frac{\sqrt[n]{u_1u_2\dots u_n}}{n}\cdot\frac{v_n-1}{\ln v_n}\cdot\ln v_n^n, \forall n>2, \text{ where}$$

$$v_n = \frac{\sqrt[n+1]{u_1u_2\dots u_nu_{n+1}}}{\sqrt[n]{u_1u_2\dots u_n}}, \forall n\geq 2$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{u_1u_2\dots u_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{u_1u_2\dots u_n}{n^n}} = \lim_{n\to\infty}\Big(\frac{u_1u_2\dots u_nu_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{u_1u_2\dots u_n}\Big) = \lim_{n\to\infty}\Big(\frac{u_{n+1}}{n+1}\cdot e_n^{-1}\Big) =$$

$$= e^{-1}\cdot\lim_{n\to\infty}\frac{(n+3)^{n+2}}{(n+2)^{n+1}(n+1)} = e^{-1}\cdot\lim_{n\to\infty}e_{n+2}\cdot\frac{n+2}{n+1} = e^{-1}\cdot e\cdot 1 = 1,$$

where we denote $e_n = \Big(1+\dfrac{1}{n}\Big)^n$; so $\lim\limits_{n\to\infty} v_n = 1$, so $\lim\limits_{n\to\infty}\dfrac{v_n-1}{\ln v_n} = 1$.

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty}\Big(\frac{u_1u_2\dots u_nu_{n+1}}{u_1u_2\dots u_n}\cdot\frac{1}{\sqrt[n+1]{u_1u_2\dots u_{n+1}}}\Big) = \lim_{n\to\infty}\Big(\frac{u_{n+1}}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{u_1u_2\dots u_{n+1}}}\Big) = e\cdot 1 = e.$$

We obtain $\lim\limits_{n\to\infty} B_n = 1\cdot 1\cdot\ln e = 1.$

□

Problem 6. D. M. Bătineţu-Giurgiu, Neculai Stanciu, AMM 9/2012.

$$\text{Compute } \lim_{n\to\infty}\left(x^{\sin^2 t}\left(((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - ((\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right)\right)$$

where $t\in\mathbb{R}$ and Γ is gamma function.

Solution.

$$\lim_{n\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{\text{C-D'A}}{=}$$

$$= \lim_{n\to\infty}\Big(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\Big) = \lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^n = \frac{1}{e};$$

$$f(x) = x^{\sin^2 t}\left((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - (\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right) = x^{\sin^2 t}(\Gamma(x+1))^{\frac{\cos^2 t}{x}}(u(x)-1), u:\mathbb{R}_+^*\to\mathbb{R},$$

$$u(x) = \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{\cos^2 t}$$

$$\lim_{n\to\infty}u(x) = \lim_{n\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{\cos^2 t} = \Big(\frac{1}{e}\cdot e\cdot 1\Big)^{\cos^2 t} = 1; \lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1.$$

$$\lim_{n\to\infty}(u(x))^x = \lim_{n\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{x\cos^2 t} = \lim_{n\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} =$$

$$= \lim_{n\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} = e^{\cos^2 t}. \text{ So,}$$

$$\lim_{n\to\infty}f(x) = -\lim_{n\to\infty}\left(x^{\sin^2 t}\left(\frac{\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot x\right)^{\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right) =$$

$$= -\lim_{n\to\infty}\left(\left(\frac{(\Gamma(x+1)^{\frac{1}{x}}}{x}\right)^{\cos^2 t}\cdot x^{\sin^2 t+\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$= e^{\cos^2 t}\lim_{n\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln(\lim_{n\to\infty}(u(x))^x) = e^{\cos^2 t}\cdot 1\cdot\ln e^{\cos^2 t} = e^{\cos^2 t}\cdot\cos^2 t$$

□

Problem 7. D. M. Bătineţu-Giurgiu, Neculai Stanciu, MP 2/2013.
If $x\in\mathbb{R}$, and $(L_n(x))_{n\geq 2}$ is defined by $L_n(x) = n^{\cos^2 x}\Big((\sqrt[n+1]{(n+1)!})^{\sin^2 x} - (\sqrt[n]{n!})^{\sin^2 x}\Big)$, then compute $\lim_{n\to\infty} L_n(x)$.

Solution.

$$L_n(x) = n^{\cos^2 x}\Big((\sqrt[n+1]{(n+1)!})^{\sin^2 x} - (\sqrt[n]{n!})^{\sin^2 x}\Big) = n^{\cos^2 x}(\sqrt[n]{n!})^{\sin^2 x}(u_n-1) =$$

$$=\Big(\frac{\sqrt[n]{n!}}{n}\Big)^{\sin^2 x} n^{\sin^2 x+\cos^2 x}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n =\Big(\frac{\sqrt[n]{n!}}{n}\Big)^{\sin^2 x} n\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$$

where $u_n =\Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\Big)^{\sin^2 x}$ and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\Big(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\Big)^{\sin^2 x} =\Big(\frac{1}{e}\cdot e\cdot 1\Big)^{\sin^2 x} = 1;\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1,$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\Big)^{n\sin^2 x} = \lim_{n\to\infty}\Big(\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\Big)^{\sin^2 x} =$$

$$= \lim_{n\to\infty}\Big(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\Big)^{\sin^2 x} = e^{\sin^2 x}, \text{ we obtain}$$

$$\lim_{n\to\infty} L_n(x) =\Big(\frac{1}{e}\Big)^{\sin^2 x}\cdot 1\cdot\ln e^{\sin^2 x} = \frac{\sin^2 x}{e^{\sin^2 x}}$$

□

Observation. If $x = \frac{\pi}{2}$, then $\sin x = 1, \cos x = 0$ so
$L_n\big(\frac{\pi}{2}\big) = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, i.e. we obtain the limit of Traian Lalescu:

$$\lim_{n\to\infty} L_n\Big(\frac{\pi}{2}\Big) = \lim_{n\to\infty} L_n = \frac{1}{e}$$

Problem 8. D. M. Bătineţu-Giurgiu, Neculai Stanciu, RMM 2020
Let $(x_n)_{n\geq 1}, x_n\in\mathbb{R}_+^*, \forall\in\mathbb{N}^*$ with $\lim_{n\to\infty}(x_{n+1}-x_n) = x\in\mathbb{R}_+^*$. Compute:

$$\lim_{n\to\infty}(x_{n+1}\sqrt[n+1]{n+1} - x_n\sqrt[n]{n})$$

Solution 1.

$y_n = x_{n+1}\sqrt[n+1]{n+1} - x_n\sqrt[n]{n} = (x_{n+1}-x_n)\sqrt[n+1]{n+1} + x_n(\sqrt[n+1]{n+1} - \sqrt[n]{n}), \forall n\in\mathbb{N}^* - \{1\};$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_{n+1}-x_n)\cdot\lim_{n\to\infty}\sqrt[n+1]{n+1} + \lim_{n\to\infty}\frac{x_n}{n}\cdot\lim_{n\to\infty}(n(\sqrt[n+1]{n+1} - \sqrt[n]{n})) =$$

$$= x\cdot 1 + \lim_{n\to\infty}\frac{x_{n+1}-x_n}{(n+1)-n}\cdot\lim_{n\to\infty} n\sqrt[n]{n}(u_n-1) = x + x\lim_{n\to\infty}\sqrt[n]{n}\cdot\lim_{n\to\infty}\Big(\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\Big);$$

$$\text{where } u_n = \frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}, \forall n\in\mathbb{N}^* - \{1\};\ \lim_{n\to\infty} u_n = 1 \text{ then } \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1;$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{1}{\sqrt[n+1]{n+1}}\right) = 1.$$

So, $\lim_{n\to\infty} y_n = x + x\cdot 1\cdot 1\cdot \ln(\lim_{n\to\infty} u_n^n) = x + x\cdot 1\cdot 1\cdot 0 = x$

□

Solution 2.

$y_n = x_{n+1}\sqrt[n+1]{n+1} - x_n\sqrt[n]{n} = x_n\sqrt[n]{n}(v_n - 1)$, where $v_n = \dfrac{x_{n+1}}{x_n}\cdot\dfrac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}, \forall n\in\mathbb{N}^* - \{1\}$,

then $\lim_{n\to\infty} v_n = \lim_{n\to\infty}\dfrac{x_{n+1}}{x_n}\cdot\lim_{n\to\infty}\dfrac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} = \lim_{n\to\infty}\dfrac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\cdot\dfrac{1}{1} = \dfrac{x}{x}\cdot 1 = 1$

So, $\lim_{n\to\infty} v_n = 1$, $\lim_{n\to\infty}\dfrac{v_n - 1}{\ln v_n} = 1$ and $\lim_{n\to\infty} v_n^n = \lim_{n\to\infty}\left(\dfrac{x_{n+1}}{x_n}\right)^n\cdot\lim_{n\to\infty}\left(\dfrac{n+1}{n}\cdot\dfrac{1}{\sqrt[n+1]{n+1}}\right) =$

$= \lim_{n\to\infty}\left(\dfrac{x_{n+1}}{x_n}\right)^n\cdot 1\cdot 1 = \lim_{n\to\infty}\left(\left(1+\dfrac{x_{n+1}-x_n}{x_n}\right)^{\frac{x_n}{x_{n+1}-x_n}}\right)^{(x_{n+1}-x_n)\frac{n}{x_n}} = e^{x\cdot\frac{1}{x}} = e$. We obtain

$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\dfrac{x_n\sqrt[n]{n}}{n}\cdot\lim_{n\to\infty}\dfrac{v_n-1}{\ln v_n}\cdot\ln(\lim_{n\to\infty} v_n^n) = \lim_{n\to\infty}\dfrac{x_n}{n}\cdot\lim_{n\to\infty}\sqrt[n]{n}\cdot 1\cdot\ln e = x\cdot 1\cdot\ln e = x.$

□

Problem 9. D. M. Bătineţu-Giurgiu, Neculai Stanciu, RMM 1/2018.

Let $(x_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty}(x_{n+1} - x_n) = x > 0$. Compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)$$

Solution.

Let $(y_n)_{n\geq 1}, y_n = \dfrac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}} - \dfrac{nx_n}{\sqrt[n]{(2n-1)!!}} = \dfrac{nx_n}{\sqrt[n]{(2n-1)!!}}(u_n - 1) =$

$$= \frac{x_n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\in\mathbb{N}^* - \{1\} \text{ where}$$

$$u_n = \frac{n+1}{n}\cdot\frac{x_{n+1}}{x_n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} = \frac{x_{n+1}}{n+1}\cdot\frac{n}{x_n}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}}.$$

We have $\lim_{n\to\infty}\dfrac{x_n}{n} = \lim_{n\to\infty}\dfrac{x_{n+1}-x_n}{(n+1)-n} = x$ and

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty}\left(\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}\right) = \lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}.$$

So, $\lim_{n\to\infty} u_n = x\cdot\dfrac{1}{x}\cdot 1\cdot\dfrac{2}{e}\cdot\dfrac{e}{2} = 1$, $\lim_{n\to\infty}\dfrac{u_n-1}{\ln u_n} = 1$;

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)^n\cdot\lim_{n\to\infty}\frac{(2n-1)!!}{(2n+1)!!}\sqrt[n+1]{(2n+1)!!} =$$

$$= e\cdot\lim_{n\to\infty}\left(\left(1+\frac{x_{n+1}-x_n}{x_n}\right)^{\frac{x_n}{x_{n-1}-x_n}}\right)^{\frac{n(x_{n+1}-x_n)}{x_n}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{2n+1} = e\cdot e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{(2n-1)^n}} =$$

$$= e^2\cdot\lim_{n\to\infty}\left(\frac{(2n+1)!!}{(2n+1)^{n+1}}\cdot\frac{(2n-1)^n}{(2n-1)!!}\right) = e^2\cdot\lim_{n\to\infty}\left(\frac{2n-1}{2n+1}\right)^n = e^2\cdot e^{-1} = e.$$

Then, $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt[n]{(2n-1)!!}} \cdot \lim\limits_{n\to\infty} \dfrac{u_n - 1}{\ln u_n} \cdot \ln(\lim\limits_{n\to\infty} u_n^n) = \lim\limits_{n\to\infty} \dfrac{x_n}{n} \cdot \lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{(2n-1)!!}} \cdot \ln e =$

$$= x \cdot \frac{e}{2} \cdot 1 = \frac{xe}{2}.$$

□

Problem 10. D. M. Bătineţu-Giurgiu, Neculai Stanciu, AMM 4/2014.

$$\text{Compute: } \lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}} - 1\right)\right)$$

Solution.

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e;$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} =$$

$$= \lim_{n\to\infty}\frac{2n+1}{(n+1)e_n} = \frac{2}{e}, \text{ where we denote } e_n = \left(1+\frac{1}{n}\right)^n, \forall n \in \mathbb{N}^*,$$

$$u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}, \forall n \geq 2; \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right) = 1, \lim_{n\to\infty}\frac{u_n-1}{\ln u_n} = 1;$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} = e.$$

We denote $t_n = \frac{\pi}{4}u_n, \forall n \geq 1$ and we have

$$\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}} - \right)\right) = \lim_{n\to\infty}\frac{\sqrt[n+1]{(2n-1)!!}}{n}\cdot\lim_{n\to\infty} n\left(\tan t_n - \tan\frac{\pi}{4}\right) =$$

$$= \frac{2}{e}\cdot\lim_{n\to\infty}\frac{\sin(t_n-\frac{\pi}{4})}{\cos t_n\cos\frac{\pi}{4}}\cdot n = \frac{2}{e}\lim_{n\to\infty}\left(\frac{\sin(t_n-\frac{\pi}{4})}{t_n-\frac{\pi}{4}}\cdot n(t_n-\frac{\pi}{4})\right)\frac{1}{\cos^2\frac{\pi}{4}} =$$

$$= \frac{4}{e}\cdot 1\cdot\lim_{n\to\aleph} n\left(t_n-\frac{\pi}{4}\right) = \frac{4}{e}\lim_{n\to\infty} n\cdot\frac{\pi}{4}\cdot(u_n-1) = \frac{4}{e}\cdot\frac{\pi}{4}\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right) =$$

$$= \frac{\pi}{e}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) = \frac{\pi}{e}\cdot e\cdot\lim_{n\to\infty}\sqrt[n]{n!}(u_n-1) =$$

$$= \pi\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\lim_{n\to\infty} n(u_n-1) =$$

$$= \frac{\pi}{e}\cdot\lim_{n\to\aleph}\left(\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = \frac{\pi}{e}\cdot 1\cdot\ln e = \frac{\pi}{e}.$$

□

Problem 11. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SSM 5/2014.

Let $a \in \mathbb{R}_+^*$ and $\{E\}_{n\geq 0}$ is defined by $E_n = \sum\limits_{k=0}^{n}\dfrac{1}{k!}$. Compute $\lim\limits_{n\to\infty}\sqrt[n]{n!}(a^{\sqrt[n]{E_n}-1} - 1)$.

Solution.

$$x_n = \sqrt[n]{n!}(a^{\sqrt[n]{E_n-1}} - 1) = \frac{\sqrt[n]{n!}}{n} \cdot n(a^{\sqrt[n]{E_n}-1}) = \frac{\sqrt{n!}}{n} \cdot n(\sqrt[n]{E_n} - 1) \cdot \frac{a^{\sqrt[n]{E_n}-1} - 1}{\sqrt[n]{E_n} - 1} =$$

$$= \frac{\sqrt[n]{n!}}{n} \cdot \frac{a^{\sqrt[n]{E_n-1}}}{\sqrt[n]{E_n} - 1} \cdot \frac{\sqrt[n]{E_n} - 1}{\frac{1}{n}\ln E_n} \cdot \ln E_n, \forall n \geq 2. \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}, \lim_{n\to\infty} \frac{a^{\sqrt[n]{E_n}-1} - 1}{\sqrt[n]{E_n} - 1} = \ln a,$$

$\lim\limits_{n\to\infty} \dfrac{a^{\sqrt[n]{E_n}-1}}{\frac{1}{n}\ln E_n} = 1$ and $\lim\limits_{n\to\infty} E_n = e$. We obtain $\lim\limits_{n\to\aleph} x_n = \dfrac{1}{e} \cdot \ln a \cdot 1 \cdot \ln e = \dfrac{\ln a}{e}$

□

Problem 12. D. M. Bătineţu-Giurgiu, Neculai Stanciu, RMM 2020.

Let $\{\gamma_n\}_{n\geq 1}, \gamma_n = -\ln n + \sum\limits_{k=1}^{n} \dfrac{1}{k}$, with $\lim\limits_{n\to\infty} \gamma_n = \gamma$ (i.e. γ is Euler - Mascheroni constant)

$$\text{Compute } \lim_{n\to\infty} (\sin\gamma_n - \sin\gamma)\sqrt[n]{n!}.$$

Solution.

$$x_n = (\sin\gamma_n - \sin\gamma) \cdot \sqrt[n]{n!} = 2 \cdot \sqrt[n]{n!} \cdot \sin\frac{\gamma_n - \gamma}{2} \cdot \cos\frac{\gamma_n + \gamma}{2} =$$

$$= \frac{\sqrt{n!}}{n} \cdot \cos\frac{\gamma_n + \gamma}{2} \cdot \frac{\sin\frac{\gamma_n - \gamma}{2}}{\frac{\gamma_n - \gamma}{2}} \cdot n \cdot (\gamma_n - \gamma), \forall n \geq 2; \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}, \lim_{n\to\infty} \cos\frac{\gamma_n + \gamma}{2} = \cos\gamma,$$

$$\lim_{n\to\infty} \frac{\sin\frac{\gamma_n - \gamma}{2}}{\frac{\gamma_n - \gamma}{2}} = 1 \text{ and } \lim_{n\to\infty} n(\gamma_n - \gamma) = \lim_{n\to\infty} \frac{\gamma_n - \gamma}{\frac{1}{n}} \underset{(\frac{0}{0})}{\overset{\text{Cesaro-Stolz}}{=}} \lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{\gamma_n - \gamma_{n+1}}{\frac{1}{n} - \frac{1}{n+1}} = \lim_{n\to\infty} \frac{-\frac{1}{n+1} + \ln\frac{n+1}{n}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} n^2\left(\ln\frac{n+1}{n} - \frac{1}{n+1}\right) =$$

$$= \lim_{n\to\infty} n^2\left(\ln(n+1) - \ln n - \frac{1}{n+1}\right) =$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+\frac{1}{x}) + \ln x - \frac{x}{x+1}}{x^2} = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x) - \frac{x}{x+1}}{x^2} \overset{\text{L'Hospital}}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{1+x} - \frac{1}{(x+1)^2}}{2x} =$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{x+1-1}{2x(x+1)^2} = \frac{1}{2}. \text{ It follows } \lim_{n\to\infty} x_n = \frac{1}{e} \cdot \frac{1}{2} \cdot \cos\gamma = \frac{\cos\gamma}{2e}.$$

□

Problem 13. D. M. Bătineţu-Giurgiu, Neculai Stanciu, REOIM 2013.
Let $s, t \in \mathbb{R}$ and $\{L_n(s,t)\}_{n\geq 2}$ be a sequence defined by
$L_n(s,t) = (n+1)^s \cdot \sqrt[n+1]{((n+1)!)^t} - n^s \cdot \sqrt[n]{(n!)^t}$. Evaluate:

$$\lim_{n\to\infty} L_n(s,t) = L(s,t)$$

Solution.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e; L_n(s,t) = n^s \cdot (\sqrt[n]{n!})^t \cdot (u_n - 1) = n^s \cdot (\sqrt[n]{n!})^t \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= n^{s+t-1} \cdot \left(\frac{\sqrt[n]{n!}}{n}\right)^t \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \in \mathbb{N}^* - \{1\}, \text{ where}$$

$$u_n = \Big(\frac{n+1}{n}\Big)^s \cdot \Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\Big), \forall n \in \mathbb{N}^* - \{1\};$$

$$\lim_{n\to\infty} u_n = 1 \Rightarrow \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1;$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \Big(\frac{n+1}{n}\Big)^{ns} \cdot \lim_{n\to\infty} \Big(\frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}}\Big)^t = e^s \cdot \lim_{n\to\infty} \Big(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\Big)^t = e^{s+t}.$$

So, $L(s,t) = \lim_{n\to\infty} L_n(s,t) = s \cdot \ln(\lim_{n\to\infty} u_n^n) \cdot \lim_{n\to\infty} \Big(\frac{\sqrt[n]{n!}}{n}\Big)^t \cdot \lim_{n\to\infty} n^{s+t-1} =$

$$= \ln e^{s+t} \cdot e^{-t} \cdot \lim_{n\to\infty} n^{s+t-1} = (s+t) \cdot \frac{1}{e^t} = \begin{cases} 0, \text{ if } s+t<1 \\ e^{-t}, \text{ if } s+t=1 \\ \infty, s+t>1 \end{cases}$$

□

Problem 14. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SM 1/2015

If $E_n = \sum_{k=0}^{n} \frac{1}{k!}$, compute $\lim_{n\to\infty} (e - E_n) \cdot (n+1)!$.

Solution.

$$\lim_{n\to\infty} (e - E_n) \cdot (n+1)! \overset{\text{C-S}}{=} \lim_{n\to\infty} \frac{(e - E_{n+1}) - (e - E_n)}{\frac{1}{(n+2)!} - \frac{1}{(n+1)!}} =$$

$$= \lim_{n\to\infty} \frac{E_{n+1} - E_n}{\frac{n+1}{(n+2)!}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{n+1}{(n+2)!}} = \lim_{n\to\infty} \frac{n+2}{n+1} = 1.$$

□

Problem 15. D. M. Bătineţu-Giurgiu, Neculai Stanciu, P 2/2014.

Compute $\lim_{n\to\infty} \sqrt{n}(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!})$.

Solution 1.

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty} \Big(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\Big) = e$$

$$x_n = \sqrt{n} \cdot (\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}) = \sqrt{n} \cdot \sqrt[2n]{n!} \cdot (u_n - 1) = \frac{\sqrt[2n]{n!}}{\sqrt{n}} \cdot (u_n - 1) \cdot n =$$

$$= \sqrt{\frac{\sqrt[n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2, \text{ where}$$

$$u_n = \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}} = \frac{\sqrt[2(n+1)]{(n+1)}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\sqrt[2n]{n!}} \cdot \sqrt{\frac{n+1}{n}}$$

$$\text{So, } u_n = \sqrt{e} \cdot \sqrt{\frac{1}{e}} \cdot 1 = 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \sqrt{\Big(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\Big)^n} = \sqrt{\lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}}} \cdot \sqrt{\lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}}} = \sqrt{e}$$

$$\text{We obtain } \lim_{n\to\infty} x_n = \sqrt{\frac{1}{e}} \cdot 1 \cdot \ln(\lim_{n\to\infty} u_n^n) = \frac{1}{\sqrt{e}} = \frac{1}{2\sqrt{e}}$$

□

Solution 2.

$$x_n = \sqrt{n}(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}) = \sqrt{n} \cdot \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt[2(n+1)]{(n+1)!} + \sqrt[2n]{n!}} =$$

$$= \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n}} + \sqrt{\frac{\sqrt[n]{n!}}{n}}}, \text{ so}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) \cdot \frac{1}{\lim_{n\to\infty}\left(\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n}} + \sqrt{\frac{\sqrt[n]{n!}}{n}}\right)} =$$

$$= \frac{1}{e} \cdot \frac{1}{\sqrt{\frac{1}{e}} + \sqrt{\frac{1}{e}}} = \frac{1}{2\sqrt{e}}$$

□

Solution 3.

$$x_n = \sqrt{n} \cdot (\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}) = \sqrt{n} \cdot \frac{\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}}{1+n-n} =$$

$$= \frac{\sqrt[2n]{n!}}{n} \cdot \frac{u_n - 1}{v_n - 1} \cdot \sqrt{n} = \sqrt{\frac{\sqrt[n]{n!}}{n}} \cdot \frac{u_n - 1}{v_n - 1} = \sqrt{\frac{\sqrt[n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n} =$$

$$= \sqrt{\frac{\sqrt[n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n^n}{\ln v_n^n}, \text{ where } u_n = \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}}, v_n = 1 + \frac{1}{n},$$

so $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 1$, then $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = \lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$. We have

$\lim_{n\to\infty} u_n^n = \sqrt{e}$, $\lim_{n\to\infty} v_n^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. So, $\lim_{n\to\infty} x_n = \frac{1}{\sqrt{e}} \cdot 1 \cdot 1 \cdot \frac{\ln \sqrt{e}}{\ln e} = \frac{1}{2\sqrt{e}}$.

□

Problem 16. D. M. Bătineţu-Giurgiu, Neculai Stanciu, AMM 7/2016.

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function such that $\lim_{n\to\infty} \frac{f(x)}{x^m} = a \in \mathbb{R}_+^*$ with $m \in [1, \infty)$.

$$\text{Compute } \lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)}\right).$$

Solution.

$$B_n = \sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)} = \sqrt[n]{\prod_{k=1}^{n} f(k)} \cdot (u_n - 1) =$$

$$= \sqrt[n]{\prod_{k=1}^{n} f(k)} \cdot \frac{1}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = \frac{1}{n^{m+1}} \cdot \sqrt[n]{\prod_{k=1}^{n} f(k)} \cdot n^m \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2, \text{ where}$$

$$u_n = \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}}{\sqrt[n]{\prod_{k=1}^{n} f(k)}}, \forall k \in \mathbb{N}-\{1\}; \lim_{n\to\infty} u_n = \left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}}{(n+1)^m} \cdot \frac{n^m}{\sqrt[n]{\prod_{k=1}^{n} f(k)}} \cdot \left(\frac{n+1}{n}\right)^m \right);$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{\prod_{k=1}^{n} f(k)}}{n^m} = \lim_{n\to\infty} \sqrt[n]{\frac{\prod_{k=1}^{n} f(k)}{n^{nm}}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty} \frac{\prod_{k=1}^{n+1} f(k)}{\prod_{k=1}^{n} f(k)} \cdot \frac{n^{nm}}{(n+1)^{(n+1)m}} =$$

$$\lim_{n\to\infty} \frac{f(n+1)}{(n+1)^m} \cdot \left(\frac{n}{n+1}\right)^{nm} = \frac{a}{e^m}; \lim_{n\to\infty} u_n = \frac{a}{e^m} \cdot \frac{e^m}{a} \cdot 1 = 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n+1} f(k)}{\prod_{k=1}^{n} f(k)} \cdot \frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}} \right) = \lim_{n\to\infty} \frac{f(n+1)}{(n+1)^m} \cdot \frac{(n+1)^m}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}} = a \cdot \frac{e^m}{a} = e^m$$

We obtain $\lim_{n\to\infty} B_n = \frac{a}{e^m} \cdot 1 \cdot \ln e^m \cdot \lim_{n\to\infty} n^{m-1} = \frac{am}{e^m} \cdot \lim_{n\to\infty} n^{m-1} = \begin{cases} \frac{a}{e}, \text{ for } m = 1 \\ \infty, \text{ for } m \in (1, \infty) \end{cases}$.

□

Problem 17. D. M. Bătineţu-Giurgiu, Neculai Stanciu, P 1/2018.

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_n}{n!} = a > 0$.

Compute $\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} \right)$

Solution.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \lim_{n\to\infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \left(\frac{n}{n+1}\right)^n \right) =$$

$$= a \cdot \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{e}. \text{ So } x_n = \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} = \frac{n^2}{\sqrt[n]{a_n}}(u_n - 1) = \frac{n^2}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \frac{n}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2. \text{ We denote } u_n = \left(\frac{n+1}{n}\right)^2 \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}, \forall n \geq 2, \text{ so}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{\sqrt[n]{a_n}}{n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot e \cdot 1 = 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1;$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^{2n} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} \right) =$$

$$= \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^{2n} \cdot \frac{a_n}{n!} \cdot \frac{(n+1)!}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \right) =$$

$$= e^2 \cdot a \cdot \frac{1}{a} \cdot \frac{1}{e} = e. \text{ We obtain } \lim_{n\to\infty} x_n = e \cdot 1 \cdot \ln(\lim_{n\to\infty} u_n^n) = e \cdot \ln e = e \cdot 1 = e.$$

□

Problem 18. D. M. Bătineţu-Giurgiu, Neculai Stanciu, PME 1/2015

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_n}{n!} = a > 0$. Compute:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right)$$

Solution.

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\overset{\text{C-D'A}}{=}\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}\Big)=\lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\Big(\frac{n}{n+1}\Big)^n\right)=$$

$$=a\cdot\frac{1}{a}\cdot\frac{1}{e}=\frac{1}{e}.\text{ So, }x_n=\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}(u_n-1)=\sqrt[n]{a_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$$=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\forall n\geq 2.\text{ We denote }u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}},\forall n\geq 2;$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\Big(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}\Big)=\frac{1}{e}\cdot e\cdot 1=1,\ \lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1;$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\Big)=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\Big)=a\cdot\frac{1}{a}\cdot e=e.$$

$$\text{We obtain }\lim_{n\to\infty}x_n=\frac{1}{e}\cdot 1\cdot\ln(\lim_{n\to\infty}u_n^n)=\frac{1}{e}\cdot\ln e=\frac{1}{e}\cdot 1=\frac{1}{e}.$$

□

Problem 19. D. M. Bătineţu-Giurgiu, Neculai Stanciu, CM 7/2014.

Let $(a_n)_{n\geq 1}$ be a positive real sequence and $a>0$ such that $\lim_{n\to\infty}(a_n-a\cdot n!)=b>0$.

$$\text{Compute }\lim_{n\to\infty}(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}).$$

Solution.

$$\lim_{n\to\infty}(a_n-a\cdot n!)=b,\ \lim_{n\to\infty}\Big(\frac{a_n}{n!}-a\Big)=\lim_{n\to\infty}\frac{b}{n!}=0,\ \lim_{n\to\infty}\frac{a_n}{n!}=a>0$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\overset{\text{C-D'A}}{=}\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n}\Big)=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\Big(\frac{n}{n+1}\Big)^n\Big)=$$

$$=a\cdot\frac{1}{a}\cdot\frac{1}{e}=\frac{1}{e}.\text{ So, }x_n=\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}(u_n-1)=\sqrt[n]{a_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$$=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\forall n\geq 2\text{ where }u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}},\forall,\forall n\geq 2;$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\Big(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}\Big)=\frac{1}{e}\cdot e\cdot 1,\text{ so }\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1.$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}\Big)=\lim_{n\to\infty}\Big(\frac{a_{n+1}}{(n+1)!}\cdot\frac{n!}{a_n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\Big)=a\cdot\frac{1}{a}\cdot e=e.$$

$$\text{We obtain }\lim_{n\to\infty}x_n=\frac{1}{e}\cdot 1\cdot\ln(\lim_{n\to\infty}u_n^n)=\frac{1}{e}\cdot\ln e=\frac{1}{e}\cdot\frac{1}{e}\cdot 1=\frac{1}{e}$$

□

Problem 20. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SSM 4/2018.

Let $(x_n)_{n\geq 1}, x_1=1, x_n=1\cdot\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}$. Compute

$$\lim_{n\to\infty}\Big(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\Big).$$

Solution.

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{x_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{x_n}}\overset{\text{Cauchy-D'Alembert}}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{x_{n+1}}\cdot\frac{x_n}{n^n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{1}{n^n}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}}=e\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}=e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{\text{C-D'A}}{=}$$

$$\overset{\text{C-D'A}}{=}e\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=e\cdot\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{e^2}{2};$$

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}=\frac{n^2}{\sqrt[n]{x_n}}\cdot(u_n-1)=\frac{n^2}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}=\frac{n}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n;$$

where we denote $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}$; $\lim_{n\to\infty}u_n=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$;

$$\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{\text{C-D'A}}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{2n+1}=\frac{e}{2}.$$

So, $\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{x_n}{x_{n+1}}\cdot\sqrt[n+1]{x_{n+1}}=e^2\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(2n+1)!!}}\cdot\sqrt[n+1]{x_{n+1}}=$

$$=e^2\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{x_{n+1}}}{n+1}=e^2\cdot\frac{e}{2}\cdot\frac{2}{e^2}=e.$$ We obtain

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\right)=\frac{e^2}{2}\cdot1\cdot\ln e=\frac{e^2}{2}$$

□

Problem 21. D. M. Bătineţu-Giurgiu, Neculai Stanciu, FQ 3/2014.

Compute $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}}-\sqrt[n]{n!F_n}\right)$.

Solution.

$$F_{n+2}-F_{n+1}-F_n=0,\forall n\in\mathbb{N}^*\Leftrightarrow\frac{F_{n+2}}{F_n}-1=0\Leftrightarrow\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1=0\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1\right)=0\Leftrightarrow x^2-x-1=0 \text{ where } x=\lim_{n\to\infty}\frac{F_{n+1}}{F_n},\text{ so } x=\frac{1\pm\sqrt{5}}{2}.$$

$$\frac{F_{n+1}}{F_n}>0; x=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{\sqrt{5}+1}{2}=\alpha.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!F_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}$$

$$\sqrt[n+1]{(n+1)!F_{n+1}}-\sqrt[n]{n!F_n}=\sqrt[n]{n!F_n}(u_n-1)=\sqrt[n]{n!F_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$$=\frac{\sqrt[n]{n!F_n}}{n}\cdot\frac{u_n-1}{\ln u_n},\forall n\geq2,\text{ where } u_n=\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{\sqrt[n]{n!F_n}}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{n!F_n}}\cdot\frac{n+1}{n}\right)=\frac{\alpha}{e}\cdot\frac{e}{\alpha}\cdot1=1,\text{ so }\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1,\text{ and}$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{n!F_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!F_{n+1}}}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}}}=\alpha\cdot\frac{e}{\alpha}=e.$$

So, $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}}-\sqrt[n]{n!F_n}\right)=\frac{\alpha}{e}\cdot 1\cdot\ln(\lim\limits_{n\to\infty}u_n^n)=\frac{\alpha}{e}\cdot\ln e=\frac{\alpha}{e}\cdot 1=\frac{\alpha}{e}$.
□

Problem 22. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SM 2/2019

Let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+^*$. Compute

$$\lim_{n\to\infty}\sqrt[n^2]{\frac{(n+1)^n}{n!}}.$$

Solution 1.

$$\lim_{n\to\infty}\sqrt[n^2]{\frac{(n+1)^n}{n!}}=\lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{\sqrt[n^2]{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n+1}{\sqrt[n]{n!}}}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{\sqrt[n]{n!}}\cdot\frac{n}{n+1}}=\lim_{n\to\infty}\sqrt[n]{e\cdot 1}=1.$$

□

Solution 2.

$$\sqrt[n^2]{\frac{(n+1)^n}{n!}}=\sqrt[n]{\frac{n+1}{\sqrt[n]{n!}}};$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{n+1}{\sqrt[n]{n!}}}\overset{\text{C-D'A}}{=}\lim_{n\to\infty}\frac{(n+2)\sqrt[n]{n!}}{(n+1)\sqrt[n+1]{(n+1)!}}=1\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\overset{\text{C-S}}{=}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}}{\sqrt[n+2]{(n+2)!}-\sqrt[n+1]{(n+1)!}}\overset{\text{Lalescu}}{=}\frac{\frac{1}{e}}{\frac{1}{e}}=1$$

□

2. Exercises proposals

E1. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SSM 4/2016
If $(2n-1)!!=1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\forall n\in\mathbb{N}^*$, then compute:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1}-\frac{\sqrt[n]{n!(2n-1)!!}}{n}\right)$$

E2. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SSM 4/2012

If $(a_n)_{n\geq 1}, a_n\in\mathbb{R}_+^*$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=b\in\mathbb{R}_+^*$, compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right)$$

E3. D. M. Bătineţu-Giurgiu, Neculai Stanciu, P 1/2012

Let $(a_n)_{n\geq 1},(b_n)_{n\geq 1}, a_n, b_n\in\mathbb{R}_+^*$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a\in\mathbb{R}_+^*$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n^3\cdot b_n}=b\in\mathbb{R}_+^*$

Compute $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}}-\sqrt[n]{\frac{b_n}{a_n}}\right)$

E4. D. M. Bătineţu-Giurgiu, Neculai Stanciu, CM 2/2012.

Let $e_n=\left(1+\frac{1}{n}\right)^n$ and $\gamma_n=-\ln n+\sum\limits_{k=1}^{n}\frac{1}{k}$. Compute

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!\gamma_n}}-\frac{n^2}{\sqrt[n]{(2n-1)!!e_n}}\right)$$

E5. D. M. Bătineţu-Giurgiu, Neculai Stanciu, LG 3/2013

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that $\lim\limits_{n\to\infty} \frac{f(x)}{x} = c \in \mathbb{R}_+^*$ and $(a_n)_{n\geq 1}$ a positive real sequence such that $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = a \in \mathbb{R}_+^*$. Compute

$$\lim_{n\to\infty}\Big(\frac{(n+1)^2}{\sqrt[n+1]{f(a_1)f(a_2)\dots f(a_n)f(a_{n+1})}} - \frac{n^2}{\sqrt[n]{f(a_1)f(a_2)\dots f(a_n)}}\Big)$$

E6. D. M. Bătineţu-Giurgiu, Neculai Stanciu, SSM 5/2016

If $a, b \in \mathbb{R}, a + b = 1$, compute $\lim\limits_{n\to\infty}\left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b}\right)$

where $e_n = \left(1 + \frac{1}{n}\right)^n$ and $c_n = -\ln n + \sum\limits_{k=1}^{n} \frac{1}{k}$.

E7. D. M. Bătineţu-Giurgiu, Neculai Stanciu, P 2/2012

If $x_n(t) = n^{1-t}\Big(\frac{(\sqrt[n+1]{(n+1)!})^{2t}}{(n+1)^t} - \frac{(\sqrt[n]{n!})^{2t}}{n^t}\Big)$ with $t > 0$, then compute $\lim\limits_{n\to\infty} x_n(t)$.

E8. D. M. Bătineţu-Giurgiu, Neculai Stanciu, P 2/2011

If $x_n = \sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$, then compute $\lim\limits_{n\to\infty}\Big(\frac{(n+1)^2}{x_{n+1}} - \frac{n^2}{x_n}\Big)$

E9. D. M. Bătineţu-Giurgiu, Neculai Stanciu, MP 3/2013

Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that $\lim\limits_{x\to\infty}(f(x+1) - f(x)) = a \in \mathbb{R}_+^*$, $\lim\limits_{n\to\infty} \frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^*$

and exists the limitis $\lim\limits_{x\to\infty} \frac{f(x)}{x}$ and $\lim\limits_{x\to\infty} \frac{(g(x))^{\frac{1}{x}}}{x}$.

For $t \in \mathbb{R}$, compute $\lim\limits_{x\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}}\right)$

E10. D. M. Bătineţu-Giurgiu, Anastasios Kotronis, Neculai Stanciu, SSM 1/2020.

Let $F : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a continue function such that $\lim\limits_{x\to\infty} \frac{f(x)}{x^2} = a \in \mathbb{R}_+^*$

Compute $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{\prod\limits_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod\limits_{k=1}^{n} \frac{f(k)}{k}}\right)$

E11. D. M. Bătineţu-Giurgiu, Neculai Stanciu, FQ 3/2019

Compute $\lim\limits_{n\to\infty}\Big(\lim\limits_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)F_{n+1}}} - (f(x))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\Big)$ where

$f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ verify $\lim\limits_{x\to\infty} \frac{f(x+1)}{xf(x)} = a \in \mathbb{R}_+^*$

E12. D. M. Bătineţu-Giurgiu, Neculai Stanciu, FQ 4/2014.

Let $e_n = \left(1+\frac{1}{n}\right)^n$, with $\lim\limits_{n\to\infty} e_n = e$. Compute $\lim\limits_{n\to\infty}\left(e\cdot \sqrt[n+1]{(n+1)!F_{n+1}} - e_n \cdot \sqrt[n]{n!F_n}\right)$.

E13. D. M. Bătineţu-Giurgiu, Neculai Stanciu, FQ 4/2014.

Let $e_n = \left(1+\frac{1}{n}\right)^n$, $\lim\limits_{n\to\infty} e_n = e$. Compute $\lim\limits_{n\to\infty}\left(e_{n+1}\cdot \sqrt[n+1]{(n+1)!F_{n+1}} - e_n\cdot \sqrt[n]{n!F_n}\right)$.

E14. D. M. Bătineţu-Giurgiu, Neculai Stanciu, AMM 10/2015.

$$\text{Let } e_n = \left(1+\frac{1}{n}\right)^n, \text{ with } \lim_{n\to\infty} e_n = e. \text{ Compute}$$
$$\lim_{n\to\infty}\left(e_{n+1}\cdot \sqrt[n+1]{(2n+1)!!L_{n+1}} - e_n\cdot \sqrt[n]{(2n-1)!!L_n}\right).$$

E15. D. M. Bătineţu-Giurgiu, Neculai Stanciu, FQ 3/2018.

$$\text{Let } (a_n)_{n\geq 0}, a_n \in \mathbb{R}_+^* \text{ such that } \lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a \in \mathbb{R}_+^*. \text{ Compute}$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left(\left(\sqrt[n+1]{a_{n+1}}\right)^{\frac{F_m}{F_{m+1}}} - \left(\sqrt[n]{a_n}\right)^{\frac{F_m}{F_{m+1}}}\right)n^{\frac{F_{m-1}}{F_{m+1}}}\right)\right)$$

E16. D. M. Bătineţu-Giurgiu, Neculai Stanciu

$$\text{Compute } \lim_{m\to\infty}\left(\sqrt[3n+3]{(n+1)!F_{n+1}} - \sqrt[3n]{n!F_n}\right)\sqrt[3]{n^2}$$

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