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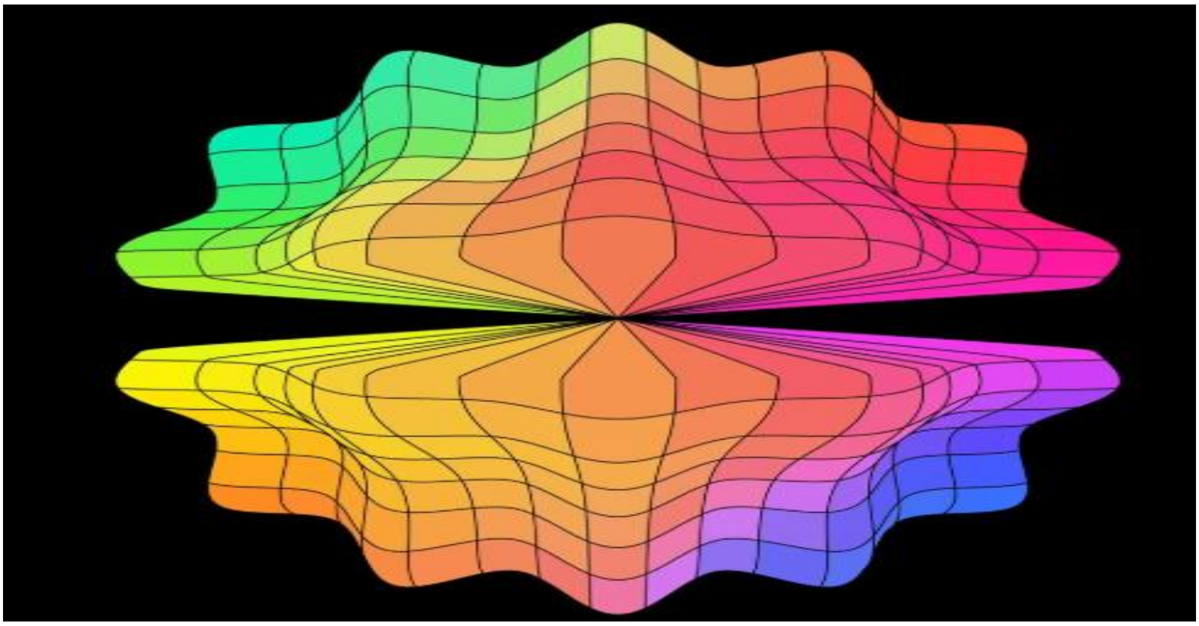

In ΔABC the following relationship holds:

$$\sqrt{\frac{2R}{r}}\sum\frac{b+c}{a}\geq\sum\left(\frac{m_a}{w_a}+\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{m_a}{w_a}\overset{\text{Tsintsifas}}{\leq}\sum\left(\frac{b^2+c^2}{2bc}\right)=\frac{1}{2}\sum\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\sum\left(\frac{c}{a}+\frac{b}{a}\right)=\frac{1}{2}\sum\frac{b+c}{a}$$

$$=\frac{1}{2}\sum\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\leq\frac{1}{2}\sum\frac{1}{\sin\frac{A}{2}}$$

$$\left(\because 0\leq\cos\frac{B-C}{2}\leq 1\text{ and analogs}\right)=\frac{1}{2r}\sum AI\Rightarrow\sum\frac{m_a}{w_a}\overset{(i)}{\leq}\frac{1}{2r}\sum AI$$

$$\sum\sqrt{\frac{m_a}{r_a}} \overset{\text{Panaitopol}}{\leq} \sum\sqrt{\frac{Rh_a}{2rr_a}} = \sum\sqrt{\frac{2Rrs(s-a)}{2r^2as}}$$

$$= \sum\sqrt{\frac{R(s-a)}{ra}} \overset{\text{by (1)}}{=} \sum\sqrt{\frac{4R^2\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}} = \frac{1}{2}\sum\frac{1}{\sin\frac{A}{2}}$$

$$= \frac{1}{2r}\sum AI \Rightarrow \sum\sqrt{\frac{m_a}{r_a}} \overset{(ii)}{\leq} \frac{1}{2r}\sum AI$$

ow, $x^4+1 \geq \frac{1}{2}(x^2+1)^2 \overset{A-G}{\geq} \frac{1}{2}(2x)(x^2+1) \Rightarrow x^4+1 \geq x(x^2+1) \Rightarrow x^2+\frac{1}{x^2} \overset{(1)}{\geq} x+\frac{1}{x}$

Choosing $x=\sqrt{\frac{a}{b}}$ in (1), we get : $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}} \leq \frac{a}{b}+\frac{b}{a}$ and analogs $\Rightarrow \sum\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$

$$\leq \sum\left(\frac{a}{b}+\frac{b}{a}\right) = \sum\left(\frac{c}{a}+\frac{b}{a}\right) = \sum\frac{b+c}{a}$$

$$= \sum\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}} \leq \sum\frac{1}{\sin\frac{A}{2}}\left(\because 0 \leq \cos\frac{B-C}{2} \leq 1 \text{ and analogs}\right) = \frac{1}{r}\sum AI$$

$$\Rightarrow \sum\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right) \overset{(iii)}{\leq} \frac{1}{r}\sum AI$$

$$\therefore \sum\frac{m_a}{w_a}+\sum\sqrt{\frac{m_a}{r_a}}+\sum\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right) \leq \frac{1}{2r}\sum AI+\frac{1}{2r}\sum AI+\frac{1}{r}\sum AI = \frac{2}{r}\sum AI$$

$$= 2\sum\frac{1}{\sin\frac{A}{2}} = 2\sum\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}}$$

$$\overset{CBS}{\leq} \frac{2}{r\sqrt{s}}\sqrt{\sum ab}\sqrt{\sum(s-a)} = \frac{2\sqrt{s}}{r\sqrt{s}}\sqrt{s^2+4Rr+r^2} = \frac{2\sqrt{s^2+4Rr+r^2}}{r}$$

$$\therefore \sum\frac{m_a}{w_a}+\sum\sqrt{\frac{m_a}{r_a}}+\sum\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right) \overset{(iv)}{\leq} \frac{2\sqrt{s^2+4Rr+r^2}}{r}$$

Now, $\sqrt{\frac{2R}{r}\sum\frac{b+c}{a}}=\sqrt{\frac{2R}{r}\frac{\sum bc(2s-a)}{4Rrs}}$

$$=\sqrt{\frac{2R}{r}\left(\frac{s^2+4Rr+r^2-6Rr}{2Rr}\right)}\overset{(v)}{=}\sqrt{\frac{2R}{r}\left(\frac{s^2-2Rr+r^2}{2Rr}\right)}$$

(iv), (v) $\Rightarrow$ it suffices to prove : $\sqrt{\frac{2R}{r}\left(\frac{s^2-2Rr+r^2}{2Rr}\right)}\geq\frac{2\sqrt{s^2+4Rr+r^2}}{r}$

$$\Leftrightarrow (s^2-2Rr+r^2)^2\geq 8Rr(s^2+4Rr+r^2)$$

$\Leftrightarrow s^4\overset{(a)}{\geq}(12Rr-2r^2)s^2+8Rr(4Rr+r^2)-(2Rr-r^2)^2$

Now, LHS of (a) $\overset{\text{Gerretsen}}{\geq}$ $(16Rr-5r^2)s^2\overset{?}{\geq}(12Rr-2r^2)s^2+8Rr(4Rr+r^2)$
$-(2Rr-r^2)^2$

$\Leftrightarrow (4Rr-3r^2)s^2\underset{(b)}{\overset{?}{\geq}}8Rr(4Rr+r^2)-(2Rr-r^2)^2$

Again, LHS of (b) $\overset{\text{Gerretsen}}{\geq}$ $(4Rr-3r^2)(16Rr-5r^2)\overset{?}{\geq}8Rr(4Rr+r^2)-(2Rr-r^2)^2$

$$\Leftrightarrow 9R^2-20Rr+4r^2\overset{?}{\geq}0$$

$\Leftrightarrow (R-2r)(9R-2r)\overset{?}{\geq}0\to$ true $\because R\overset{\text{Euler}}{\geq}2r\Rightarrow$ (b) $\Rightarrow$ (a) is true $\therefore\sqrt{\frac{2R}{r}\sum\frac{b+c}{a}}$

$$\geq\sum\left(\frac{m_a}{w_a}+\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)\text{ (Proved)}$$