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ROMANIAN MATHEMATICAL MAGAZINE
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Find $a \in \mathbb{R}, n \in \mathbb{N}$ if:

$$\lim_{x\to 0}\frac{\sin\left(\frac{3\pi}{2}\left(\frac{\sin x}{x}\right)\right)+\sin\left(\frac{\pi}{2}\left(\frac{\sin x}{x}\right)\right)}{ax^n}=1$$

Proposed by Jalil Hajimir-Toronto-Canada
Solution by Yen Tung Chung-Taichung-Taiwan

$$\lim_{x\to 0}\frac{\sin\left(\frac{3\pi}{2}\left(\frac{\sin x}{x}\right)\right)+\sin\left(\frac{\pi}{2}\left(\frac{\sin x}{x}\right)\right)}{ax^n}$$

$$=\lim_{x\to 0}\frac{\sin\left(\frac{3\pi}{2}\left(\frac{x-\frac{1}{6}x^3+\cdots}{x}\right)\right)+\sin\left(\frac{\pi}{2}\left(\frac{x-\frac{1}{6}x^3+\cdots}{x}\right)\right)}{ax^n}$$

$$=\lim_{x\to 0}\frac{\sin\left(\frac{3\pi}{2}\left(1-\frac{1}{6}x^2+\cdots\right)\right)+\sin\left(\frac{\pi}{2}\left(1-\frac{1}{6}x^2+\cdots\right)\right)}{ax^n}$$

$$=\lim_{x\to 0}\frac{-\cos\left(\frac{\pi}{4}x^2+O(x^4)\right)+\cos\left(\frac{\pi}{12}x^2+O(x^4)\right)}{ax^n}$$

$$= \lim_{x\to 0} \frac{-\left(1 - \frac{\left(\frac{\pi}{4}x^2 + O(x^4)\right)^2}{2!} + \cdots\right) + \left(1 - \frac{\left(\frac{\pi}{12}x^2 + O(x^4)\right)^2}{2!} + \cdots\right)}{ax^n}$$

$$= \lim_{x\to 0} \frac{\left(\frac{\pi^2}{32} - \frac{\pi^2}{288}\right)x^4 + O(x^6)}{ax^n} = 1$$

So, $n = 4$, $a = \frac{\pi^2}{36}$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.