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ROMANIAN MATHEMATICAL MAGAZINE

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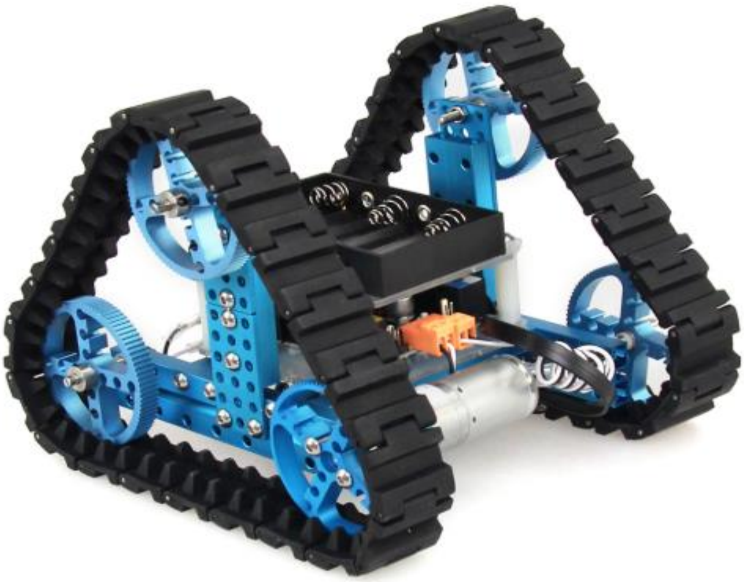

Find:

$$\Omega=\lim_{n\to\infty}\int_a^b\left(\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n+x^n}\right)dx\,,0\le a<b$$

Proposed by Vasile Mircea Popa-Romania

Solution by Kamel Benaicha-Algiers-Algerie

Put:

$$f(x)=\frac{2x^2-2(a+b)x+b(a+b)}{b-a}-x$$

$$=\frac{2x^2-(a+3b)x+b(a+b)}{b-a}=0\Leftrightarrow$$

$$2x^2-(a+3b)x+b(a+b)=0\Leftrightarrow x\in\left\{\frac{a+b}{2};b\right\}$$

$$\therefore\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\ge x\ for\ x\in\left[a;\frac{a+b}{2}\right]$$

$$\therefore\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\le x\ for\ x\in\left[\frac{a+b}{2};b\right]$$

$$\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n+x^n}\le\frac{2x^2-2(a+b)x+b(a+b)}{b-a}+x$$

Put:

$$g(x)=\ 2x^2-(a+3b)x+b(a+b)$$

$$g'(x)=4x-(3a+b)$$

$$g'(x)=0\Leftrightarrow x=\frac{3a+b}{4}\in[a,b]$$

$$\therefore g'(x)\geq 0\ for\ x\in\left[\frac{3a+b}{4};b\right]\Rightarrow g(x)\leq g(b)$$

$$\therefore g'(x)\leq 0\ for\ x\in\left[a;\frac{3a+b}{4}\right]\Rightarrow g(x)\leq g(a)$$

$$\therefore g(x)\leq max\{g(a),g(b)\}$$

$$g(a)=2a^2-3a^2-ab+ab+b^2=b^2-a^2$$

$$g(b)=2b(b-a)$$

$$g(b)-g(a)=(b-a)(2b-a-b)=(b-a)^2\Rightarrow g(a)\leq g(b)$$

$$\therefore\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n+x^n}\leq 2b;\ \forall x\in[a;b]$$

$$\Omega=\int_a^b\left(\lim_{n\to\infty}\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n+x^n}\right)dx$$

We know that:

$$\lim_{n\to\infty}(\alpha^n+\beta^n)^{\frac{1}{n}}=max\{\alpha;\beta\}$$

Then:

$$\Omega=\int_a^{\frac{a+b}{2}}\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)dx+\int_{\frac{a+b}{2}}^b xdx$$

$$=\frac{1}{b-a}\left(\frac{2x^3}{3}-(a+b)x^2+b(a+b)x\right)\Bigg|_a^{\frac{a+b}{2}}+\frac{x^2}{2}\Bigg|_{\frac{a+b}{2}}^b$$

$$=\frac{1}{24}(b-a)(7a+17b)$$

So:

$$\Omega=\lim_{n\to\infty}\int_a^b\left(\sqrt[n]{\left(\frac{2x^2-2(a+b)x+b(a+b)}{b-a}\right)^n+x^n}\right)dx=\frac{1}{24}(b-a)(7a+17b)$$

Note by Editor:
Many thanks to Florica Anastase-Romania for typed solution.