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Prove:

$$\frac{\tan^{-1}x}{x}+\frac{\tan^{-1}y}{y}\ge\frac{\pi}{2};0<x,y\le 1$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Daniel Sitaru-Romania, Solution 2 by Marian Voinea-Romania

Solution 1 by Daniel Sitaru-Romania

$$f:(0,1]\to\mathbb{R}, f(x)=\frac{\tan^{-1}x}{x}, f'(x)=\frac{\frac{x}{1+x^2}-\tan^{-1}x}{x^2}$$

$$g:(0,1]\to\mathbb{R}, g(x)=\frac{x}{1+x^2}-\tan^{-1}x, g'(x)=\frac{-2x^2}{(1+x^2)^2}<0$$

$$\sup g(x)=\lim_{\substack{x\to 0\\ x>0}} g(x)=0\Rightarrow g(x)<0\Rightarrow f'(x)<0$$

$$f\ -\text{decreasing}, \min f(x)=f(1)=\frac{\pi}{4}\Rightarrow f(x)\ge\frac{\pi}{4}$$

$$\frac{\tan^{-1}x}{x}+\frac{\tan^{-1}y}{y}=f(x)+f(y)\ge\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$$

Equality holds for $x = y = 1$

Solution 2 by Marian Voinea-Romania

$$x, y \in \left(0, \frac{\pi}{2}\right] \Rightarrow x = \frac{1}{a}; y = \frac{1}{b}; a, b \in [1, \infty)$$

Then: $\frac{\tan^{-1}x}{x} + \frac{\tan^{-1}y}{y} = a \cdot \tan^{-1}\frac{1}{a} + b \cdot \tan^{-1}\frac{1}{b} = a \cdot \tan^{-1}a + b \cdot \tan^{-1}b$

The function $f: [1, \infty) \to \mathbb{R}, f(t) = t \cdot \tan^{-1}t$ *is increasing, then* $f(t) \geq f(1) = \frac{\pi}{4}$.

So,

$$\frac{\tan^{-1}x}{x} + \frac{\tan^{-1}y}{y} = f(x) + f(y) \geq \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Equality for $x = y = 1$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.