

SPECIAL SEQUENCES (I)

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ABSTRACT. In this paper we present some certain limits of sequences.

Theorem 1. Let $(a_n)_{n \geq 1}, a_1 = 1, a_{n+1} = (2n+1)!!a_n, \forall n \in \mathbb{N}^*$, then:

$$\lim_{n \rightarrow \infty} \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}} = \sqrt[4]{e}$$

Proof. If $(x_n)_{n \geq 1}, x_n > 0, \forall n \in \mathbb{N}^*$ and $\lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} = x > 0$, then:

$$(1) \quad \lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} = \sqrt{x}$$

$$\begin{aligned} \text{Indeed, } \ln \left(\lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} \right) &= \lim_{n \rightarrow \infty} \left(\ln \sqrt[n^2]{x_n} \right) = \lim_{n \rightarrow \infty} \frac{\ln x_n}{n^2} \stackrel{\text{Cesaro-Stolz}}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1)^2 - n^2} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\ln x_{n+1} - \ln x_n}{n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\ln x_{n+1} - \ln x_n}{n} = \\ &\stackrel{\text{Cesaro-Stolz}}{=} \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\ln x_{n+2} - 2\ln x_{n+1} + \ln x_n}{(n+1) - n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} = \\ &= \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} \right) = \frac{1}{2} \ln x = \ln \sqrt{x}, \text{ so } \lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} = \sqrt{x} \end{aligned}$$

Since, $a_{n+1} = (2n+1)!!a_n, \forall n \in \mathbb{N}^* \Rightarrow a_{n+2} = (2n+3)!!a_{n+1} = (2n+3)!!(2n+1)!!a_n$,

$\forall n \in \mathbb{N}^*$, we have $\frac{a_{n+2}a_n}{a_{n+1}^2} = \frac{(2n+3)!!(2n+1)!!a_n^2}{((2n+1)!!a_n)^2} = \frac{(2n+3)!!}{(2n+1)!!} = 2n+3, \forall n \in \mathbb{N}^*$

$$\begin{aligned} \text{If } y_n = \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}} &= \sqrt[2n^2]{\frac{((2n-1)!!)^n}{a_n^2}} = \left(\sqrt[n^2]{\frac{((2n-1)!)^n}{a_n^2}} \right)^{\frac{1}{2}}, \forall n \geq 2 \Rightarrow \\ \lim_{n \rightarrow \infty} y_n &= \sqrt{\lim_{n \rightarrow \infty} \sqrt[n^2]{\frac{((2n-1)!)^n}{a_n^2}}} \stackrel{(1)}{=} \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((2n+3)!!)^{n+2}}{a_{n+2}^2} \cdot \frac{((2n-1)!)^n}{a_n^2} \cdot \frac{a_{n+1}^4}{((2n+1)!!)^{2n+2}}} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((2n+3)!!)^{n+2}}{((2n+1)!!)^{n+2}} \cdot \frac{((2n-1)!)^n}{((2n+1)!)^n} \cdot \frac{((2n+1)!!)^4 a_{n+1}^4}{((2n+3)!!(2n+1)!!)^2 a_{n+2}^2 a_n^2}} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \left(\frac{2n+3}{2n+1} \right)^n} = \sqrt[4]{e} \end{aligned}$$

□

Key words and phrases. Lalescu, sequences, limits.

Theorem 2. Let $(a_n)_{n \geq 1}, a_1 = 1, a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^*$, $(b_n)_{n \geq 1}, b_n > 0, \forall n \in \mathbb{N}^*$ such that:

$$\lim_{n \rightarrow \infty} \frac{b_n}{n!} = b > 0, \text{ then } \lim_{n \rightarrow \infty} \frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}} = \sqrt[4]{e}.$$

Proof. If $(x_n)_{n \geq 1}, x_n > 0, \forall n \in \mathbb{N}^*$ and

$$(1) \quad \lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} = x > 0, \text{ then } \lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} = \sqrt{x}$$

$$\begin{aligned} \text{Indeed, } \ln \left(\lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} \right) &= \lim_{n \rightarrow \infty} \left(\ln \sqrt[n^2]{x_n} \right) = \lim_{n \rightarrow \infty} \frac{x_n}{n^2} \stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1)^2 - n^2} = \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\ln x_{n+1} - \ln x_n}{n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{x_{n+1} - \ln x_n}{n} \stackrel{\text{Cesaro-Stolz}}{=} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\ln x_{n+2} - 2\ln x_{n+1} + \ln x_n}{(n+1) - n} = \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \ln \frac{x_{n+2}x_n}{x_{n+1}^2} = \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} \right) = \frac{1}{2} \ln x = \ln \sqrt{x}, \text{ so } \lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} = \sqrt{x} \end{aligned}$$

We have $a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^* \Rightarrow a_{n+2} = (n+2)!a_{n+1} = (n+2)!(n+1)!a_n, \forall n \in \mathbb{N}^*$

$$\begin{aligned} \text{If, } \frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}} &= \sqrt[2n]{\frac{b_n}{n!}} \cdot \frac{\sqrt[2n]{n!}}{\sqrt[n^2]{n!}} = \sqrt[2n]{\frac{b_n}{n!}} \cdot \sqrt[2n^2]{\frac{(n!)^n}{a_n^2}} \Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}} = b^0 \sqrt{\lim_{n \rightarrow \infty} \sqrt[n^2]{\frac{(n!)^n}{a_n^2}}} \stackrel{(1)}{=} \\ &\stackrel{(1)}{=} \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}}{a_{n+2}^2} \cdot \frac{(n!)^n}{a_n^2} \left(\frac{a_{n+1}^2}{((n+1)!)^{n+1}} \right)^2} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{a_{n+1}^4}{a_{n+2}^2 a_n^2}} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{((n+1)!a_n)^4}{((n+2)(n+1)!a_n)^2 a_n^2}} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{((n+1)!)^4}{((n+2)(n+1)!)^2}} = \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^n(n!)^n}{((n+1)!)^n((n+1)!)^n}} = \\ &= \sqrt[4]{\lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right)^n} = \sqrt[4]{e}, \text{ and we are done!} \end{aligned}$$

□

Theorem 3. Let $(a_n)_{n \geq 1}, a_1 = 1, a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^*$ then:

$$\lim_{n \rightarrow \infty} \frac{\sqrt[2n]{n!}}{\sqrt[n^2]{a_n}} = \sqrt[4]{e}.$$

Proof. If $(x_n)_{n \geq 1}, x_n > 0, \forall n \in \mathbb{N}^*$ and

$$(1) \quad \lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} = x > 0, \text{ then } \lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} = \sqrt{x}$$

$$\begin{aligned} \text{Indeed, } \ln \left(\lim_{n \rightarrow \infty} \sqrt[n^2]{x_n} \right) &= \lim_{n \rightarrow \infty} \left(\ln \sqrt[n^2]{x_n} \right) = \lim_{n \rightarrow \infty} \frac{\ln x}{n^2} \stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{\ln x_{n+1} - \ln x_n}{(n+1)^2 - n^2} = \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\ln x_{n+1} - \ln x_n}{n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\ln x_{n+1} - \ln x_n}{n} \stackrel{\text{Cesaro-Stolz}}{=} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\ln x_{n+2} - 2 \ln x_{n+1} + \ln x_n}{(n+1) - n} = \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} = \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{x_{n+2}x_n}{x_{n+1}^2} \right) = \frac{1}{2} \ln x = \ln \sqrt{x}, \text{ so } \lim_{n \rightarrow \infty} \sqrt[n]{x_n} = \sqrt{x}
\end{aligned}$$

Because $a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^* \Rightarrow a_{n+2} = (n+2)!a_{n+1} = (n+2)!(n+1)!a_n,$

$$\begin{aligned}
\forall n \in \mathbb{N}^*, \text{ we have } \lim_{n \rightarrow \infty} \sqrt[n^2]{a_n} &= \lim_{n \rightarrow \infty} \sqrt[2n^2]{\frac{(n!)^n}{a_n^2}} = \sqrt{\lim_{n \rightarrow \infty} \sqrt[n^2]{\frac{(n!)^n}{a_n^2}}} \stackrel{(1)}{=} \\
&\stackrel{(1)}{=} \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}}{a_{n+2}^2} \cdot \frac{(n!)^n}{a_n^2} \left(\frac{a_{n+1}^2}{((n+1)!)^{n+1}} \right)^2} = \\
&= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{a_{n+1}^4}{a_{n+2}^2 a_n^2}} = \\
&= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{((n+1)!a_n)^4}{((n+2)!(n+1)!a_n)^2 a_n^2}} = \\
&= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{((n+1)!a_n)^4}{((n+2)!(n+1)!a_n)^2 a_n^2}} = \\
&= \sqrt[4]{\lim_{n \rightarrow \infty} \frac{((n+2)!)^{n+2}(n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{((n+1)!a_n)^4}{((n+2)!(n+1)!a_n)^2 a_n^2}} = \\
&= \sqrt[4]{\lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right)^n} = \sqrt[4]{e}
\end{aligned}$$

□

Theorem 4. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}, a_n = \sum_{k=1}^n \frac{1}{k}, b_n > 0$ and $b_{n+3}b_{n+1}^3n^3 = b_{n+2}^3b_n e^{3a_n}, \forall n \in \mathbb{N}^*$, then:

$$\lim_{n \rightarrow \infty} \sqrt[n^3]{b_n} = e^{\frac{\gamma}{2}}$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{b_{n+3}b_{n+1}^3}{b_{n+2}^3b_n} = \lim_{n \rightarrow \infty} \frac{e^{3a_n}}{e^{3\ln n}} = \lim_{n \rightarrow \infty} e^{3(a_n - \ln n)} = e^{3\lim_{n \rightarrow \infty} \gamma_n} = e^{3\gamma},$$

$$\begin{aligned}
\text{where } \gamma \text{ is Euler-Mascheroni's constant. So, } \lim_{n \rightarrow \infty} \ln \sqrt[n^3]{b_n} &= \lim_{n \rightarrow \infty} \frac{\ln b_n}{n^3} \stackrel{\text{Cesaro-Stolz}}{=} \\
&= \lim_{n \rightarrow \infty} \frac{\ln b_{n+1} - \ln b_n}{(n+1)^3 - n^3} = \lim_{n \rightarrow \infty} \frac{n^2}{3n^2 + 3n + 1} \cdot \frac{\ln b_{n+1} - \ln b_n}{n^2} = \\
&= \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{\ln b_{n+1} - \ln b_n}{n^2} \stackrel{\text{Cesaro-Stolz}}{=} \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{\ln b_{n+2} - 2\ln b_{n+1} + \ln b_n}{(n+1)^2 - n^2} = \\
&= \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\ln b_{n+2} - 2\ln b_{n+1} + \ln b_n}{n} = \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \frac{\ln b_{n+2} - 2\ln b_{n+1} + \ln b_n}{n} = \\
&\stackrel{\text{Cesaro-Stolz}}{=} \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \frac{\ln b_{n+3} - 2\ln b_{n+2} + \ln b_{n+1} - \ln b_{n+2} + 2\ln b_{n+1} - \ln b_n}{(n+1) - n} = \\
&= \frac{1}{6} \cdot \lim_{n \rightarrow \infty} (\ln b_{n+3} - 3\ln b_{n+2} + 3\ln b_{n+1} - \ln b_n) = \frac{1}{6} \ln \left(\lim_{n \rightarrow \infty} \frac{b_{n+3}b_{n+1}^3}{b_{n+2}^3b_n} \right) = \frac{1}{6} \ln e^{3y} = \frac{\gamma}{2}.
\end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \sqrt[n^3]{b_n} = e^{\frac{\gamma}{2}}$$

□

Theorem 5. Let $m > 0$, $(a_n)_{n \geq 1}$, $a_n = \sqrt[n]{n!}$, $(x_n)_{n \geq 1}$,

$$x_n = \sum_{k=1}^n a_k^n, \text{ then } \lim_{n \rightarrow \infty} \frac{x_n}{n^{m+1}} = \frac{1}{(m+1)e^m}$$

Proof.

$$\begin{aligned} \text{We have } \lim_{n \rightarrow \infty} \frac{x_n}{n^{m+1}} &\stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{(n+1)^{m+1} - n^{m+1}} = \lim_{n \rightarrow \infty} \frac{a_{n+1}^m}{(n+1)^{m+1} - n^{m+1}} = \\ &= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{m+1} - n^{m+1}} \cdot \frac{a_{n+1}^m}{n^m} = \frac{1}{m+1} \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{n+1} \right)^m \left(\frac{n+1}{n} \right)^n = \\ &= \frac{1}{m+1} \cdot 1 \cdot \lim_{n \rightarrow \infty} \left(\frac{a_n}{n} \right)^m = \frac{1}{m+1} \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{n!}}{n} \right)^m = \frac{1}{m+1} \cdot \left(\frac{1}{e} \right)^m, \text{ where we've} \\ &\text{used the well-known fact } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \text{ Hence } \lim_{n \rightarrow \infty} \frac{x_n}{n^{m+1}} = \frac{1}{(m+1)e^m} \end{aligned}$$

□

Theorem 6. Let $(a_n)_{n \geq 1}$, $a_n = \prod_{k=1}^n (k!)^2$, then

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n^2]{a_n}} = e^{\frac{3}{2}}$$

Proof.

$$\begin{aligned} \text{It is well-known that } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e. \text{ So,} \\ \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n^2]{a_n}} &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n^2]{a_n}} = e \cdot \lim_{n \rightarrow \infty} \sqrt[n^2]{\frac{(n!)^n}{a_n}} = e \cdot \lim_{n \rightarrow \infty} \sqrt[n^2]{u_n}, \text{ where we} \\ &\text{denote } u_n = \frac{(n!)^n}{a_n}. \text{ We have } \lim_{n \rightarrow \infty} \ln \sqrt[n^2]{u_n} = \lim_{n \rightarrow \infty} \frac{1}{n^2} \ln u_n \stackrel{\text{Cesaro-Stolz}}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\ln u_{n+1} - \ln u_n}{(n+1)^2 - n^2} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\ln u_{n+1} - \ln u_n}{n} \stackrel{\text{Cesaro-Stolz}}{=} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} (\ln u_{n+2} - 2 \ln u_{n+1} + \ln u_n) = \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{u_{n+2} u_n}{u_{n+1}^2} \right) = \\ &= \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{\frac{((n+2)!)^{n+2} (n!)^n}{((n+1)!)^{2n+2}} \cdot \frac{a_{n+1}^2}{a_{n+2} a_n}}{1} \right) = \\ &= \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{(n+2)^{n+2}}{(n+1)^n} \cdot \frac{(1! \cdot 2! \cdot \dots \cdot n! \cdot (n+1)!)^4}{(1! \cdot 2! \cdot \dots \cdot (n+2)!)^2 (1! \cdot 2! \cdot \dots \cdot n!)^2} \right) = \\ &= \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \frac{(n+2)^{n+2}}{(n+1)^n} \cdot \left(\frac{(n+1)!}{(n+2)!} \right)^2 \right) = \frac{1}{2} \ln \left(\lim_{n \rightarrow \infty} \left(\frac{n+2}{n+1} \right)^n \right) = \frac{1}{2} \ln e = \frac{1}{2} \\ &\text{so, } \lim_{n \rightarrow \infty} \sqrt[n^2]{u_n} = e^{\frac{1}{2}}. \text{ Hence } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n^2]{a_n}} = e \cdot e^{\frac{1}{2}} = e^{\frac{3}{2}}. \end{aligned}$$

□

Theorem 7. If $a \in (0, \frac{\pi}{2})$, then

$$\lim_{n \rightarrow \infty} \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{a \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - \sin a \right) = \frac{2a \cos a}{e}$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \right) =$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e}$$

$$\text{If we denote } u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}};$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{n} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} =$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e.$$

$$\text{Therefore, } x_n = \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{a \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - \sin a \right) =$$

$$= \sqrt[n]{(2n-1)!!} (\sin a u_n - \sin a) =$$

$$= 2 \cdot \sqrt[n]{(2n-1)!!} \cdot \sin \frac{a}{2} (u_n - 1) \cdot \cos \frac{a}{2} (u_n + 1) =$$

$$= a \cdot \sqrt[n]{(2n-1)!!} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot (u_n - 1) =$$

$$= a \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

$$\text{Hence, } \lim_{n \rightarrow \infty} x_n = a \cdot \frac{2}{e} \cdot 1 \cdot \cos a \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{2a \cos a}{e}$$

□

Theorem 8. If $a \in (0, \frac{\pi}{2})$, then

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} \left(\sin \left(\frac{a \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - \sin a \right) = \frac{a \cos a}{e}$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e \text{ and } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \right) = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e}$$

$$\text{If we denote } u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}};$$

$$= \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{n} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} =$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e.$$

$$\begin{aligned} \text{Therefore, } x_n &= \sqrt[n]{n!} \left(\sin \left(\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - \sin a \right) = \sqrt[n]{n!} (\sin au_n - \sin a) = \\ &= 2 \cdot \sqrt[n]{n!} \cdot \sin \frac{a}{2} (u_n - 1) \cdot \cos \frac{a}{2} (u_n + 1) = a \cdot \sqrt[n]{n!} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= a \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n \\ \text{Hence, } \lim_{n \rightarrow \infty} x_n &= a \cdot \frac{1}{e} \cdot 1 \cdot \cos a \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a \cos a}{e} \end{aligned}$$

□

Theorem 9. If $a \in (0, \frac{\pi}{2})$ and $b = \arcsin a$, then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} \left(\sin \left(\frac{b \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - a \right) = \frac{\sqrt{1-a^2}}{e} \arcsin a.$$

Proof.

$$\begin{aligned} \text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= e \text{ and } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!}{n^n}} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \right) = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e}. \\ \text{If we denote } u_n &= \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}; \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{n} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so} \\ \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} &= 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} = \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } x_n &= \sqrt[n]{n!} \left(\sin \left(\frac{b \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - a \right) = \sqrt[n]{n!} (\sin bu_n - \sin b) = \\ &= 2 \cdot \sqrt[n]{n!} \cdot \sin \frac{b}{2} (u_n - 1) \cdot \cos \frac{b}{2} (u_n + 1) = \\ &= b \cdot \sqrt[n]{n!} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \cos \frac{b}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= b \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \cos \frac{b}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n. \end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} x_n = b \cdot \frac{1}{e} \cdot 1 \cdot \cos b \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{b \cos b}{e} = \frac{\sqrt{1-a^2}}{e} \arcsin a$$

□

Theorem 10. If $a \in (0, \frac{\pi}{2})$ and $b = \arcsin a$, then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{b \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - a \right) = \frac{2\sqrt{1-a^2}}{e} \arcsin a.$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e \text{ and } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \right) = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e}.$$

$$\text{If we denote } u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}};$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{n} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} &= 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} = \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } x_n &= \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{b \cdot \sqrt[n]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \right) - a \right) = \sqrt[n]{(2n-1)!!} (\sin bu_n - \sin b) = \\ &= 2 \cdot \sqrt[n]{(2n-1)!!} \cdot \sin \frac{b}{2} (u_n - 1) \cdot \cos \frac{b}{2} (u_n + 1) = \\ &= b \cdot \sqrt[n]{(2n-1)!!} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \cos \frac{b}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= b \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \cos \frac{b}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n. \end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} x_n = b \cdot \frac{2}{e} \cdot 1 \cdot \cos b \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{2b \cos b}{e} = \frac{2\sqrt{1-a^2}}{e} \arcsin a$$

□

Theorem 11. If $a \in (0, \frac{\pi}{2})$, then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} \left(\sin \left(\frac{a \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right) - \sin a \right) = \frac{a \cos a}{e}.$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e.$$

$$\text{If we denote } u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}};$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e.$$

$$\begin{aligned} \text{Therefore, } x_n &= \sqrt[n]{n!} \left(\sin \left(\frac{a \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right) - \sin a \right) = \sqrt[n]{n!} (\sin a u_n - \sin a) = \\ &= 2 \sqrt[n]{n!} \cdot \sin \frac{a}{2} (u_n - 1) \cdot \cos \frac{a}{2} (u_n + 1) = a \sqrt[n]{n!} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= a \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} x_n = a \cdot \frac{1}{e} \cdot 1 \cdot \cos a \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a \cos a}{e}.$$

□

Theorem 12. If $a \in (0, 1)$ and $b = \arcsin a$, then:

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} \left(\sin \left(\frac{b \cdot \sqrt[n]{(n+1)!}}{\sqrt[n]{n!}} \right) - b \right) = \frac{\sqrt{1-a^2}}{e} \arcsin a$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e. \text{ If we denote } u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}},$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1; \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e.$$

$$\begin{aligned} \text{Therefore, } x_n &= \sqrt[n]{n!} \left(\sin \left(\frac{b \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right) - b \right) = \sqrt[n]{n!} (\sin b u_n - \sin b) = \\ &= 2 \cdot \sqrt[n]{n!} \cdot \sin \frac{b}{2} (u_n - 1) \cdot \cos \frac{b}{2} (u_n + 1) = \\ &= b \cdot \sqrt[n]{n!} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \cos \frac{b}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= b \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{\sin \frac{b}{2} (u_n - 1)}{\frac{b}{2} (u_n - 1)} \cdot \frac{b}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n. \end{aligned}$$

$$\text{Hence, } \lim_{x_n} = b \cdot \frac{1}{e} \cdot 1 \cdot \cos b \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{b \cos b}{e} = \frac{\sqrt{1-a^2}}{e} \cdot \arcsin a$$

□

Theorem 13. If $a \in (0, \frac{\pi}{2})$, then

$$\lim_{n \rightarrow \infty} \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{a \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right) - \sin a \right) = \frac{2a \cos a}{e}.$$

Proof.

We have $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1} \right)^n = \frac{2}{e}. \end{aligned}$$

If we denote $u_n = \frac{\sqrt[n]{(n+1)!}}{\sqrt[n]{n!}}$; $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot \frac{e}{2} \cdot 1 = 1$, so
 $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$; $\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e$.

$$\begin{aligned} \text{Therefore } x_n &= \sqrt[n]{(2n-1)!!} \left(\sin \left(\frac{a \cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right) - \sin a \right) = \\ &= \sqrt[n]{(2n-1)!!} (\sin a u_n - \sin a) = \\ &= 2 \cdot \sqrt[n]{(2n-1)!!} \cdot \sin \frac{a}{2} (u_n - 1) \cdot \cos \frac{a}{2} (u_n + 1) = \\ &= a \cdot \sqrt[n]{(2n-1)!!} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot (u_n - 1) = \\ &= a \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot \frac{\sin \frac{a}{2} (u_n - 1)}{\frac{a}{2} (u_n - 1)} \cdot \cos \frac{a}{2} (u_n + 1) \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n. \end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} x_n = a \cdot \frac{2}{e} \cdot 1 \cdot \cos a \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{2a \cos a}{e}$$

□

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