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ROMANIAN MATHEMATICAL MAGAZINE

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Find:

$$\Omega = \lim_{x\to 0}\left(\frac{\sin(\sin 2x - \sin x) - \sin(\tan 2x - \tan x)}{x(\sin(\cos^{-1}x) - 1)}\right)$$

Proposed by Qusay Yousef-Amman-Jordan

Solution 1 by Igor Soposki-Skopje-Macedonia; Solution 2 by Rajeev Rastogi-India, Solution 3 by Yen Tung Chung-Taichung-Taiwan

Solution 1 by Igor Soposki-Skopje-Macedonia

$$\Omega = \lim_{x\to 0}\left(\frac{\sin(\sin 2x - \sin x) - \sin(\tan 2x - \tan x)}{x(\sin(\cos^{-1}x) - 1)}\right)$$

$$= \lim_{x\to 0}\frac{2\cos\frac{\sin 2x - \sin x + \tan 2x - \tan x}{2}\cdot \sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{x(\sin(\cos^{-1}x) - 1)}$$

$$= 2\lim_{x\to 0}\frac{\sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}{\frac{\sin 2x - \sin x - \tan 2x + \tan x}{2}}\cdot\frac{1}{2}\cdot\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x(\sin(\cos^{-1}x) - 1)}$$

$$= \lim_{x\to 0}\frac{\sin 2x - \sin x - \tan 2x + \tan x}{x\left(\sqrt{1 - x^2} - 1\right)}$$

$$=\lim_{x\to 0}\frac{\sin 2x\left(1-\frac{1}{\cos 2x}\right)-\sin x\left(1-\frac{1}{\cos x}\right)}{x\left(\sqrt{1-x^2}-1\right)}\cdot\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}-1}$$

$$=2\lim_{x\to 0}\frac{\sin 2x\left(\frac{\cos 2x-1}{\cos 2x}\right)-\sin x\left(\frac{\cos x-1}{\cos x}\right)}{x(1-x^2-1)}$$

$$=-2\lim_{x\to 0}\frac{\sin 2x\left(-\frac{2\sin^2 x}{\cos 2x}\right)-\sin x\left(-\frac{2\sin^2\frac{x}{2}}{\cos x}\right)}{x^3}$$

$$=4\lim_{x\to 0}\frac{2\sin x\cos x\sin^3 x-2\sin\frac{x}{2}\cos\frac{x}{2}\frac{\sin^2\frac{x}{2}}{\cos x}}{x^3}$$

$$=8\left(\lim_{x\to 0}\frac{\cos x\sin^3 x}{x^3}-\lim_{x\to 0}\frac{\cos^3\frac{x}{2}}{\cos x}\cdot\frac{\sin^3\frac{x}{2}}{\left(\frac{x}{2}\right)^3\cdot 8}\right)=7$$

Solution 2 by Rajeev Rastogi-India

$$\Omega=\lim_{x\to 0}\left(\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x(\sin(\cos^{-1}x)-1)}\right)$$

$$=\lim_{x\to 0}\frac{2\cos\left(\frac{\sin 2x-\sin x+\tan 2x-\tan x}{2}\right)\cdot\sin\left(\frac{\sin 2x-\tan 2x+\tan x-\sin x}{2}\right)}{x\left(\sqrt{1-x^2}-1\right)}$$

$$=2\lim_{x\to 0}\frac{\left(\frac{\sin 2x-\tan 2x+\tan x-\sin x}{2}\right)\left(\sqrt{1-x^2}-1\right)}{x(1-x^2-1)}$$

$$=-2\lim_{x\to 0}\left(\frac{\sin 2x-\tan 2x}{x^3}-\frac{\tan x-\sin x}{x^3}\right)$$

$$=-2\lim_{x\to 0}\left(-\frac{\sin 2x}{x}\left(\frac{1-\cos 2x}{x^2}\right)\frac{1}{\cos 2x}+\frac{\sin x}{x}\left(\frac{1-\cos 2x}{x^2}\right)\frac{1}{\cos x}\right)=7$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\lim_{x\to 0}\left(\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x(\sin(\cos^{-1}x)-1)}\right)$$

$$=\lim_{x\to 0}\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x\left(\sqrt{1-x^2}-1\right)}$$

$$= \lim_{x\to 0}\frac{\sin\left(\left(2x-\frac{(2x)^3}{3!}+O(x^5)\right)-\left(x-\frac{x^3}{3!}+O(x^5)\right)\right)}{x\left(1-\frac{x^2}{2}+O(x^4)-1\right)}-$$

$$-\lim_{x\to 0}\frac{\sin\left(\left(2x+\frac{(2x)^3}{3!}+O(x^5)\right)-\left(x+\frac{x^3}{3!}+O(x^5)\right)\right)}{x\left(1-\frac{x^2}{2}+O(x^4)-1\right)}$$

$$= \lim_{x\to 0}\frac{\sin\left(x-\frac{7}{6}x^3+O(x^5)\right)-\sin\left(x+\frac{7}{3}x^3+O(x^5)\right)}{-\frac{1}{2}x^3+O(x^5)}$$

$$= \lim_{x\to 0}\frac{\left\{\left(x-\frac{7}{6}x^3+O(x^5)\right)-\frac{1}{3!}\left(x-\frac{7}{6}x^3+O(x^5)\right)^3+\cdots\right\}}{-\frac{1}{2}x^3+O(x^5)}-$$

$$-\lim_{x\to 0}\frac{\left\{\left(x+\frac{7}{3}x^3+O(x^5)\right)-\frac{1}{3!}\left(x+\frac{7}{3}x^3+O(x^5)\right)^3+\cdots\right\}}{-\frac{1}{2}x^3+O(x^5)}$$

$$= \lim_{x\to 0}\frac{\left(-\frac{7}{6}-\frac{1}{6}-\frac{7}{3}+\frac{1}{6}\right)x^3+O(x^5)}{-\frac{1}{2}x^3+O(x^5)}=7$$

Note by Editor:
Mant thanks to Florică Anastase-Romania for typed solutions.