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**SP.265. In acute  $\triangle ABC$  the following relationship holds:**  
 $(\cos A)^A \cdot (\cos B)^B \cdot (\cos C)^C \leq 2^{-\pi}$

*Proposed by Florentin Vişescu-Romania*

*Solution 1 by Radu Butelcă-Romania; Solution 2 by Adrian Popa-Romania*

***Solution 1 by Radu Butelcă-Romania***

*As the members are positive, we can apply the function  $x \rightarrow \log(x)$*

$$\log((\cos A)^A \cdot (\cos B)^B \cdot (\cos C)^C) \leq \log(2^{-\pi}) \Leftrightarrow$$

$$A \log(A) + B \log(B) + C \log(C) \leq -\pi \log 2 \quad (1)$$

$$\text{Let } f: [0, \pi] \rightarrow \mathbb{R}, f(x) = x \log(x)$$

$$f'(x) = \log(\cos x) - x \tan x$$

$$f''(x) = -2 \tan x - \frac{x}{\cos^2 x} < 0, \forall x \in [0, \pi] \text{ then } f \text{ is concave}$$

$$\text{By Jensen, we have } f\left(\frac{A+B+C}{3}\right) \geq \frac{f(A)+f(B)+f(C)}{3} \Leftrightarrow$$

$$\frac{A+B+C}{3} \log\left(\cos\left(\frac{A+B+C}{3}\right)\right) \geq \frac{A \cdot \log(\cos A) + B \cdot \log(\cos B) + C \cdot \log(\cos C)}{3}$$

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$$\begin{aligned} \pi \log\left(\cos\frac{\pi}{3}\right) &\geq A \cdot \log(\cos A) + B \cdot \log(\cos B) + C \cdot \log(\cos C) \\ -\pi \log 2 &\geq A \cdot \log(\cos A) + B \cdot \log(\cos B) + C \cdot \log(\cos C) \quad (2) \end{aligned}$$

From (1)+(2) proved.

**Solution 2 by Adrian Popa-Romania**

$$P = (\cos A)^A \cdot (\cos B)^B \cdot (\cos C)^C \leq 2^{-\pi}$$

$$\begin{aligned} P &= (\cos A)^A \cdot (\cos B)^B \cdot (\cos C)^C \stackrel{\text{Am-Gm}}{\leq} \left( \frac{A \cdot \cos A + B \cdot \cos B + C \cdot \cos C}{A + B + C} \right)^{A+B+C} \\ &= \left( \frac{A \cdot \cos A + B \cdot \cos B + C \cdot \cos C}{\pi} \right)^\pi \\ &\quad \text{Let: } f(x) = x \cos x, x \in \left(0, \frac{\pi}{2}\right) \end{aligned}$$

$$f'(x) = -2 \sin x - x \cos x < 0, \text{ because } \sin x > 0; \cos x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

So,  $f$  is concave  $\xrightarrow{\text{Jensen}} \frac{f(A)+f(B)+f(C)}{3} \leq f\left(\frac{A+B+C}{3}\right)$  then

$$\begin{aligned} A \cdot \cos A + B \cdot \cos B + C \cdot \cos C &\leq 3 \left( \frac{A + B + C}{3} \right) \cos \left( \frac{A + B + C}{3} \right) = \frac{\pi}{2} \Rightarrow \\ P &< 2^{-\pi} \end{aligned}$$

Note by Editor:

Many thanks to Florică Anastase-Romania for typed solution.