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SP.282. If in ΔABC; H –orthocentre; HD, HE, HF bisectors of angles BHC, CHA respectively AHB; $D \in (BC)$; $E \in (CA)$; $F \in (AB)$ then the following relationship holds:

$$\frac{[DEF]}{[ABC]} \geq 13\left(\frac{r}{R}\right)^2 - 3$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\Delta BGC \Rightarrow \frac{BD}{DC} = \frac{HB}{HC} \text{ and analogs}$$

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{HB}{HC} \cdot \frac{HC}{HA} \cdot \frac{HA}{HB} = 1; (1)$$

$$\text{Let: } \frac{HA}{HB} = m, \frac{HB}{HC} = k, \frac{HC}{HA} = p$$

$$S_{AEF} = \frac{AF \cdot AE \cdot sinA}{2} = \frac{m}{(m+1)(p+1)} \cdot \frac{bc \cdot sinA}{2} = \frac{m}{(m+1)(p+1)} \cdot S_{ABC} \text{ and analogs}$$

$$S_{DEF} = \frac{1 + kmp}{(1+k)(1+m)(1+p)} \overset{(1)}{=} \frac{2}{(1+k)(1+m)(1+p)} =$$

$$= \frac{2AH \cdot BH \cdot CH}{(AH+BH)(AH+CH)(CH+BH)} \cdot S_{ABC}$$

$$\text{But: } AH = 2RsinA \Rightarrow$$

$$S_{DEF}=\frac{2cosAcosBcosC}{(cosA+cosB)(cosB+cosC)(cosC+cosA)}\cdot S_{ABC};(2)$$

But: $cosAcosBcosC=\frac{s^2-(2R+r)^2}{4R^2};(3)$

$$(cosA+cosB)(cosB+cosC)(cosC+cosA)=\frac{r(s^2+r^2+2Rr)}{4R^3};(4)$$

$$s^2\le 27R^2\cdot s^2\ge\frac{27}{4}\cdot r^2,R\ge 2r;(5)$$

From (1)+(2)+(3)+(4)+(5) proved.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.