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JP.274. If $x, y, z \geq 0, x + y + z = 1; n \geq 2$ then:

$$(n + 1)(xy + yz + zx) \leq n(x^2 + y^2 + z^2) + 9xyz$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam; Solution 2 by Marian Dincă-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

By Schur's inequality:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x) \Leftrightarrow$$

$$x^2 + y^2 + z^2 + \frac{9xyz}{x + y + z} \geq 2(xy + yz + zx) \ \ (*)$$

Now, because: $x + y + z = 1$

Inequality becomes as: $(n + 1)(xy + yz + zx) \leq n(x^2 + y^2 + z^2) + \frac{9xyz}{x+y+z} \Leftrightarrow$

$$2(xy + yz + zx) + (n - 1)(xy + yz + zx) \leq$$

$$(n - 1)(x^2 + y^2 + z^2) + x^2 + y^2 + z^2 + \frac{9xyz}{x + y + z} \Leftrightarrow$$

$$(n - 1)(x^2 + y^2 + z^2 - xy - yz - zx) +$$

$$+\left(x^2 + y^2 + z^2 + \frac{9xyz}{x+y+z} - 2(xy + yz + zx)\right) \geq 0$$

Which is true because: $n \geq 2 \Rightarrow n - 1 > 0, x^2 + y^2 + z^2 \geq xy + yz + zx$ ***and by (*)***

Proved.

Solution 2 by Marian Dincă-Romania

Let: $x=\frac{a}{a+b+c}, y=\frac{b}{a+b+c}, z=\frac{c}{a+b+c}$

$$(n+1)(ab+bc+ca)(a+b+c)\le n(a^2+b^2+c^2)(a+b+c)+9abc$$

Let: $a+b+c=p, ab+bc+ca=q, abc=r$

$$(n+1)qp\le n(p^2-2q)p+9r$$

$$f(r)=n(p^2-2q)p+9r-(n+1)qp, 0\le r\le\frac{pq}{9}$$

Because it is of te first degree in the variable r, it will be necessary and sufficient to:

$$f\left(\frac{pq}{9}\right)\ge 0\Leftrightarrow n(p^2-2q)p+9r-(n+1)qp\ge\Leftrightarrow np(p^2-2q)-npq\ge 0\Leftrightarrow$$

$np(p^2-2q-q)\ge 0\Leftrightarrow np(p^2-3q)\ge 0$, **true.**

$f(0)\ge 0$ **for** $r=0\Rightarrow abc=0$

Let: $c=0\Rightarrow p=a+b, q=ab\Rightarrow q\le\frac{p^2}{4}$

$$f(0)=n(p^2-2q)p-(n+1)qp=n\left(p^2-\frac{p^2}{2}\right)p-\frac{(n+1)p^3}{4}=\frac{np^3}{2}-\frac{(n+1)p^3}{4}$$

$$=p^3\left(\frac{n-1}{4}\right)>0$$

Note by Editor:

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