

SOME LIMITS OF TRAIAN LALESCU TYPE (III)

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ABSTRACT. In this paper we present some limits of Lalescu type.

Theorem 1.

If $E_n = \sum_{k=0}^n \frac{1}{k!}$, then $\lim_{n \rightarrow \infty} (e - E_n) \cdot (n+1)! = 1$.

Proof. By Cesàro - Stolz, the case $\frac{0}{0}$, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} (e - E_n) \cdot (n+1)! &= \lim_{n \rightarrow \infty} \frac{(e - E_{n+1}) - (e - E_n)}{\frac{1}{(n+2)!} - \frac{1}{(n+1)!}} = \\ &= \lim_{n \rightarrow \infty} \frac{E_{n+1} - E_n}{\frac{n+1}{(n+2)!}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{n+1}{(n+2)!}} = \lim_{n \rightarrow \infty} \frac{n+2}{n+1} = 1. \end{aligned}$$

□

Theorem 2.

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \frac{1}{2\sqrt{e}}$$

Proof 1.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = e. \\ x_n &= \sqrt{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \sqrt{n} \cdot \sqrt[2n]{n!} \cdot (u_n - 1) = \frac{\sqrt[2n]{n!}}{\sqrt{n}} \cdot (u_n - 1) \cdot n = \\ &= \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \forall n \geq 2, \text{ where} \\ u_n &= \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}} = \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\sqrt[2n]{n!}} \cdot \frac{n+1}{n} \end{aligned}$$

So, $\lim_{n \rightarrow \infty} u_n = \sqrt{e} \cdot \sqrt{\frac{1}{e}} \cdot 1 = 1$, which yields that $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$. Also, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}} \right)^n} = \\ &= \sqrt[n]{\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[2(n+1)]{(n+1)!}}} \cdot \sqrt[n]{\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}} = \sqrt{e} \end{aligned}$$

Key words and phrases. Lalescu, limits.

$$\text{Hence: } \lim_{n \rightarrow \infty} x_n = \sqrt{\frac{1}{e}} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{1}{\sqrt{e}} \cdot \ln \sqrt{e} = \frac{1}{2\sqrt{e}}$$

□

Proof 2.

$$\begin{aligned} x_n &= \sqrt{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \sqrt{n} \cdot \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!} + \sqrt[n+1]{(n+1)!}} = \\ &= \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n} + \sqrt{\frac{\sqrt[n]{n!}}{n}}}}, \\ \text{so, } \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) \cdot \frac{1}{\lim_{n \rightarrow \infty} \left(\sqrt{\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n}} + \sqrt{\frac{\sqrt[n]{n!}}{n}} \right)} = \\ &= \frac{1}{e} \cdot \frac{1}{\sqrt{\frac{1}{e} + \sqrt{\frac{1}{e}}}} = \frac{1}{2\sqrt{e}} \end{aligned}$$

□

Proof 3.

$$\begin{aligned} x_n &= \sqrt{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \sqrt{n} \cdot \frac{\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}}{1+n-n} = \\ &= \frac{\sqrt[2n]{n!}}{n} \cdot \frac{u_n - 1}{v_n - 1} \cdot \sqrt{n} = \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{v_n - 1} = \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n} = \\ &= \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n}, \text{ where } u_n = \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}}, v_n = 1 + \frac{1}{n} \\ \text{which yields } \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} v_n = 1 \text{ then } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = \lim_{n \rightarrow \infty} \frac{v_n - 1}{\ln v_n} = 1 \\ \text{Also, we have } \lim_{n \rightarrow \infty} u_n^n &= \sqrt{e}, \lim_{n \rightarrow \infty} v_n^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e. \text{ Therefore} \\ \lim_{n \rightarrow \infty} x_n &= \frac{1}{\sqrt{e}} \cdot 1 \cdot 1 \cdot \frac{\ln \sqrt{e}}{\ln e} = \frac{1}{2\sqrt{e}} \end{aligned}$$

□

Theorem 3.

$$\lim_{n \rightarrow \infty} \sqrt[3]{n^2} \cdot \left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!} \right) = \frac{1}{3\sqrt[3]{e}}$$

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e \\ \lim_{n \rightarrow \infty} \frac{\sqrt[3n]{n!}}{\sqrt[3]{n}} &= \sqrt[3]{\lim_{n \rightarrow \infty} \frac{\sqrt[3n]{n!}}{n}} = \frac{1}{\sqrt[3]{e}} \\ x_n &= \sqrt[3]{n^2} \cdot \left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!} \right) = \frac{\sqrt[3n]{n!}}{\sqrt[3]{n}} \cdot n \cdot (u_n - 1) = \sqrt[3]{\frac{\sqrt[3n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2, \end{aligned}$$

$$u_n = \frac{\sqrt[3]{(n+1)!}}{\sqrt[3]{n!}} = \frac{\sqrt[3]{(n+1)!}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{\sqrt[3]{n!}} \cdot \sqrt[3]{\frac{n+1}{n}}. \text{ So, } \lim_{n \rightarrow \infty} u_n = \sqrt{e} \cdot \sqrt{\frac{1}{e}} \cdot 1 = 1$$

then $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$

$$\text{We have } u_n^n = \sqrt[3]{\left(\frac{\sqrt[3]{(n+1)!}}{\sqrt[3]{n!}}\right)^n} = \sqrt[3]{\frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[3]{(n+1)!}}}$$

$$\lim_{n \rightarrow \infty} u_n^n = \sqrt[3]{\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[3]{(n+1)!}}} = \sqrt[3]{e}.$$

Hence: $\lim_{n \rightarrow \infty} x_n = \frac{1}{\sqrt[3]{e}} \cdot 1 \cdot \ln \sqrt[3]{e} = \frac{1}{3\sqrt[3]{e}}$

□

Proof 2.

$$x_n = \sqrt[3]{n^2} \cdot \left(\sqrt[3]{(n+1)!} - \sqrt[3]{n!} \right) = \sqrt[3]{n^2} \cdot \frac{\sqrt[3]{(n+1)!} - \sqrt[3]{n!}}{1+n-n} =$$

$$= \frac{\sqrt[3]{n!}}{n} \cdot \frac{u_n - 1}{v_n - 1} \cdot \sqrt[3]{n^2} = \sqrt[3]{\frac{\sqrt[3]{n!}}{n}} \cdot \frac{u_n - 1}{v_n - 1} = \sqrt[3]{\frac{\sqrt[3]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n} =$$

$$= \sqrt[3]{\frac{\sqrt[3]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n}, v_n = 1 + \frac{1}{n}, \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = 1 \text{ then:}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = \lim_{n \rightarrow \infty} \frac{v_n - 1}{\ln v_n} = 1$$

$$\lim_{n \rightarrow \infty} u_n^n = \sqrt[3]{e}, \lim_{n \rightarrow \infty} v_n^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n \rightarrow \infty} x_n = \sqrt[3]{\frac{1}{e}} \cdot 1 \cdot 1 \cdot \frac{\ln \sqrt[3]{e}}{\ln e} = \frac{1}{3\sqrt[3]{e}}.$$

□

Theorem 4. If $m \in [1, \infty)$ then

$$\lim_{n \rightarrow \infty} \left(((n+1)!)^{\frac{1}{m(n+1)}} - (n!)^{\frac{1}{mn}} \right) \cdot n^{\frac{m-1}{m}} = \frac{1}{m \cdot e^{\frac{1}{m}}}$$

Proof. We have:

$$x_n = \left(((n+1)!)^{\frac{1}{m(n+1)}} - (n!)^{\frac{1}{mn}} \right) \cdot n^{\frac{m-1}{m}} = \frac{\left(((n+1)!)^{\frac{1}{m(n+1)}} - (n!)^{\frac{1}{mn}} \right)}{1+n-n} \cdot n^{\frac{m-1}{m}} =$$

$$= \frac{(n!)^{\frac{1}{mn}}}{n} \cdot \frac{u_n - 1}{v_n - 1} \cdot n^{\frac{m-1}{m}} = \left(\frac{\sqrt[3]{n!}}{n} \right)^{\frac{1}{m}} \cdot \frac{u_n - 1}{v_n - 1}, \forall n \geq 2, \text{ where}$$

$$u_n = \frac{((n+1)!)^{\frac{1}{m(n+1)}}}{(n!)^{\frac{1}{mn}}}, v_n = 1 + \frac{1}{n}, \forall n \geq 1$$

Because, $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[3]{n!}} = \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) =$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e, \text{ yields } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[3]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[3]{n!}} \right)^{\frac{1}{m}} =$$

$= \left(\frac{1}{e} \cdot e \cdot 1 \right)^{\frac{1}{m}} = 1$, so $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$. Also, we have: $\lim_{n \rightarrow \infty} v_n = 1$, so $\lim_{n \rightarrow \infty} \frac{v_n - 1}{\ln v_n} = 1$

$$\text{We have: } \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^n \right)^{\frac{1}{m}} = \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} \right)^{\frac{1}{m}} = \\ = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} \right)^{\frac{1}{m}} = e^{\frac{1}{m}} \text{ and } \lim_{n \rightarrow \infty} v_n^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

$$\text{Therefore, } x_n = \left(\frac{\sqrt[n]{n!}}{n} \right)^{\frac{1}{m}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n}, \forall n \geq 2, \text{, so:}$$

$$\lim_{n \rightarrow \infty} x_n = \frac{1}{e^{\frac{1}{m}}} \cdot 1 \cdot 1 \cdot \frac{\ln e^{\frac{1}{m}}}{\ln e} = \frac{1}{m \cdot e^{\frac{1}{m}}}$$

□

Theorem 5. If $m \in [1, \infty)$ and Γ is the Euler's gamma function then:

$$\lim_{x \rightarrow \infty} \left(\left(\Gamma(x+2) \right)^{\frac{1}{m(x+1)}} - \left(\Gamma(x+1) \right)^{\frac{1}{mx}} \right) \cdot x^{\frac{m-1}{m}} = \frac{1}{m \cdot e^{\frac{1}{m}}}$$

Proof. We have:

$$y(x) = \left(\left(\Gamma(x+2) \right)^{\frac{1}{m(x+1)}} - \left(\Gamma(x+1) \right)^{\frac{1}{mx}} \right) \cdot x^{\frac{m-1}{m}} = \\ = \frac{\left(\Gamma(x+2) \right)^{\frac{1}{m(x+1)}} - \left(\Gamma(x+1) \right)^{\frac{1}{mx}}}{1 + x - x} \cdot x^{\frac{m-1}{m}} = \\ = \frac{\left(\Gamma(x+1) \right)^{\frac{1}{mx}}}{x} \cdot \frac{u(x) - 1}{v(x) - 1} \cdot x^{\frac{m-1}{m}} = \left(\frac{\left(\Gamma(x+1) \right)^{\frac{1}{x}}}{x} \right)^{\frac{1}{m}} \cdot \frac{u(x) - 1}{v(x) - 1} = \\ = \left(\frac{\left(\Gamma(x+1) \right)^{\frac{1}{x}}}{x} \right)^{\frac{1}{m}} \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \frac{\ln v(x)}{v(x) - 1} \cdot \frac{\ln(u(x))^x}{\ln(v(x))^x}, \text{ where } u(x) = \left(\frac{\left(\Gamma(x+2) \right)^{\frac{1}{x+1}}}{\left(\Gamma(x+1) \right)^{\frac{1}{x}}} \right)^{\frac{1}{m}}$$

$$v(x) = 1 + \frac{1}{x}. \text{ Since, } \lim_{x \rightarrow \infty} \frac{x}{\left(\Gamma(x+1) \right)^{\frac{1}{x}}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{\Gamma(n+1)}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = \\ = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e$$

$$\text{yields } \lim_{x \rightarrow \infty} u(x) = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \right)^{\frac{1}{m}} = \left(\frac{1}{e} \cdot e \cdot 1 \right)^{\frac{1}{m}} = 1, \text{ so}$$

$$\lim_{x \rightarrow \infty} \frac{u(x) - 1}{\ln u(x)} = 1. \text{ Also, we have: } \lim_{x \rightarrow \infty} v(x) = 1, \text{ so } \lim_{x \rightarrow \infty} \frac{v(x) - 1}{\ln v(x)} = 1.$$

$$\text{We have: } \lim_{x \rightarrow \infty} (u(x))^x = \lim_{n \rightarrow \infty} \left(\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \right)^n \right)^{\frac{1}{m}} = \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} \right)^{\frac{1}{m}} = \\ = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} \right)^{\frac{1}{m}} = e^{\frac{1}{m}} \text{ and } \lim_{x \rightarrow \infty} (v(x))^x = e$$

$$\text{Therefore: } \lim_{x \rightarrow \infty} y(x) = \frac{1}{e^{\frac{1}{m}}} \cdot 1 \cdot 1 \cdot \frac{\ln e^{\frac{1}{m}}}{\ln e} = \frac{1}{m \cdot e^{\frac{1}{m}}}.$$

□

Theorem 6.

$$\lim_{n \rightarrow \infty} \left(\sqrt[2(n+1)]{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot (n+1)} - \sqrt[2n]{1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n} \right) \sqrt{n} = \frac{1}{2\sqrt{e}}$$

Proof 1. Denoting $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = e \\ x_n &= \sqrt{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \sqrt{n} \cdot \sqrt[2n]{n!} \cdot (u_n - 1) = \frac{\sqrt[2n]{n!}}{\sqrt{n}} \cdot (u_n - 1) \cdot n = \\ &= \sqrt{\frac{\sqrt[n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2, \text{ where} \\ u_n &= \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}} = \frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\sqrt[2n]{n!}} \cdot \sqrt{\frac{n+1}{n}} \\ \text{So, } \lim_{n \rightarrow \infty} u_n &= \sqrt{e} \cdot \frac{1}{e} \cdot 1 = 1, \text{ it follows } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1. \text{ We have:} \\ \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \sqrt{\left(\frac{\sqrt[2(n+1)]{(n+1)!}}{\sqrt[2n]{n!}} \right)^n} = \\ &= \sqrt{\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[2(n+1)]{(n+1)!}}} \cdot \sqrt{\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[2(n+1)]{(n+1)!}}} = \sqrt{e} \\ \text{Therefore, } \lim_{n \rightarrow \infty} x_n &= \sqrt{\frac{1}{e}} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{1}{\sqrt{e}} \cdot \ln \sqrt{e} = \frac{1}{2\sqrt{e}}. \end{aligned}$$

□

Proof 2. We have:

$$\begin{aligned} x_n &= \sqrt{n} \cdot \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) = \sqrt{n} \cdot \frac{\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}}{\frac{\sqrt[2(n+1)]{(n+1)!} + \sqrt[2n]{n!}}{2\sqrt{n}}} = \\ &= \frac{\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!}}{\sqrt{\frac{\sqrt[2(n+1)]{(n+1)!}}{n+1} \cdot \frac{n+1}{n} + \sqrt{\frac{\sqrt[2n]{n!}}{n}}}}, \text{ so:} \\ \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left(\sqrt[2(n+1)]{(n+1)!} - \sqrt[2n]{n!} \right) \cdot \frac{1}{\lim_{n \rightarrow \infty} \left(\sqrt{\frac{\sqrt[2(n+1)]{(n+1)!}}{n+1} \cdot \frac{n+1}{n}} + \sqrt{\frac{\sqrt[2n]{n!}}{n}} \right)} = \\ &= \frac{1}{e} \cdot \frac{1}{\sqrt{\frac{1}{e}} + \sqrt{\frac{1}{e}}} = \frac{1}{2\sqrt{e}} \end{aligned}$$

□

Proof 3. We have: $x_n = \sqrt{n} \cdot (\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}) = \sqrt{n} \cdot \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{1+n-n} =$

$$= \frac{\sqrt[2n]{n!}}{n} \cdot \frac{u_n - 1}{v_n - 1} \cdot \sqrt{n} = \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{v_n - 1} = \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n}{\ln v_n} =$$

$$= \sqrt{\frac{\sqrt[2n]{n!}}{n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \frac{\ln v_n}{v_n - 1} \cdot \frac{\ln u_n^n}{\ln v_n^n}, \text{ where like in solution 1 } u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[2n]{n!}},$$

$$v_n = 1 + \frac{1}{n}, \text{ and so } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = 1 \text{ and it follows}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = \lim_{n \rightarrow \infty} \frac{v_n - 1}{\ln v_n} = 1. \text{ We have:}$$

$$\lim_{n \rightarrow \infty} u_n^n = \sqrt{e}, \lim_{n \rightarrow \infty} v_n^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\text{Therefore: } \lim_{n \rightarrow \infty} x_n = \frac{1}{\sqrt{e}} \cdot 1 \cdot 1 \cdot \frac{\ln \sqrt{e}}{\ln e} = \frac{1}{2\sqrt{e}}.$$

□

Theorem 7. Let $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a continue function such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^m} = a \in \mathbb{R}_+^*, \text{ where } m \in [1, \infty) \text{ then}$$

$$\lim_{n \rightarrow \infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^n f(k)} \right) = \begin{cases} \frac{a}{e}, & \text{for } m = 1 \\ \infty, & \text{for } m \in (1, \infty) \end{cases}$$

Proof. We have:

$$B_n = \sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^n f(k)} = \sqrt[n]{\prod_{k=1}^n f(k)} \cdot (u_n - 1) =$$

$$(1) = \sqrt[n]{\prod_{k=1}^n f(k)} \cdot \frac{1}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = \frac{1}{n^{m+1}} \cdot \sqrt[n]{\prod_{k=1}^n f(k)} \cdot n^m \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2$$

Where $u_n = \frac{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}}{\sqrt[n]{\prod_{k=1}^n f(k)}}, \forall k \in \mathbb{N}^* - \{1\}$. We have:

$$(2) \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}}{(n+1)^m} \cdot \frac{n^m}{\sqrt[n]{\prod_{k=1}^n f(k)}} \cdot \left(\frac{n+1}{n}\right)^m \right)$$

$$\text{But, } \lim_{n \rightarrow \infty} \frac{\sqrt[n]{\prod_{k=1}^n f(k)}}{n^m} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\prod_{k=1}^n f(k)}{n^{nm}}} \stackrel{\text{Cauchy-D'Alembert}}{=} \\ = \lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n+1} f(k)}{\prod_{k=1}^n f(k)} \cdot \frac{n^{nm}}{(n+1)^{(n+1)m}} =$$

$$(3) \quad = \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^m} \cdot \left(\frac{n}{n+1}\right)^{nm} = \frac{a}{e^m}$$

So, $\lim_{n \rightarrow \infty} u_n = \frac{a}{e^m} \cdot \frac{e^m}{a} \cdot 1 = 1$, thus $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$. Also, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{\prod_{k=1}^{n+1} f(k)}{\prod_{k=1}^n f(k)} \cdot \frac{1}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^m} \cdot \frac{(n+1)^m}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)}} = a \cdot \frac{e^m}{a} = e^m \end{aligned}$$

and then by (1) we deduce that: $\lim_{n \rightarrow \infty} B_n = \frac{a}{e^m} \cdot 1 \cdot \ln e^m \cdot \lim_{n \rightarrow \infty} n^{m-1} = \frac{am}{e^m} \cdot \lim_{n \rightarrow \infty} n^{m-1} =$
 $= \begin{cases} \frac{a}{e}, & \text{for } m = 1 \\ \infty, & \text{for } m \in (1, \infty) \end{cases}$, and we are done.

□

Theorem 8. Let $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a continue function such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = a \in \mathbb{R}_+^*, \text{ then:}$$

$$\lim_{n \rightarrow \infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k}} \right) = \frac{a}{e}$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{1}{n} \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^n} \cdot \prod_{k=1}^n \frac{f(k)}{k}} \stackrel{\text{Cauchy-D'Alembert}}{=} \frac{1}{e}$$

$$\begin{aligned} &\stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^{n+1}} \prod_{k=1}^{n+1} \frac{f(k)}{k} \cdot n^n \prod_{k=1}^n \frac{k}{f(k)} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\left(\frac{n}{n+1} \right)^n \cdot \frac{f(n+1)}{n+1} \cdot \frac{1}{n+1} \right) = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^2} = \frac{a}{e}. \end{aligned}$$

$$\text{Let } B_n = \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} - \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k}} = \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k}} \cdot (u_n - 1) =$$

$$= \frac{1}{n} \cdot \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k}} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2, \text{ where } u_n = \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} \cdot \sqrt[n]{\prod_{k=1}^n \frac{k}{f(k)}}, \forall n \geq 2$$

$$\text{We have } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k}} \cdot n \cdot \sqrt[n]{\prod_{k=1}^n \frac{k}{f(k)}} \cdot \frac{n+1}{n} \right) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1$$

thus $\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$. Also, we have:

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\prod_{k=1}^{n+1} \frac{f(k)}{k} \cdot \prod_{k=1}^n \frac{k}{f(k)} \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k}{f(k)}} \right) = \lim_{n \rightarrow \infty} \frac{f(n+1)}{n+1} \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k}{f(k)}} =$$

$$= \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^2} \cdot \lim_{n \rightarrow \infty} \left((n+1) \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k}{f(k)}} \right) = a \cdot \frac{e}{a} = e.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} B_n = \frac{a}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a}{e} \cdot \ln e = \frac{a}{e}.$$

□

Theorem 9. Let $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a continue function such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^{t+1}} = a \in \mathbb{R}_+^*, \text{ where } t \in \mathbb{R}_+^* \text{ then } \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^t}} - \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k^t}} \right) = \frac{a}{e}.$$

Proof.

$$\text{We have } \lim_{n \rightarrow \infty} \frac{1}{n} \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k^t}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^n} \prod_{k=1}^n \frac{f(k)}{k^t}} \stackrel{\text{Cauchy-D'Alembert}}{=} \frac{1}{e}.$$

$$\begin{aligned} & \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^{n+1}} \prod_{k=1}^{n+1} \frac{f(k)}{k^t} \cdot n^n \prod_{k=1}^n \frac{k^t}{f(k)} \right) = \\ & = \lim_{n \rightarrow \infty} \left(\left(\frac{n}{n+1} \right)^n \cdot \frac{f(n+1)}{(n+1)^{t+1}} \right) = \frac{a}{e}. \end{aligned}$$

$$\text{Let } B_n = \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^t}} - \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k^t}} = \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k^t}} \cdot (u_n - 1) =$$

$$= \frac{1}{n} \cdot \sqrt[n]{\prod_{k=1}^n \frac{f(k)}{k^t}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \forall n \geq 2, \text{ where } u_n = \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^t}} \cdot \sqrt[n]{\prod_{k=1}^n \frac{k^t}{f(k)}}, \forall n \geq 2.$$

$$\text{We have } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{f(k)}{k^t}} \cdot n \cdot \sqrt[n]{\prod_{k=1}^n \frac{k^t}{f(k)}} \cdot \frac{n+1}{n} \right) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1$$

$$\text{thus } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1. \text{ Also, we have}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\prod_{k=1}^{n+1} \frac{f(k)}{k^t} \cdot \prod_{k=1}^n \frac{k^t}{f(k)} \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k^t}{f(k)}} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^{t+1}} \cdot (n+1) \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k^t}{f(k)}} = \\ &= \lim_{n \rightarrow \infty} \frac{f(n+1)}{(n+1)^{t+1}} \cdot \lim_{n \rightarrow \infty} \left((n+1) \cdot \sqrt[n+1]{\prod_{k=1}^{n+1} \frac{k^t}{f(k)}} \right) = a \cdot \frac{e}{a} = e. \end{aligned}$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} B_n = \frac{a}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a}{e} \cdot \ln e = \frac{a}{e}.$$

□

Theorem 10. Let $f, g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be continuous functions such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^{t+1} f(x)} = a \in \mathbb{R}_+^*, \lim_{x \rightarrow \infty} \frac{g(x+1)}{x^t g(x)} = b \in \mathbb{R}^*, \text{ where } t \in \mathbb{R}_+^* \text{ then}$$

$$\lim_{x \rightarrow \infty} \left(\left(\frac{f(x+1)}{g(x+1)} \right)^{\frac{1}{x+1}} - \left(\frac{f(x)}{g(x)} \right)^{\frac{1}{x}} \right) = \frac{a}{be}.$$

Proof.

$$\begin{aligned} \text{We have: } & \lim_{x \rightarrow \infty} \frac{1}{x} \left(\frac{f(x)}{g(x)} \right)^{\frac{1}{x}} = \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{1}{n} \sqrt[n]{\frac{f(n)}{g(n)}} \stackrel{\text{Cauchy-D'Alembert}}{=} \\ & \lim_{n \rightarrow \infty} \left(\frac{f(n+1)}{(n+1)^{n+1} g(n+1)} \cdot \frac{n^n g(n)}{f(n)} \right) = \\ & = \lim_{n \rightarrow \infty} \left(\frac{f(n+1)}{n^{t+1} f(n)} \cdot \frac{n^t g(n)}{g(n+1)} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = \frac{a}{be} \\ (1) \quad & B(x) = \left(\frac{f(x)}{g(x)} \right)^{\frac{1}{x}} (u(x) - 1) = \frac{1}{x} \cdot \left(\frac{f(x)}{g(x)} \right)^{\frac{1}{x}} \cdot \frac{u(x) - 1}{\ln u(x)} \cdot \ln(u(x))^x \\ & \text{where } u(x) = \left(\frac{f(x+1)}{g(x+1)} \right)^{\frac{1}{x+1}} \left(\frac{g(x)}{f(x)} \right)^{\frac{1}{x}}. \text{ So, } \lim_{x \rightarrow \infty} u(x) = \frac{be}{a} \cdot \frac{a}{be} = 1, \text{ thus} \\ & \lim_{x \rightarrow \infty} \frac{u(x) - 1}{\ln u(x)} = 1. \text{ Also, we have} \\ & \lim_{x \rightarrow \infty} (u(x))^x = \lim_{x \rightarrow \infty} \left(\frac{f(x+1)}{g(x+1)} \cdot \frac{g(x)}{f(x)} \cdot \left(\frac{g(x+1)}{f(x+1)} \right)^{\frac{1}{x+1}} \right) = \\ & = \lim_{x \rightarrow \infty} \frac{f(x+1)}{x^{t+1} f(x)} \cdot \lim_{x \rightarrow \infty} \frac{x^t g(x)}{g(x+1)} \cdot \lim_{x \rightarrow \infty} (x+1) \left(\frac{g(x+1)}{f(x+1)} \right)^{\frac{1}{x+1}} \cdot \lim_{x \rightarrow \infty} \frac{x}{x+1} = \\ & = \frac{a}{b} \cdot \frac{be}{a} \cdot 1 = e. \text{ Then, taking to limit in (1) with } x \rightarrow \infty \text{ we obtain that:} \\ & \lim_{x \rightarrow \infty} B(x) = \frac{a}{be} \cdot 1 \cdot \ln \left(\lim_{x \rightarrow \infty} (u(x))^x \right) = \frac{a}{be} \cdot \ln e = \frac{a}{be}, \text{ and we are done.} \end{aligned}$$

□

Theorem 11. Let $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a continuous function such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = a \in \mathbb{R}_+^* \text{ then } \lim_{x \rightarrow \infty} \left(\left(\frac{f(x+1)}{\Gamma(x+2)} \right)^{\frac{1}{x+1}} - \left(\frac{f(x)}{\Gamma(x+1)} \right)^{\frac{1}{x}} \right) = \frac{a}{e}$$

where Γ is gamma function.

Proof. We have:

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(\frac{f(x)}{\Gamma(x+1)} \right)^{\frac{1}{x}} = \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}^*}} \frac{1}{n} \cdot \sqrt[n]{\frac{f(n)}{\Gamma(n+1)}} \stackrel{\text{Cauchy-D'Alembert}}{=} \\ & \stackrel{\text{C-D'A}}{=} \lim_{n \rightarrow \infty} \left(\frac{f(n+1)}{(n+1)!(n+1)^{n+1}} \cdot \frac{n! \cdot n^n}{f(n)} \right) = \lim_{n \rightarrow \infty} \left(\frac{f(n+1)}{n^2 f(n)} \cdot \frac{n^{n+2}}{(n+1)^{n+2}} \right) = \\ & = a \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{n+2} = \frac{a}{e} \end{aligned}$$

We denote $B(x) = \left(\frac{f(x+1)}{\Gamma(x+2)}\right)^{\frac{1}{x+1}} - \left(\frac{f(x)}{\Gamma(x+1)}\right)^{\frac{1}{x}} = \left(\frac{f(x)}{\Gamma(x+1)}\right)^{\frac{1}{x}}(u(x) - 1) = \frac{1}{x} \cdot \left(\frac{f(x)}{\Gamma(x+1)}\right)^{\frac{1}{x}} \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln(u(x))^x$, where
 $u(x) = \left(\frac{f(x+1)}{\Gamma(x+2)}\right)^{\frac{1}{x}} \cdot \left(\frac{\Gamma(x+1)}{f(x)}\right)^{\frac{1}{x}} = \frac{1}{x+1} \cdot \left(\frac{f(x+1)}{\Gamma(x+2)}\right)^{\frac{1}{x+1}} \cdot x \cdot \left(\frac{\Gamma(x+1)}{f(x)}\right)^{\frac{1}{x}} \cdot \frac{x+1}{x}$
So, $\lim_{x \rightarrow \infty} u(x) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1$ and thus $\lim_{x \rightarrow \infty} \frac{u(x)-1}{\ln u(x)} = 1$

Also, we have $\lim_{x \rightarrow \infty} (u(x))^x = \lim_{x \rightarrow \infty} \left(\frac{\Gamma(x+1)}{f(x)} \cdot \frac{f(x+1)}{\Gamma(x+2)} \cdot \left(\frac{\Gamma(x+2)}{f(x+1)}\right)^{\frac{1}{x+1}} \right) = \lim_{x \rightarrow \infty} \left(\frac{f(x+1)}{x^2 f(x)} \cdot (x+1) \cdot \left(\frac{\Gamma(x+2)}{f(x+1)}\right)^{\frac{1}{x+1}} \cdot \left(\frac{x}{x+1}\right)^2 \right) = a \cdot \frac{e}{a} \cdot 1 = e$

Therefore, taking to the limit with $x \rightarrow \infty$ we obtain:

$$\lim_{x \rightarrow \infty} B(x) = \frac{a}{e} \cdot 1 \cdot \ln \left(\lim_{x \rightarrow \infty} (u(x))^x \right) = \frac{a}{e} \cdot \ln e = \frac{a}{e}, \text{ and we are done.}$$

□

Theorem 12. Let $a, b \in (-\infty, +\infty)$ then

$$\lim_{n \rightarrow \infty} \left(\frac{(n+3)^{n+2+a}}{(n+2)^{n+1+b}} - \frac{(n+2)^{n+1+a}}{(n+1)^{n+b}} \right) = \begin{cases} 0, & \text{if } a < b \\ e, & \text{if } a = b \\ +\infty, & \text{if } a > b \end{cases}$$

Proof. We denote $G_n(a, b) = \frac{(n+3)^{n+2+a}}{(n+2)^{n+1+b}} - \frac{(n+2)^{n+1+a}}{(n+1)^{n+b}}$, and we have that:

$$G_n(a, b) = (n+2) \left(\frac{n+3}{n+2} \right)^{n+2} \cdot \frac{(n+3)^a}{(n+2)^b} - (n+1) \left(\frac{n+2}{n+1} \right)^{n+1} \cdot \frac{(n+2)^a}{(n+1)^b} = (n+2)^{1-b} (n+3)^a e_{n+2} - (n+1)^{1-b} (n+2)^a e_{n+1}, \text{ where we denote}$$

$$e_n = \left(1 + \frac{1}{n}\right)^n \cdot G_n(a, b) = (n+1)^{1-b} (n+2)^a e_{n+1} \left(\left(\frac{n+2}{n+1}\right)^{1-b} \left(\frac{n+3}{n+2}\right)^a \frac{e_{n+2}}{e_{n+1}} - 1 \right) =$$

$$(1) = (n+1)^{1-b} (n+2)^a e_{n+1} (u_n - 1) = \frac{n+1}{n} \cdot (n+1)^{-b} \cdot (n+2)^a \cdot e_{n+1} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

where we denote $u_n = \left(\frac{n+2}{n+1}\right)^{1-b} \left(\frac{n+3}{n+2}\right)^a \frac{e_{n+2}}{e_{n+1}}$. We have:

$$\lim_{n \rightarrow \infty} e_n = e, \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\left(\frac{n+2}{n+1}\right)^{1-b} \left(\frac{n+3}{n+2}\right)^a \frac{e_{n+2}}{e_{n+1}} \right) = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\begin{aligned} \text{Also, we have: } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\left(\frac{n+2}{n+1}\right)^{n(1-b)} \left(\frac{n+3}{n+2}\right)^{na} \left(\frac{e_{n+2}}{e_{n+1}}\right)^n \right) = e^{1-b} \cdot e^a \cdot \lim_{n \rightarrow \infty} \left(\frac{e_{n+2}}{e_{n+1}}\right)^n = \\ &= e^{1-b+a} \cdot \lim_{n \rightarrow \infty} \left(\frac{e_{n+1}}{e_n}\right)^n = e^{1-b+a} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{e_{n+1} - e_n}{e_n}\right)^{\frac{e_n}{e_{n+1}-e_n} \cdot \frac{n(e_{n+1}-e_n)}{e_n}} = \end{aligned}$$

$$= e^{1-b+a} \cdot e^{\lim_{n \rightarrow \infty} \frac{n(e_{n+1}-e_n)}{e_n}} = e^{1+a-b} \cdot e^0 = e^{1+a-b}$$

By (1) and the above we deduce that:

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n(a, b) &= 1 \cdot e \cdot 1 \cdot \ln e^{1+a-b} \cdot \lim_{n \rightarrow \infty} ((n+1)^{-b}(n+2)^a) = e(1+a-b) \lim_{n \rightarrow \infty} ((n+1)^{-b}(n+2)^a) = \\ &= e(1+a-b) \cdot \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right)^n \cdot n^{-b} \cdot (n+2)^a \right) = e(1+a-b) \cdot 1 \cdot \lim_{n \rightarrow \infty} \left(\left(\frac{n+2}{n} \right)^a \cdot n^a \cdot n^{-b} \right) = \\ &= e(1+a-b) \cdot 1 \cdot \lim_{n \rightarrow \infty} n^{a-b} = \begin{cases} 0, & \text{if } a < b \\ e, & \text{if } a = b \\ +\infty, & \text{if } a > b \end{cases} \end{aligned}$$

□

Theorem 13. Let $(a_n)_{n \geq 1}$ be a positive real sequence such that

$$\lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} \right) = e$$

Proof. By the theorem of Cauchy-D'Alembert we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \left(\frac{n}{n+1} \right)^n \right) = \\ &= a \cdot \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{e} \\ \text{So, } x_n &= \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} = \frac{n^2}{\sqrt[n]{a_n}} (u_n - 1) = \frac{n^2}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \\ (1) \quad &= \frac{n}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2 \end{aligned}$$

We denote $u_n = \left(\frac{n+1}{n} \right)^2 \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}, \forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a_n}}{n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot e \cdot 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right)^{2n} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right)^n \cdot \frac{a_n}{n!} \cdot \frac{(n+1)!}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \right) = e^2 \cdot a \cdot \frac{1}{a} \cdot \frac{1}{e} = e. \end{aligned}$$

Therefore taking to limit in (1) we obtain $\lim_{n \rightarrow \infty} x_n = e \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = e \cdot \ln e = e \cdot 1 = e$.

□

Theorem 14. Let $(a_n)_{n \geq 1}$ be a positive real sequence such that:

$$\lim_{n \rightarrow \infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right) = e$$

Proof. By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \left(\frac{n}{n+1} \right)^n \right) = \\ &= a \cdot \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{e}. \end{aligned}$$

So, $x_n = \sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \sqrt[n]{a_n}(u_n - 1) = \sqrt[n]{a_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$

$$(1) \quad = \frac{\sqrt[n]{a_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2.$$

We denote $u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$, $\forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{a_n}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot e \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \right) = \\ &= a \cdot \frac{1}{a} \cdot e = e. \end{aligned}$$

Therefore taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{1}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{1}{e} \cdot \ln e = \frac{1}{e} \cdot 1 = \frac{1}{e}.$$

□

Theorem 15. Let $(a_n)_{n \geq 1}$ be a positive real sequence and $a > 0$ such that:

$$\lim_{n \rightarrow \infty} (a_n - a \cdot n!) = b > 0 \text{ then } \lim_{n \rightarrow \infty} (\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{1}{e}.$$

Proof.

Since $\lim_{n \rightarrow \infty} (a_n - a \cdot n!) = b$ we have that $\lim_{n \rightarrow \infty} \left(\frac{a_n}{n!} - a \right) = \lim_{n \rightarrow \infty} \frac{b}{n!} = 0$, so:

$\lim_{n \rightarrow \infty} \frac{a_n}{n!} = a > 0$. By the theorem of Cauchy-D'Alembert we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \left(\frac{n}{n+1} \right)^n \right) = \\ &= a \cdot \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{e}. \end{aligned}$$

$$\text{So, } x_n = \sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \sqrt[n]{a_n}(u_n - 1) = \sqrt[n]{a_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$(1) \quad = \frac{\sqrt[n]{a_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2$$

We denote $u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$, $\forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{a_n}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot e \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\begin{aligned} \text{We have: } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \right) = \\ &= a \cdot \frac{1}{a} \cdot e = e. \end{aligned}$$

Therefore taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{1}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{1}{e} \cdot \ln e = \frac{1}{e} \cdot 1 = \frac{1}{e}.$$

□

Theorem 16. Let $(a_n)_{n \geq 1}$ be a positive real sequence and $a > 0$ such that:

$$\lim_{n \rightarrow \infty} (a_n - a \cdot n!) = b > 0 \text{ then } \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} \right) = e.$$

Proof.

Since $\lim_{n \rightarrow \infty} (a_n - a \cdot n!) = b$ we deduce that $\lim_{n \rightarrow \infty} \left(\frac{a_n}{n!} - a \right) = \lim_{n \rightarrow \infty} \frac{b}{n!} = 0$, so

$\lim_{n \rightarrow \infty} \frac{a_n}{n!} = a > 0$. By the theorem of Cauchy-D'Alembert we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{(n+1)!} \cdot \frac{n!}{a_n} \cdot \left(\frac{n}{n+1} \right)^n \right) = \\ &= a \cdot \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{e} \end{aligned}$$

$$\text{So, } x_n = \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} = \frac{n^2}{\sqrt[n]{a_n}} (u_n - 1) = \frac{n^2}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$(1) \quad = \frac{n}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \geq 2$$

We denote $u_n = \left(\frac{n+1}{n} \right)^n \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}, \forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a_n}}{n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n+1}{n} \right) = \frac{1}{e} \cdot e \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$$

$$\begin{aligned} \text{We have: } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right)^{2n} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right)^{2n} \cdot \frac{a_n}{n!} \cdot \frac{(n+1)!}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \right) = e^2 \cdot a \cdot \frac{1}{a} \cdot \frac{1}{e} = e. \end{aligned}$$

Therefore, taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = e \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = e \cdot \ln e = e \cdot 1 = e.$$

□

Theorem 17. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be positive real sequences such that:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0 \text{ and } \lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v, \text{ where } u, v > 0. \text{ If there exists:}$$

$\lim_{n \rightarrow \infty} a_n \in \bar{R}_+$, then calculate:

$$\text{a)} \quad \lim_{n \rightarrow \infty} \left(\sqrt[n]{b_{n+1}} - \sqrt[n]{b_n} \right) = \frac{a}{e}$$

$$\text{b)} \quad \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} \right) = \frac{e}{a}.$$

Proof for a).

We suppose that $\lim_{n \rightarrow \infty} a_n = c > 0$, then because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$ we get:

$$\frac{c}{c \cdot \infty} = a = 0, \text{ which is false. So, } \lim_{n \rightarrow \infty} a_n = +\infty. \text{ Therefore, from}$$

$$\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v \text{ we deduce that } \lim_{n \rightarrow \infty} \left(\frac{b_n}{a_n} - u \right) = \lim_{n \rightarrow \infty} \frac{v}{a_n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = u.$$

By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{1}{e} = \frac{a}{e}. \text{ So,} \end{aligned}$$

$$(1) \quad x_n = \sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n} = \sqrt[n]{b_n} \cdot (u_n - 1) = \frac{\sqrt[n]{b_n}}{n} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2$$

We denote $u_n = \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}$, $\forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{b_n}} \cdot \frac{n+1}{n} \right) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1$$

$$\text{so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1. \text{ Also, we have:}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{b_n} \cdot \frac{1}{\sqrt[n+1]{b_{n+1}}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \frac{n+1}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{n}{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{e}{a} \cdot 1 = e. \end{aligned}$$

Therefore taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{a}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a}{e} \cdot \ln e = \frac{a}{e}.$$

□

Proof for b).

We suppose that $\lim_{n \rightarrow \infty} a_n = c > 0$, then, because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$, we get:

$\frac{c}{c \cdot \infty} = a = 0$, which is false. So, $\lim_{n \rightarrow \infty} a_n = +\infty$. Therefore, from

$$\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v \text{ we deduce } \lim_{n \rightarrow \infty} \left(\frac{b_n}{a} - u \right) = \lim_{n \rightarrow \infty} \frac{v}{a} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{a} = u.$$

By Cauchy-D'Alembert theorem, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{1}{e} = \frac{a}{e}. \text{ So,} \end{aligned}$$

$$(1) \quad x_n = \frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} = \frac{n^2}{\sqrt[n]{b_n}} \cdot (u_n - 1) = \frac{n}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2$$

We denote $u_n = \left(\frac{n+1}{n} \right)^2 \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}, \forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{\sqrt[n]{b_n}}{n} \cdot \frac{n+1}{n} \right) = \frac{e}{a} \cdot \frac{a}{e} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

Also, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{b_n}{b_{n+1}} \cdot \left(\frac{n+1}{n} \right)^{2n} \cdot \sqrt[n+1]{b_{n+1}} \right) = \\ &= e^2 \cdot \lim_{n \rightarrow \infty} \left(\frac{b_n}{a_n} \cdot \frac{a_{n+1}}{b_{n+1}} \cdot \frac{a_n \cdot n}{a_{n+1}} \cdot \frac{n+1}{n} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n+1} \right) = e^2 \cdot \left(u \cdot \frac{1}{u} \cdot a^{-1} \cdot 1 \cdot \frac{a}{e} \right) = e. \end{aligned}$$

Therefore, taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{e}{a} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{e}{a} \cdot \ln e = \frac{e}{a}.$$

□

Theorem 18. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be positive real sequences such that there exists:

$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n}$ and $\lim_{n \rightarrow \infty} (b_n - u \cdot a_n)$, then the following limits exists:

$$\text{a)} \quad \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n} \right)$$

$$\text{b)} \quad \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} \right)$$

Proof for a).

Let $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$ and $\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v$. We suppose that

$\lim_{n \rightarrow \infty} a_n = c > 0$, then because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$, we get $\frac{c}{c \cdot \infty} = a = 0$. So,

$\lim_{n \rightarrow \infty} a_n = +\infty$. Therefore, from $\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v$, we deduce

$$\lim_{n \rightarrow \infty} \left(\frac{b_n}{a_n} - u \right) = \lim_{n \rightarrow \infty} \frac{v}{a_n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = u$$

By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{1}{e} = \frac{a}{e}. \text{ So,} \end{aligned}$$

$$(1) \quad x_n = \sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n} = \sqrt[n]{b_n} \cdot (u_n - 1) = \frac{\sqrt[n]{b_n}}{n} \cdot \frac{u_n - 1}{\ln u_n}, \forall n \geq 2$$

We denote $u_n = \frac{\sqrt[n]{b_{n+1}}}{\sqrt[n]{b_n}}$, $\forall n \geq 2$ and we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{b_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{b_n}} \cdot \frac{n+1}{n} \right) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1 \\ \text{so, } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} &= 1. \text{ Also, we have:} \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{b_n} \cdot \frac{1}{\sqrt[n+1]{b_{n+1}}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \frac{n+1}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{n}{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{e}{a} \cdot 1 = e. \end{aligned}$$

Therefore taking to limit in (1) we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{a}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{a}{e} \cdot \ln e = \frac{a}{e}.$$

□

Proof for b).

Let $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$ and $\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v$. We suppose that $\lim_{n \rightarrow \infty} a_n = c > 0$,

then, because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$ we get $\frac{c}{c \cdot \infty} = a = 0$, which is false. So,

$\lim_{n \rightarrow \infty} = +\infty$. Therefore, from $\lim_{n \rightarrow \infty} (b_n - u \cdot a_n) = v$ we deduce

$$\lim_{n \rightarrow \infty} \left(\frac{b_n}{a_n} - u \right) = \lim_{n \rightarrow \infty} \frac{v}{a_n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = u.$$

By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{b_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{b_n} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \frac{a_{n+1}}{n \cdot a_n} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = u \cdot \frac{1}{u} \cdot a \cdot \frac{1}{e} = \frac{a}{e}. \text{ So,} \end{aligned}$$

$$(1) \quad x_n = \frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} = \frac{n^2}{\sqrt[n]{b_n}} \cdot (u_n - 1) = \frac{n}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\sqrt[n]{\ln u_n}} \cdot \ln u_n^n, \forall n \geq 2$$

We denote $u_n = \left(\frac{n+1}{n}\right)^n \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n]{b_{n+1}}}$, $\forall n \geq 2$ and we have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[n+1]{b_{n+1}}} \cdot \frac{\sqrt[n]{b_n}}{n} \cdot \frac{n+1}{n} \right) = \frac{e}{a} \cdot \frac{a}{e} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\begin{aligned} \text{Also, we have: } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{b_n}{b_{n+1}} \cdot \left(\frac{n+1}{n} \right)^{2n} \cdot \sqrt[n+1]{b_{n+1}} \right) = \\ &= e^2 \cdot \lim_{n \rightarrow \infty} \left(\frac{b_n}{a_n} \cdot \frac{a_{n+1}}{b_{n+1}} \cdot \frac{a_n \cdot n}{a_{n+1}} \cdot \frac{n+1}{n} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n+1} \right) = e^2 \cdot \left(u \cdot \frac{1}{u} \cdot a^{-1} \cdot 1 \cdot \frac{a}{e} \right) = e. \end{aligned}$$

Therefore, taking to limit in (1), we obtain:

$$\lim_{n \rightarrow \infty} x_n = \frac{e}{a} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{e}{a} \cdot \ln e = \frac{a}{e}.$$

□

Theorem 19. Let $(a_n)_{n \geq 1}$ be a positive real sequence such that:

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = a > 0 \text{ and let } (x_n)_{n \geq 1} \text{ be a sequence with}$$

$$x_1 = 1, x_n = \sqrt[n]{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}}, \text{ then:}$$

$$\text{a) } \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}^3}{x_{n+1}} - \frac{a_n^3}{x_n} \right) = \frac{a^3 e^4}{4}$$

$$\text{b) } \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{a_{n+1}} - \frac{x_n}{a_n} \right) = \frac{4}{a \cdot e^4}.$$

Proof for a). Let $A_n = \frac{a_{n+1}^3}{x_{n+1}} - \frac{a_n^3}{x_n} = \frac{a_n^3}{x_n} (u_n - 1) = \frac{a_n^3}{n^3} \cdot \frac{n^2}{x_n} \cdot n \cdot (u_n - 1) =$

$$(*) \quad = \left(\frac{a_n}{n} \right)^3 \cdot \frac{n^2}{x_n} \cdot n \cdot \frac{u_n - 1}{\ln u_n} = \left(\frac{a_n}{n} \right)^3 \cdot \frac{n^2}{x_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \forall n \geq 2$$

Where $u_n = \left(\frac{a_{n+1}}{a_n} \right)^3 \cdot \frac{x_n}{x_{n+1}}$, $\forall n \geq 2$. By Cesearo-Stolz theorem we have:

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{(n+1) - n} = \lim_{n \rightarrow \infty} (a_{n+1} - a_n) = a, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{n+1} \cdot \frac{n}{a_n} \cdot \frac{n+1}{n} \right) = a \cdot \frac{1}{a} \cdot 1 = 1. \text{ We have:}$$

$$1) \quad \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{a_{n+1} - a_n}{a_n} \right)^{\frac{a_n}{a_{n+1} - a_n}} \right)^{\frac{a_n}{a_n} \cdot (a_{n+1} - a_n)} = e^{\frac{1}{a} \cdot a} = e.$$

2) By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{x_n}{n^2} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}}{n^{2n}}} \stackrel{\text{C-D'A}}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{3} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!} \cdot \sqrt[n+1]{(2n+1)!}}{\sqrt{3} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}} \cdot \frac{n^{2n}}{(n+1)^{2n+2}} = \\ &= \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{(2n+1)!}}{(n+1)^2} = \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!}{n^{2n}}} \stackrel{\text{C-D'A}}{=} \\ &= \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n-1)!} = \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{2n(2n+1)}{(n+1)^2} \left(\frac{n}{n+1} \right)^{2n} = \frac{4}{e^4}. \end{aligned}$$

$$3) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^3 \cdot \lim_{n \rightarrow \infty} \left(\frac{x_n}{n^2} \cdot \frac{(n+1)^2}{x_{n+1}} \cdot \left(\frac{n}{n+1} \right)^2 \right) = 1 \cdot \frac{4}{e^4} \cdot \frac{e^4}{4} \cdot 1 = 1, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$$

$$4) \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^{3n} \cdot \lim_{n \rightarrow \infty} \left(\frac{x_n^n}{x_{n+1}^{n+1}} \cdot x_{n+1} \right) = \\ = e^3 \cdot \lim_{n \rightarrow \infty} \frac{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}}{\sqrt[3]{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!} \cdot \sqrt[n+1]{(2n+1)!}} \cdot x_{n+1} = \\ = e^3 \cdot \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!}} \cdot \frac{x_{n+1}}{(n+1)^2} = e^3 \cdot \frac{e^2}{4} \cdot \frac{4}{e^4} = e.$$

By the above and (*) we obtain: $\lim_{n \rightarrow \infty} A_n = a^3 \cdot \frac{e^4}{4} \cdot 1 \cdot \ln e = \frac{a^3 e^4}{4}$.

□

$$\text{Proof for b). Let } B_n = \frac{x_{n+1}}{a_{n+1}} - \frac{x_n}{a_n} = \frac{x_n}{a_n}(u_n - 1) = \frac{x_n}{a_n} \cdot \frac{n}{n^2} \cdot n \cdot (u_n - 1) = \\ (***) \quad = \frac{x_n}{n^2} \cdot \frac{n}{a_n} \cdot n \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{x_n}{n^2} \cdot \frac{n}{a_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \forall n \geq 2$$

Where $u_n = \frac{x_{n+1}}{x_n} \cdot \frac{a_n}{a_{n+1}}, \forall n \geq 2$. By Cesaro-Stolz theorem we have:

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{(n+1) - n} = \lim_{n \rightarrow \infty} (a_{n+1} - a_n) = a, \text{ so}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{n+1} \cdot \frac{n}{a_n} \cdot \frac{n+1}{n} \right) = a \cdot \frac{1}{a} \cdot 1 = 1. \text{ We have:}$$

$$1) \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{a_{n+1} - a_n}{a_n} \right)^{\frac{a_n}{a_{n+1} - a_n}} \right)^{\frac{n}{a_n} \cdot (a_{n+1} - a_n)} = e^{\frac{1}{a} \cdot a} = e.$$

2) By Cauchy-D'Alembert theorem we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{x_n}{n^2} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}}{n^{2n}}} \stackrel{\text{C-D'A}}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!} \cdot \sqrt[n+1]{(2n+1)!}}{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}} \cdot \frac{n^{2n}}{(n+1)^{2n+2}} = \\ &= \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{(2n+1)!}}{(n+1)^2} = \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(2n-1)!}{n^{2n}}} \stackrel{\text{C-D'A}}{=} \\ &= \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n-1)!} = \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{2n(2n+1)}{(n+1)^2} \left(\frac{n}{n+1} \right)^{2n} = \frac{4}{e^4} \end{aligned}$$

$$3) \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{x_n} \cdot \frac{a_n}{a_{n+1}} \right) = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{(n+1)^2} \cdot \frac{n^2}{x_n} \cdot \frac{a_n}{n} \cdot \frac{n+1}{a_{n+1}} \cdot \frac{n+1}{n} \right) = \\ = \frac{4}{e^4} \cdot \frac{e^4}{4} \cdot a \cdot \frac{1}{a} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$4) \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{x_n} \right)^n \cdot \left(\frac{a_n}{a_{n+1}} \right)^n = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{x_{n+1}^{n+1}}{x_n^n} \cdot \frac{1}{x_{n+1}} = \\ = \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \left(\frac{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!} \cdot \sqrt[n+1]{(2n+1)!}}{\sqrt{3!} \cdot \sqrt[3]{5!} \cdot \dots \cdot \sqrt[n]{(2n-1)!}} \cdot \frac{1}{x_{n+1}} \right) =$$

$$= \frac{1}{e} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{(2n+1)!}}{(n+1)^2} \cdot \frac{(n+1)^2}{x_{n+1}} = \frac{1}{e} \cdot \frac{4}{e^2} \cdot \frac{e^4}{4} = e.$$

By the above and (**) we obtain $\lim_{n \rightarrow \infty} B_n = \frac{4}{e^4} \cdot \frac{1}{4} \cdot 1 \cdot \ln e = \frac{4}{a \cdot e^4}$.

□

Theorem 20. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}, (x_n)_{n \geq 1}, (y_n)_{n \geq 1}$ be positive real sequences such that:

$$\lim_{n \rightarrow \infty} a_n = a > 0, \lim_{n \rightarrow \infty} b_n = b > 0, \lim_{n \rightarrow \infty} \frac{x_{n+1}}{n \cdot x_n} = x > 0 \text{ and } y_n = a_n x_n + b_n$$

$$\text{If there exists } \lim_{n \rightarrow \infty} x_n, \text{ then } \lim_{n \rightarrow \infty} (\sqrt[n+1]{y_{n+1}} - \sqrt[n]{y_n}) = \frac{x}{e}$$

Proof.

We suppose that $\lim_{n \rightarrow \infty} x_n = t > 0$, and because $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{n \cdot x_n} = x$ yields that

$$x = \lim_{n \rightarrow \infty} \left(\frac{x_{n+1}}{x_n} \cdot \frac{1}{n} \right) = \frac{t}{t} \cdot \frac{1}{\infty} = 0, \text{ contradiction with } x > 0. \text{ So, } \lim_{n \rightarrow \infty} x_n = \infty.$$

Because $y_n - a_n x_n = b_n$ we have $\frac{y_n}{x_n} - a_n = \frac{b_n}{x_n}$ which yields that:

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} \frac{b_n}{x_n} = a + 0 = a. \text{ By Cauchy-D'Alembert theorem we have:}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{y_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{y_n}{n^{2n}}} \stackrel{\text{C-D'A}}{=} \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{y_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{n \cdot y_n} \cdot \left(\frac{n}{n+1} \right)^{n+1} \right) = \\ &= \frac{1}{e} \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{x_{n+1}} \cdot \frac{x_n}{y_n} \cdot \frac{x_{n+1}}{n \cdot x_n} \right) = \frac{1}{e} \cdot a \cdot \frac{1}{a} \cdot x = \frac{x}{e}. \text{ Also, we have:} \end{aligned}$$

(1)

$$z_n = \sqrt[n+1]{y_{n+1}} - \sqrt[n]{y_n} = \sqrt[n]{y_n} \cdot (u_n - 1) = \sqrt[n]{y_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{\sqrt[n]{y_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \forall n \geq 2$$

Where we denote $u_n = \frac{\sqrt[n+1]{y_{n+1}}}{\sqrt[n]{y_n}}$, $\forall n \geq 2$. We have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{y_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{y_n}} \cdot \frac{n+1}{n} \right) = \frac{x}{e} \cdot \frac{e}{x} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\begin{aligned} \text{Also, we have: } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{y_n} \cdot \frac{1}{\sqrt[n+1]{y_{n+1}}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{n \cdot y_n} \cdot \frac{n+1}{\sqrt[n+1]{y_{n+1}}} \cdot \frac{n}{n+1} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{y_{n+1}}{x_{n+1}} \cdot \frac{x_n}{y_n} \cdot \frac{n+1}{\sqrt[n+1]{y_{n+1}}} \cdot \frac{x_{n+1}}{n \cdot x_n} \cdot \frac{n}{n+1} \right) = a \cdot \frac{1}{a} \cdot \frac{e}{x} \cdot 1 = e. \end{aligned}$$

By the above and (1) we obtain $\lim_{n \rightarrow \infty} z_n = \frac{x}{e} \cdot 1 \cdot \ln \left(\lim_{n \rightarrow \infty} u_n^n \right) = \frac{x}{e} \cdot \ln e = \frac{x}{e}$. □

Theorem 21.

Let $(s_n)_{n \geq 1}, s_n = \sum_{k=1}^n \frac{1}{k^2}$ then $\lim_{n \rightarrow \infty} \left(s_n \cdot \sqrt[n+1]{(n+1)!} - \frac{\pi^2}{6} \cdot \sqrt[n]{n!} \right) = \frac{\pi^2 - 6}{6e}$

Proof.

$$\text{It is well-known that } \lim_{n \rightarrow \infty} s_n = \frac{\pi^2}{6}$$

$$\begin{aligned} \text{We have: } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{n!}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e. \text{ So, } B_n = s_n \sqrt[n+1]{(n+1)!} - \frac{\pi^2}{6} \sqrt[n]{n!} = \\ &= \sqrt[n+1]{(n+1)!} \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \\ (1) \quad &= \frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n+1}{n} \cdot n \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) \end{aligned}$$

We have:

$$\begin{aligned} (2) \quad \lim_{n \rightarrow \infty} n \left(s_n - \frac{\pi^2}{6} \right) &= \lim_{n \rightarrow \infty} \frac{s_n - \frac{\pi^2}{6}}{\frac{1}{n}} \stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{s_{n+1} - s_n}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n \rightarrow \infty} -\frac{\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}} = -1 \\ \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} \cdot \lim_{n \rightarrow \infty} n \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1 \right) = \frac{1}{e} \lim_{n \rightarrow \infty} n(u_n - 1) = \\ &= \frac{1}{e} \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \text{ where } u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}, \forall n \geq 2. \\ \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{n+1}{n} \right) = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1 \\ \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{1}{\sqrt[n+1]{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e, \text{ and then:} \end{aligned}$$

$$(3) \quad \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}$$

By (1), (2) and (3) we obtain that:

$$\lim_{n \rightarrow \infty} B_n = \frac{1}{e}(-1) + \frac{\pi^2}{6} \cdot \frac{1}{e} = \frac{\pi^2 - 6}{6e}$$

□

Theorem 22.

Let $(s_n)_{n \geq 1}$, $s_n = \sum_{k=1}^n \frac{1}{k^2}$ and let $(a_n)_{n \geq 1}$ be a positive real sequence such that:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*, \text{ then } \lim_{n \rightarrow \infty} \left(s_n \cdot \sqrt[n+1]{a_{n+1}} - \frac{\pi^2}{6} \cdot \sqrt[n]{a_n} \right) = \frac{a(\pi^2 - 6)}{6e}$$

Proof.

It is well-known that $\lim_{n \rightarrow \infty} s_n = \frac{\pi^2}{6}$. We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^n}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \\ &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} \left(\frac{n}{n+1} \right)^{n+1} = a \cdot \frac{1}{e} = \frac{a}{e} \end{aligned}$$

$$\text{So, } B_n = s_n \sqrt[n+1]{a_{n+1}} - \frac{\pi^2}{6} \sqrt[n]{a_n} = \sqrt[n+1]{a_{n+1}} \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right) =$$

$$(1) \quad = \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n+1}{n} \cdot n \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right), \forall n \geq 2$$

We have:

$$(2) \quad \lim_{n \rightarrow \infty} \left(s_n - \frac{\pi^2}{6} \right) = \lim_{n \rightarrow \infty} \frac{s_n - \frac{\pi^2}{6}}{\frac{1}{n}} \stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{s_{n+1} - s_n}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n \rightarrow \infty} -\frac{\frac{1}{(n+1)^2}}{\frac{1}{n(n+1)}} = -1$$

$$\lim_{n \rightarrow \infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} \cdot \lim_{n \rightarrow \infty} n \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1 \right) = \frac{a}{e} \lim_{n \rightarrow \infty} n(u_n - 1) =$$

$$= \frac{a}{e} \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} \cdot \ln u_n, \text{ where } u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}, \forall n \geq 2.$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{a_n}} \cdot \frac{n+1}{n} \right) = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{na_n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n}{n+1} \right) = a \cdot \frac{e}{a} \cdot 1 = e, \text{ then:}$$

$$(3) \quad \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right) = \frac{a}{e}$$

By (1), (2) and (3) we obtain:

$$\lim_{n \rightarrow \infty} B_n = \frac{a}{e}(-1) + \frac{\pi^2}{6} \cdot \frac{a}{e} = \frac{a(\pi^2 - 6)}{6e}$$

□

Theorem 23.

Let $(s_n)_{n \geq 1}, s_n = \sum_{k=1}^n \frac{1}{k^2}$, then

$$\lim_{n \rightarrow \infty} \left(s_n \cdot \sqrt[n+1]{(2n+1)!!} - \frac{\pi^2}{6} \cdot \sqrt[n]{(2n-1)!!} \right) = \frac{\pi^2 - 6}{3e}$$

Proof.

It is well-known that $\lim_{n \rightarrow \infty} s_n = \frac{\pi^2}{6}$

$$\text{We have: } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \stackrel{\text{Cauchy-D'Alembert}}{=}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} = \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} \left(\frac{n+1}{n} \right)^n = \frac{e}{2}$$

$$\text{So, } B_n = s_n \sqrt[n+1]{(2n+1)!!} - \frac{\pi^2}{6} \sqrt[n]{(2n-1)!!} =$$

$$= \sqrt[n+1]{(2n+1)!!} \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right) =$$

$$(1) \quad = \frac{\sqrt[n+1]{(2n+1)!!}}{n} \cdot n \left(s_n - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} \left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right)$$

We have:

$$\begin{aligned}
 (2) \quad & \lim_{n \rightarrow \infty} n \left(s_n - \frac{\pi^2}{6} \right) = \lim_{n \rightarrow \infty} \frac{s_n - \frac{\pi^2}{6}}{\frac{1}{n}} \stackrel{\text{Cesaro-Stolz}}{=} \lim_{n \rightarrow \infty} \frac{s_{n+1} - s_n}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}} = -1 \\
 & \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot \lim_{n \rightarrow \infty} n(u_n - 1) = \\
 & = \frac{2}{e} \lim_{n \rightarrow \infty} n(u_n - 1) = \frac{2}{e} \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \text{ where } u_n = \frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}, \forall n \geq 2. \\
 & \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{n} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 = 1, \text{ so } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1. \\
 & \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n \rightarrow \infty} \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = 2 \cdot \frac{e}{2} = e \\
 (3) \quad & \text{and then } \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right) = \frac{2}{e}
 \end{aligned}$$

By (1), (2), (3) we obtain that:

$$\lim_{n \rightarrow \infty} B_n = \frac{2}{e}(-1) + \frac{\pi^2}{6} \cdot \frac{2}{e} = \frac{\pi^2 - 6}{3e}$$

□

Theorem 24. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be positive real sequences such that $b_n = a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdots \sqrt[n]{a_n!}$ and

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n \cdot n} = a, \text{ then } \lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} \right) = \frac{e^2}{a}.$$

Proof.

$$\begin{aligned}
 \text{We have } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{a_n}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{a_n}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{x \rightarrow \infty} \frac{(x+1)^{x+1}}{a_{x+1}} \cdot \frac{a_x}{x^n} = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{n+1} \cdot \lim_{n \rightarrow \infty} \frac{a_n \cdot n}{a_{n+1}} = \frac{e}{a} \text{ and} \\
 \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{b_n}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{b_n}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{x \rightarrow \infty} \frac{(x+1)^{x+1}}{b_{x+1}} \cdot \frac{b_x}{x^n} = \\
 &= \lim_{x \rightarrow \infty} \left(\frac{x+1}{x} \right)^x \cdot \frac{b_x(x+1)}{b_{x+1}} = e \cdot \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{a_{n+1}}} = e \cdot \frac{e}{a} = \frac{e^2}{a}
 \end{aligned}$$

We have:

$$(1) \quad \frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} = \frac{n^2}{\sqrt[n]{b_n}} \cdot (u_n - 1) = \frac{n^2}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{n}{\sqrt[n]{b_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

Above we denote $u_n = \left(\frac{n+1}{n} \right)^2 \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}$. We have $\lim_{n \rightarrow \infty} u_n = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$

$$\begin{aligned}
 \text{Then, } \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{2n} \cdot \frac{b_n}{b_{n+1}} \cdot \sqrt[n+1]{b_{n+1}} = e^2 \cdot \lim_{n \rightarrow \infty} \left(\frac{b_n \cdot (n+1)}{b_{n+1}} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n+1} \right) = \\
 &= e^2 \cdot \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{b_{n+1}}}{n+1} = e^2 \cdot \frac{e}{a} \cdot \frac{a}{e^2} = e
 \end{aligned}$$

From (1) and above we obtain that:

$$\lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}} \right) = \frac{e^2}{a} \cdot 1 \cdot \ln e = \frac{e^2}{a}, \text{ and we are done!}$$

□

Theorem 25. Let $(x_n)_{n \geq 1}$, $x_1 = 1$, $x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdots \sqrt[n]{(2n-1)!!}$, then

$$\lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \right) = \frac{e^2}{a}.$$

Proof.

$$\text{We have: } \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{x_n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{x_n}} \stackrel{\text{Cauchy-D'Alembert}}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{x_{n+1}} \cdot \frac{x_n}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{\sqrt[n+1]{(2n+1)!!}} \cdot \frac{1}{n^n} =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n \cdot \lim_{n \rightarrow \infty} \frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}} = e \cdot \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{(2n-1)!!}} = e \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \stackrel{\text{C-D'A}}{=}$$

$$(1) \quad \stackrel{\text{C-D'A}}{=} e \cdot \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} = e \cdot \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} \cdot \left(\frac{n+1}{n} \right)^n = \frac{e^2}{2}$$

We have:

$$(2) \quad \frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} = \frac{n^2}{\sqrt[n]{x_n}} \cdot (u_n - 1) = \frac{n^2}{\sqrt[n]{x_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{n}{\sqrt[n]{x_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

Above, we denote: $u_n = \left(\frac{n+1}{n} \right)^2 \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}$. We have $\lim_{n \rightarrow \infty} u_n = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \stackrel{\text{C-D'A}}{=}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n \cdot \frac{n+1}{2n+1} = \frac{e}{2}$$

$$\text{Then, } \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{2n} \cdot \frac{x_n}{x_{n+1}} \cdot \sqrt[n+1]{x_{n+1}} = e^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n+1]{(2n+1)!!}} \cdot \sqrt[n+1]{x_{n+1}} =$$

$$= e^2 \cdot \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{x_{n+1}}}{n+1} = e^2 \cdot \frac{e}{2} \cdot \frac{2}{e^2} = e.$$

From (2) and above we obtain that:

$$\lim_{n \rightarrow \infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \right) = \frac{e^2}{2} \cdot 1 \cdot \ln e = \frac{e^2}{2}, \text{ and we are done!}$$

□

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