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Find:

$$\Omega=\int_0^{\infty}\frac{dx}{(1+x^2)(4+x^2)(2-cosx)}$$

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Solution 1 by Kamel Benaicha-Algiers-Algerie; Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$f(x)=\frac{1}{2-cosx},\ f \text{ –is } 2\pi \text{ periodic and even function.}$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}a_n cos(nx)\ ;\ f \text{ –is continuous function.}$$

Where

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{cos(nx)}{2-cosx}dx\overset{z=e^{iz}}{=}\frac{1}{\pi}Re\left(i\oint_{|z|=1}\frac{z^n}{z^2-4z+1}dz\right)$$

$$z^2-4z+1=0\to\Delta=12=\left(2\sqrt{3}\right)^2\to z_0=2-\sqrt{3},z_1=2+\sqrt{3};\ |z_0|<1,|z_1|>1$$

$$a_n=\frac{1}{\pi}Re\left(2i\oint_{|z|=1}\frac{\frac{z^n}{z-(2+\sqrt{3})}}{z-(2-\sqrt{3})}dz\right)=\frac{1}{\pi}Re\left(-4\pi\frac{(2-\sqrt{3})^n}{-2\sqrt{3}}\right)=\frac{2}{\sqrt{3}}(2-\sqrt{3})^n$$

$$\frac{1}{2-cosx}=\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\sum_{n=1}^{+\infty}(2-\sqrt{3})^n cos(nx)$$

$$\frac{1}{(1+x^2)(4+x^2)}=\frac{1}{3}\left(\frac{1}{1+x^2}-\frac{1}{4+x^2}\right)$$

$$\Omega=\frac{1}{3\sqrt{3}}\int_0^{+\infty}\left(\frac{1}{1+x^2}-\frac{1}{4+x^2}\right)dx+\frac{2}{3\sqrt{3}}\sum_{n=1}^{+\infty}(2-\sqrt{3})^n\left(\int_0^{+\infty}\frac{cos(nx)}{1+x^2}dx+\frac{1}{2}\int_0^{+\infty}\frac{cos(2nt)}{1+t^2}dt\right)$$

$$=\frac{\pi}{12\sqrt{3}}+\frac{2}{3\sqrt{3}}\sum_{n=1}^{+\infty}(2-\sqrt{3})^n\left(\frac{\pi}{2e^n}-\frac{\pi}{4e^{2n}}\right)$$

$$=\frac{\pi}{12\sqrt{3}}+\frac{\pi}{3\sqrt{3}}\left(\frac{2-\sqrt{3}}{e}\cdot\frac{1}{1-\frac{2-\sqrt{3}}{e}}-\frac{2-\sqrt{3}}{e^2}\cdot\frac{1}{1-\frac{2-\sqrt{3}}{e^2}}\right)$$

$$=\frac{\pi}{12\sqrt{3}}-\frac{\pi}{3\sqrt{3}}(\sqrt{3}-2)\left(\frac{1}{e+\sqrt{3}-2}-\frac{1}{2(e^2+\sqrt{3}-2)}\right)$$

$$\Omega=\int_0^{\infty}\frac{dx}{(1+x^2)(4+x^2)(2-cosx)}$$

$$=\frac{\pi}{36}\left(\sqrt{3}-4(3-2\sqrt{3})\right)\left(\frac{1}{e+\sqrt{3}-2}-\frac{1}{2(e^2+\sqrt{3}-2)}\right)$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

From fourier series we have the trigonometric series identity for: $|a|<1$:

$$\frac{1-acosx}{1-2acosx}=\sum_{n=0}^{\infty}a^n cos(nx)\Rightarrow\frac{1}{1-2acosx}=\frac{1}{1-a^2}\left(1+2\sum_{n=0}^{\infty}a^n cos(nx)\right)$$

Now let find:

$$\Omega=\int_0^\infty\frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a}-\cos x\right)}=\frac{2a}{1-a^2}\int_0^\infty\frac{\left(1+2\sum_{n=0}^\infty a^n\cos(nx)\right)}{(1+x^2)(4+x^2)}dx$$

$$=\frac{2a}{1-a^2}\int_0^\infty\frac{1}{(1+x^2)(4+x^2)}dx+\frac{4a}{1-a^2}\sum_{n=1}^\infty a^n\int_0^\infty\frac{\cos(nx)dx}{(1+x^2)(4+x^2)}$$

$$=\frac{2a}{1-a^2}M+\frac{4a}{1-a^2}\sum_{n=1}^\infty a^n\,\varphi(n)$$

$$M=\int_0^\infty\frac{1}{(1+x^2)(4+x^2)}dx=\frac{1}{3}\int_0^\infty\left(\frac{1}{1+x^2}-\frac{1}{4+x^2}\right)dx=\frac{1}{3}\left[\tan^{-1}x-\frac{\tan^{-1}\left(\frac{x}{2}\right)}{2}\right]_0^\infty=\frac{\pi}{12}$$

$$\varphi(n)=\int_0^\infty\frac{\cos(nx)dx}{(1+x^2)(4+x^2)}$$

$$=\pi i Re\left(Res\left(\frac{e^{inz}}{(1+z^2)(4+z^2)},i\right)+Res\left(\frac{e^{inz}}{(1+z^2)(4+z^2)},2i\right)\right)$$

$$=\pi i Re\left(\frac{e^{-n}}{6i}-\frac{e^{-2n}}{12i}\right)=\pi\left(\frac{e^{-n}}{6}-\frac{e^{-2n}}{12}\right)\Rightarrow$$

$$\sum_{n=1}^\infty a^n\,\varphi(n)\overset{|a|<e}{=}\frac{\pi a}{6(e-a)}-\frac{\pi a}{12(e^2-a)}$$

So, we get:

$$\Omega=\int_0^\infty\frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a}-\cos x\right)}$$

$$=\frac{2a}{1-a^2}\left(\frac{\pi}{12}+\frac{\pi a}{3(e-a)}-\frac{\pi a}{6(e^2-a)}\right)$$

But we have: $2-\cos x$ ***so*** $\frac{1+a^2}{2a}=2$ ***and*** $|a|<e\Rightarrow a=2-\sqrt{3}$

Finally:

$$\Omega=\int_0^\infty \frac{dx}{(1+x^2)(4+x^2)\left(\frac{1+a^2}{2a}-cosx\right)}$$

$$=\frac{\pi}{3}\cdot\frac{2(2-\sqrt{3})}{4\sqrt{3}-6}\left(\frac{1}{4}+(2-\sqrt{3})(\frac{1}{e-2+\sqrt{3}}-\frac{1}{2(e^2-2+\sqrt{3})}\right)$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.