A Elegant Developing of the Integers Series Involving Degree Two Polynomials In The Denominator

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Introdution

Faced with the various Integrals and Series problems that we face, there were some that were quite recurring and almost always boiled down to the same problem, that of rational series with second degree polynomials in the denominator. With this came the elegant idea of a generalization from my esteemed friend Professor Paulo Sergio Lino, in which he proposed the present work. We will start from two special cases of series that run all integers, one will be the alternating general term and the other non-alternating, divided into only two simple parts. We will glimpse elegant, similar results involving sequential hyperbolic and trigonometric functions being branched out with a Δ's cases analysis generating and proving some important identities that are exaggeratedly usuals in the study of Series and Integrals. We will use crucial properties of the digamma function and some trigonometric identities to make our results as simple as possible.

Part 1.0.

Let a, b and $c \in \mathbb{C}$ then,

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{an^2+bn+c} = \frac{1}{\sqrt{\Delta}} \frac{4\pi \sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}$$

where $\Delta = b^2 - 4ac$.

Proof:
First, note that

$$\sum_{n=-N}^{+N} s_n = \sum_{n=-N}^{0} s_n + \sum_{n=0}^{+N} s_n - a_0$$

where $s_n = \{a_{-N}, ..., a_{-1}, a_0, a_1, ..., a_N\}$ is a sequence of the terms associated to summations above. Thus

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{an^2+bn+c} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{an^2+bn+c} + \sum_{n=0}^{+\infty} \frac{(-1)^n}{an^2+bn+c} - \frac{1}{c}$$

Shifting $n \to -n$ on the second sum from the right side, we have

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{an^2+bn+c} = \sum_{n=0}^{\infty} \frac{(-1)^n}{an^2+bn+c} + \sum_{n=0}^{\infty} \frac{(-1)^n}{an^2-bn+c} - \frac{1}{c}$$

Now, let's call $\mathbf{\Omega}$ the series from the our problem and $\mathbf{\Omega_1}$, $\mathbf{\Omega_2}$ denoting the first and second series from the right side

$$\mathbf{\Omega} = \mathbf{\Omega_1} + \mathbf{\Omega_2} - \frac{1}{c} \quad (1)$$

Now, let's evaluating $\mathbf{\Omega_1}$. Let x_1 and x_2 roots of an^2+bn+c, then:

$$\mathbf{\Omega_1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{an^2+bn+c} = \sum_{n=0}^{\infty} \frac{(-1)^n}{a(n-x_1)(n-x_2)}$$

Applying Partial Fractions, we have that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{a(n-x_1)(n-x_2)} = \frac{1}{a(x_1-x_2)} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{n-x_2} - \frac{(-1)^n}{n-x_1} \right\}$$

$$\Rightarrow \mathbf{\Omega_1} = \frac{1}{a(x_1-x_2)} \left\{ \frac{1}{x_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{-\frac{n}{x_1}+1} - \frac{1}{x_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{-\frac{n}{x_2}+1} \right\}$$

From the Digamma Function, we have the following property:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{zn+1} = \frac{1}{2z}\left\{\psi\left(\frac{1+z}{2z}\right) - \psi\left(\frac{1}{2z}\right)\right\} \quad (2)$$

we have

$$\mathbf{\Omega_1} = \frac{1}{a(x_1-x_2)}\left\{\frac{1}{2}\left[\psi\left(\frac{1-x_1}{2}\right) - \psi\left(-\frac{x_1}{2}\right)\right] - \frac{1}{2}\left[\psi\left(\frac{1-x_2}{2}\right) - \psi\left(-\frac{x_2}{2}\right)\right]\right\}$$

Again, from the Digamma Function, we have

$$\psi(-z) - \psi(z) = \frac{1}{z} + \pi\cot(\pi z)$$

Then

$$\mathbf{\Omega_1} = \frac{1}{2a(x_1-x_2)}\left\{\psi\left(\frac{1-x_1}{2}\right) - \psi\left(\frac{1-x_2}{2}\right) + \frac{2}{x_2} - \frac{2}{x_1} + \pi\cot\left(\frac{\pi x_2}{2}\right) - \pi\cot\left(\frac{\pi x_1}{2}\right) + \psi\left(\frac{x_2}{2}\right) - \psi\left(\frac{x_1}{2}\right)\right\}$$

Similarly, for $\mathbf{\Omega_2}$, let x_3 and x_4 roots of $an^2 - bn + c$, note that $x_4 = -x_1$ and $x_3 = -x_2$ then using (2) we have directly:

$$\mathbf{\Omega_2} = \frac{1}{2a(x_1-x_2)}\left\{\psi\left(\frac{1+x_2}{2}\right) - \psi\left(\frac{1+x_1}{2}\right) + \psi\left(\frac{x_1}{2}\right) - \psi\left(\frac{x_2}{2}\right)\right\}$$

realizing the sum of $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$, we obtain the expression

$$\mathbf{\Omega_1}+\mathbf{\Omega_2} = \frac{1}{2a(x_1-x_2)}\left\{\psi\left(\frac{1+x_2}{2}\right) - \psi\left(\frac{1-x_2}{2}\right) + \psi\left(\frac{1-x_1}{2}\right) - \psi\left(\frac{1+x_1}{2}\right) + \frac{2}{x_2} - \frac{2}{x_1} + \pi\cot\left(\frac{\pi x_2}{2}\right) - \pi\cot\left(\frac{\pi x_1}{2}\right)\right\}$$

Note that:

$$\psi\left(\frac{1+x_2}{2}\right) - \psi\left(\frac{1-x_2}{2}\right) = \psi\left(1 - \frac{1-x_2}{2}\right) - \psi\left(\frac{1-x_2}{2}\right)$$

and

$$\psi\left(\frac{1-x_1}{2}\right) - \psi\left(\frac{1+x_1}{2}\right) = \psi\left(\frac{1-x_1}{2}\right) - \psi\left(1 - \frac{1+x_2}{2}\right)$$

then by the property:

$$\psi(z) - \psi(1-z) = -\pi\cot(\pi z)$$

we have

$$\Rightarrow\ \mathbf{\Omega_1} + \mathbf{\Omega_2} = \frac{1}{2a(x_1 - x_2)}\left\{\frac{2}{x_2} - \frac{2}{x_1} + \pi\cot\left(\frac{\pi}{2} - \frac{\pi x_2}{2}\right) - \pi\cot\left(\frac{\pi}{2} - \frac{\pi x_1}{2}\right) + \pi\cot\left(\frac{\pi x_2}{2}\right) - \pi\cot\left(\frac{\pi x_1}{2}\right)\right\}$$

$$\Rightarrow\ \mathbf{\Omega_1} + \mathbf{\Omega_2} = \frac{1}{2a(x_1 - x_2)}\left\{\frac{2}{x_2} - \frac{2}{x_1} + \pi\tan\left(\frac{\pi x_2}{2}\right) + \pi\cot\left(\frac{\pi x_2}{2}\right) - \pi\tan\left(\frac{\pi x_1}{2}\right) - \pi\cot\left(\frac{\pi x_1}{2}\right)\right\}$$

Note also that:

$$\tan(a) + \cot(a) = 2\csc(2a)$$

then

$$\Rightarrow\ \mathbf{\Omega_1} + \mathbf{\Omega_2} = \frac{1}{a(x_1 - x_2)}\left\{\frac{1}{x_2} - \frac{1}{x_1} + \pi\csc(\pi x_2) - \pi\csc(\pi x_1)\right\}$$

Since $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$, we have hence

$$\Rightarrow\ \mathbf{\Omega_1}+\mathbf{\Omega_2} = \frac{1}{\sqrt{\Delta}}\left\{\frac{\sqrt{\Delta}}{c} + \pi\csc\left(\frac{\pi b}{2a} - \frac{\pi\sqrt{\Delta}}{2a}\right) - \pi\csc\left(\frac{\pi b}{2a} + \frac{\pi\sqrt{\Delta}}{2a}\right)\right\} \quad (3)$$

where we using fact: $\csc(-a) = -\csc(a)$. Now, let's use following trigonometric identity:

$$\csc(x-y) - \csc(x+y) = \frac{4\sin(y)\cos(x)}{\cos(2y) - \cos(2x)}$$

So, taking $x = \frac{\pi b}{2a}$ and $y = \frac{\pi\sqrt{\Delta}}{2a}$ and plugging in (3)

$$\Rightarrow\ \mathbf{\Omega_1} + \mathbf{\Omega_2} = \frac{1}{c} + \frac{1}{\sqrt{\Delta}}\frac{4\pi\sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}$$

Replacing the above expression in the equation (1) we have

$$\mathbf{\Omega} = \frac{1}{\sqrt{\Delta}} \frac{4\pi \sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}$$

:

Then, we conclude that:

$$\therefore \boxed{\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{an^2+bn+c} = \frac{1}{\sqrt{\Delta}} \frac{4\pi \sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}, \text{ where } \Delta = b^2 - 4ac.} \blacksquare$$

Delta's cases

We observe to $\Delta > 0$, we don't have alteration in the theorem result. So, we stay with just two cases:

1.1 Same roots $(\Delta = 0)$

If $b^2 = 4ac$ with $a, c > 0$, then $\Delta = 0$, we have that $\mathbf{\Omega}$ takes the form

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{a\left(n \pm \frac{b}{2a}\right)^2} = \lim_{\Delta \to 0} \frac{1}{\sqrt{\Delta}} \frac{4\pi \sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}.$$

And discovering the RHS

$$\lim_{\Delta\to 0}\frac{1}{\sqrt{\Delta}}\frac{4\pi\sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}=4\pi\underbrace{\lim_{\Delta\to 0}\frac{\sin\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)}{\sqrt{\Delta}}}_{\lim_{x\to 0}\frac{\sin(x)}{x}=1}.\lim_{\Delta\to 0}\frac{\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}=$$

$$=4\pi.\frac{\pi}{2a}.\frac{\cos\left(\dfrac{\pi b}{2a}\right)}{1-\cos\left(\dfrac{\pi b}{a}\right)}=\frac{2\pi^2}{a}.\frac{1}{2}\cot\left(\frac{\pi b}{2a}\right)\csc\left(\frac{\pi b}{2a}\right)=\frac{\pi^2}{a}\cot\left(\frac{\pi b}{2a}\right)\csc\left(\frac{\pi b}{2a}\right).$$

Therefore, we conclude that for the case $\Delta = 0$

$$\therefore\ \boxed{\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{\left(n\pm\frac{b}{2a}\right)^2}=\pi^2\cot\left(\frac{\pi b}{2a}\right)\csc\left(\frac{\pi b}{2a}\right)}$$

1.2 Complex roots ($\Delta < 0$)

If $b^2 < 4ac$ with $c < 0$, then $\Delta < 0$, we have that $\mathbf{\Omega}$ takes the form

$$\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{an^2+bn+c}=\frac{1}{i\sqrt{\Delta}}\frac{4\pi\sin\left(\dfrac{\pi i\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi i\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}$$

Using the identities $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$ in RHS, we have

$$\Rightarrow\ \frac{1}{i\sqrt{\Delta}}\frac{4\pi\sin\left(\dfrac{\pi i\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cos\left(\dfrac{\pi i\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}=\frac{1}{\sqrt{\Delta}}\frac{4\pi\sinh\left(\dfrac{\pi\sqrt{\Delta}}{2a}\right)\cos\left(\dfrac{\pi b}{2a}\right)}{\cosh\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}.$$

Then, considering $|\Delta|$, we have finally for the case $\Delta < 0$

$$\therefore \boxed{\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{an^2+bn+c} = \frac{1}{\sqrt{|\Delta|}} \frac{4\pi \sinh\left(\frac{\pi\sqrt{|\Delta|}}{2a}\right)\cos\left(\frac{\pi b}{2a}\right)}{\cosh\left(\frac{\pi\sqrt{|\Delta|}}{a}\right) - \cos\left(\frac{\pi b}{a}\right)} \text{ for } \Delta < 0}$$

From the above result we can set $b = 0$, $c = x^2$ and $a = 1$ to we obtain type's series

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+x^2} = \frac{1}{x}.\frac{2\pi \sinh(\pi x)}{\cosh(2\pi x) - 1}$$

Note that: $\frac{\sinh(z)}{\cosh(2z)-1} = \frac{\operatorname{csch}(z)}{2}$, we have the identities

$$\therefore \boxed{\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{n^2+x^2} = \frac{\pi \operatorname{csch}(\pi x)}{x}}$$

$$\therefore \boxed{\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+x^2} = \frac{\pi \operatorname{csch}(\pi x)}{2x} + \frac{1}{2x^2}}$$

$$\therefore \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2-x^2} = -\frac{\pi \csc(\pi x)}{2x} - \frac{1}{2x^2}}$$

where in the middle identity i shifting $x \to ix$ to we get the third identity.

Part 2.0.

Let a, b and $c \in \mathbb{C}$ then,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{2\pi}{\sqrt{\Delta}}.\frac{\sin\left(\frac{\pi\sqrt{\Delta}}{a}\right)}{\cos\left(\frac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\frac{\pi b}{a}\right)}$$

where $\Delta = b^2 - 4ac$.

Proof:
Similarly to part **Part 1**, we broken the summation

$$\sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \sum_{n=0}^{\infty} \frac{1}{an^2+bn+c} + \sum_{n=0}^{\infty} \frac{1}{an^2-bn+c} - \frac{1}{c} = \mathbf{\Omega_1} + \mathbf{\Omega_2} - \frac{1}{c}$$

Evaluating $\mathbf{\Omega_1}$ we have that by Partial Fractions:

$$\mathbf{\Omega_1} = \frac{1}{a(x_1-x_2)} \sum_{n=0}^{\infty} \left\{ \frac{1}{n-x_1} - \frac{1}{n-x_2} \right\} = \frac{1}{a(x_1-x_2)} \left\{ \sum_{n=0}^{\infty} \frac{1}{n-x_1} - \sum_{n=0}^{\infty} \frac{1}{n-x_2} \right\}$$

$$\implies \therefore\ \mathbf{\Omega_1} = \frac{1}{a(x_1-x_2)} \{\psi(-x_2) - \psi(-x_1)\} \quad (1)$$

Similarly to $\mathbf{\Omega_2}$ we have directly that

$$\implies \therefore\ \mathbf{\Omega_2} = \frac{1}{a(x_1-x_2)} \{\psi(x_1) - \psi(x_2)\} \quad (2)$$

plugging (1) and (2)

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{1}{a(x_1-x_2)} \{\psi(x_1) - \psi(-x_1) + \psi(-x_2) - \psi(x_2)\} - \frac{1}{c}$$

Using again the identity:

$$\psi(-z) - \psi(z) = \frac{1}{z} + \pi\cot(\pi z)$$

We have

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{1}{a(x_1-x_2)} \left\{ \frac{1}{x_2} - \frac{1}{x_1} + \pi\cot(\pi x_2) - \pi\cot(\pi x_1) \right\} - \frac{1}{c}$$

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{1}{ax_1x_2} + \frac{1}{a(x_1-x_2)}(\pi\cot(\pi x_2) - \pi\cot(\pi x_1)) - \frac{1}{c}$$

Note that $\dfrac{1}{ax_1x_2} = \dfrac{1}{c}$, so

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{1}{a(x_1-x_2)}(\pi\cot(\pi x_2) - \pi\cot(\pi x_1))$$

Since $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$, where $\Delta = b^2-4ac$, we have hence

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{\pi}{\sqrt{\Delta}}\left(\cot\left(-\frac{\pi b}{2a} - \frac{\pi\sqrt{\Delta}}{2a}\right) - \cot\left(-\frac{\pi b}{2a} + \frac{\pi\sqrt{\Delta}}{2a}\right)\right)$$

as $\cot(-z) = -\cot(z)$ we have

$$\implies \sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{\pi}{\sqrt{\Delta}}\left(\cot\left(\frac{\pi b}{2a} - \frac{\pi\sqrt{\Delta}}{2a}\right) - \cot\left(\frac{\pi b}{2a} + \frac{\pi\sqrt{\Delta}}{2a}\right)\right)$$

Now, let's use the following trigonometric identity

$$\cot(x-y) - \cot(x+y) = \frac{2\sin(2y)}{\cos(2y)-\cos(2x)}$$

we have finally

$$\therefore \boxed{\sum_{n=-\infty}^{+\infty} \frac{1}{an^2+bn+c} = \frac{2\pi}{\sqrt{\Delta}}.\frac{\sin\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)}{\cos\left(\dfrac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\dfrac{\pi b}{a}\right)}} \blacksquare$$

where $\Delta = b^2-4ac$.

Delta's cases

2.1 Same roots $(\Delta = 0)$

If $b^2 = 4ac$ with $a, c > 0$, then $\Delta = 0$, we have that $\mathbf{\Omega}$ takes the form

$$\implies \mathbf{\Omega} = \sum_{n=-\infty}^{\infty} \frac{1}{a\left(n \pm \frac{b}{2a}\right)^2} = \lim_{\Delta \to 0} \frac{2\pi}{\sqrt{\Delta}} . \frac{\sin\left(\frac{\pi\sqrt{\Delta}}{a}\right)}{\cos\left(\frac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\frac{\pi b}{a}\right)} .$$

Developing RHS

$$\Rightarrow \; \underbrace{2\pi \lim_{\Delta \to 0} \frac{\sin\left(\frac{\pi\sqrt{\Delta}}{a}\right)}{\sqrt{\Delta}}}_{\lim_{x\to 0}\frac{\sin(x)}{x}=1} . \lim_{\Delta \to 0} \frac{1}{\cos\left(\frac{\pi\sqrt{\Delta}}{a}\right) - \cos\left(\frac{\pi b}{a}\right)} = 2\pi . \frac{\pi}{a} . \frac{1}{1 - \cos\left(\frac{\pi b}{a}\right)} =$$

$$= \frac{2\pi^2}{a} . \left(\frac{1}{2}\csc^2\left(\frac{\pi b}{2a}\right)\right) = \frac{\pi^2}{a}\csc^2\left(\frac{\pi b}{2a}\right) .$$

Therefore, we conclude that for the case $\Delta = 0$

$$\therefore \; \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{\left(n \pm \frac{b}{2a}\right)^2} = \pi^2 \csc^2\left(\frac{\pi b}{2a}\right) .}$$

2.2 Complex roots $(\Delta < 0)$

If $b^2 < 4ac$ with $c < 0$, then $\Delta < 0$, we have that $\mathbf{\Omega}$ takes the form

$$\implies \mathbf{\Omega} = \sum_{n=-\infty}^{\infty} \frac{1}{an^2 + bn + c} = \frac{2\pi}{i\sqrt{\Delta}} . \frac{\sin\left(\frac{i\pi\sqrt{\Delta}}{a}\right)}{\cos\left(\frac{i\pi\sqrt{\Delta}}{a}\right) - \cos\left(\frac{\pi b}{a}\right)}$$

Using the identities $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$ in RHS, we have

$$\implies \frac{2\pi}{i\sqrt{\Delta}}.\frac{i\sinh\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)}{\cosh\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)} = \frac{1}{\sqrt{\Delta}}.\frac{2\pi\sinh\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)}{\cosh\left(\dfrac{\pi\sqrt{\Delta}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)}.$$

Then, considering $|\Delta|$, we have finally for the case $\Delta < 0$

$$\therefore \boxed{\sum_{n=-\infty}^{\infty}\frac{1}{an^2+bn+c} = \frac{1}{\sqrt{|\Delta|}}.\frac{2\pi\sinh\left(\dfrac{\pi\sqrt{|\Delta|}}{a}\right)}{\cosh\left(\dfrac{\pi\sqrt{|\Delta|}}{a}\right)-\cos\left(\dfrac{\pi b}{a}\right)},\text{ where }\Delta = b^2-4ac.}$$

Note that from above result we can to extract the following identities taking $b = 0$, $a = 1$ and $c = x^2$, where $x > 0$ we have

$$\implies \sum_{n=-\infty}^{\infty}\frac{1}{n^2+x^2} = \frac{1}{\sqrt{|-4x^2|}}.\frac{2\pi\sinh(\pi\sqrt{|-4x^2|})}{\cosh(\pi\sqrt{|-4x^2|})-1} = \frac{1}{2x}.\frac{2\pi\sinh(2\pi x)}{\cosh(2\pi x)-1} = \frac{\pi\coth(\pi x)}{x}$$

so

$$\therefore \sum_{n=-\infty}^{\infty}\frac{1}{n^2+x^2} = \frac{\pi\coth(\pi x)}{x}$$

Further

$$\Rightarrow \sum_{n=-\infty}^{\infty}\frac{1}{n^2+x^2} = 2\sum_{n=0}^{\infty}\frac{1}{n^2+x^2} - \frac{1}{x^2}$$

Then

$$\Rightarrow \therefore \sum_{n=0}^{\infty}\frac{1}{n^2+x^2} = \frac{\pi\coth(\pi x)}{2x} + \frac{1}{2x^2}$$

Observe that, shiffiting x by ix and using $\coth(iz) = -i\cot(z)$

$$\Rightarrow \therefore \sum_{n=0}^{\infty}\frac{1}{n^2-x^2} = -\frac{\pi\cot(\pi x)}{2x} - \frac{1}{2x^2}$$

Which are famous identities and very usuals both for solving integral problems and for solving series problems. So we have

$$\therefore \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{n^2+x^2} = \frac{\pi \coth(\pi x)}{x}}$$

$$\therefore \boxed{\sum_{n=0}^{\infty} \frac{1}{n^2+x^2} = \frac{\pi \coth(\pi x)}{2x} + \frac{1}{2x^2}}$$

$$\therefore \boxed{\sum_{n=0}^{\infty} \frac{1}{n^2-x^2} = -\frac{\pi \cot(\pi x)}{2x} - \frac{1}{2x^2}}$$

Discussions

It is quite obvious that the identities proved in the present work are only starting points to obtain a closed form for a number of other more complex problems that are just derivations of the problem addressed here, simply using from the Calculus techniques, i.e. Derivations or Integrations of the presented identities. Given the objectivity of this work, we will be able to skip many calculation processes making the use and manipulation of the obtained formulas sufficient. We are open to suggestions and constructive criticism with respect to the our work.

Grateful by the attention!