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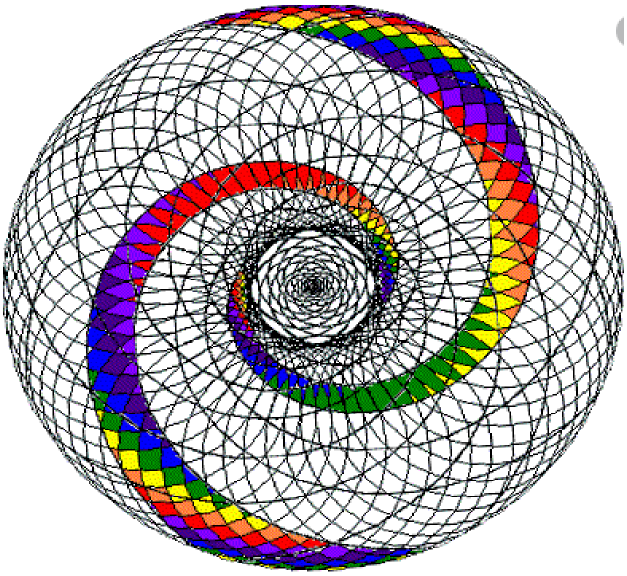

JP.286. If $a,b,c>0, ab+bc+ca=3$ then:

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Daniel Văcaru-Romania,Solution 2, generalizations and extensions by Marin Chirciu-Romania

Solution 1 by Daniel Văcaru-Romania

We have:

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq 6$$

$$\Leftrightarrow \sum_{cyc}\frac{a^2+b^2}{a+b}+\sum_{cyc}\frac{(a^2+b^2)ab}{a+b}\geq 6$$

We have:

$$\frac{a^2+b^2}{a+b}\geq\frac{a+b}{2}\Rightarrow\sum_{cyc}\frac{a^2+b^2}{a+b}\geq a+b+c$$

On the other hand

$$\frac{(a^2+b^2)ab}{a+b}\geq\frac{2a^2b^2}{a+b}\Rightarrow\sum_{cyc}\frac{(a^2+b^2)ab}{a+b}\geq 2\sum_{cyc}\frac{a^2b^2}{a+b}\geq 2\frac{(\sum ab)^2}{\sum(a+b)}=\frac{9}{a+b+c}$$

It follows

$$\frac{(a^2+b^2)(ab+1)}{a+b}+\frac{(b^2+c^2)(bc+1)}{b+c}+\frac{(c^2+a^2)(ca+1)}{c+a}\geq a+b+c+\frac{9}{a+b+c}$$

Solution 2, generalizations and extensions by Marin Chirciu-Romania

Use** inequality* $x^2+y^2\geq\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\geq 0$, ***equality for $x=y$, ***we get:***

$$M_s=\sum\frac{(b^2+c^2)(bc+1)}{b+c}\geq\sum\frac{\frac{(b+c)^2}{2}(bc+1)}{b+c}=\sum\frac{(b+c)(bc+1)}{2}\overset{(1)}{\geq}6=M_d,$$

where (1) $\Leftrightarrow\sum(b+c)(bc+1)\geq 12$.

From $ab+bc+ca=3$, ***and*** $a,b,c>0$, ***we can let trigonometric substitutions:***

$$a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}\text{, because:}$$

$$ab+bc+ca=3\Leftrightarrow\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1\Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1\text{, (true in any triangle }\Delta XYZ\text{).}$$

With substitutions $a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}$,

Inequality $\sum(b+c)(bc+1)\geq 12$ ***becomes:***

$$\sum\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+1\right)\geq 12\Leftrightarrow\sum\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+1\right)\geq 4\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow 3\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)+2\sum tg\frac{X}{2}\geq 4\sqrt{3}\text{, (2).}$$

Following relationship holds in any triangle:

$$\sum tg\frac{X}{2}=\frac{4R+r}{p}\text{ and }\prod tg\frac{X}{2}=\frac{r}{p}\text{, we get: }\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p},$$

and ineqality (2) becomes:

$$3\cdot\frac{2(2R-r)}{p}+2\cdot\frac{4R+r}{p}\geq 4\sqrt{3}\Leftrightarrow 5R-r\geq p\sqrt{3},$$

True, from Doucet inequality $4R+r\geq p\sqrt{3}$.

We must show that:

$5R-r\geq 4R+r\Leftrightarrow R\geq 2r$, ***(Euler inequality).***

Equality if only if triangle is equilateral.

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Inequality can be extended:

1) If $a,b,c>0$ **such that** $ab+bc+ca=3$ **și** $n\geq 0$**.Prove:**

$$\frac{(a^2+b^2)(ab+n)}{a+b}+\frac{(b^2+c^2)(bc+n)}{b+c}+\frac{(c^2+a^2)(ca+n)}{c+a}\geq 3(n+1).$$

Proposed by Marin Chirciu-Romania

Solution:

Applying inequality $x^2+y^2\geq\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\geq 0$***, equality for*** $x=y$***, we get:***

$$M_s=\sum\frac{(b^2+c^2)(bc+n)}{b+c}\geq\sum\frac{\frac{(b+c)^2}{2}(bc+n)}{b+c}=\sum\frac{(b+c)(bc+n)}{2}\overset{(1)}{\geq}3(n+1)=M_d,$$

where (1) $\Leftrightarrow\sum(b+c)(bc+n)\geq 6(n+1)$.

From $ab+bc+ca=3$***, where*** $a,b,c>0$***, we can let trigonometric substitutions:***

$$a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}\text{, because:}$$

$$ab+bc+ca=3\Leftrightarrow\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1\Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1,\text{(true in any }\Delta XYZ\text{).}$$

With substitutions: $a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}$***, inequality***

$$\sum(b+c)(bc+n)\geq 6(n+1)$$

becomes:

$$\sum\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+n\right)\geq 6(n+1)\Leftrightarrow$$

$$\Leftrightarrow\sum\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\left(3tg\frac{Y}{2}tg\frac{Z}{2}+n\right)\geq 2(n+1)\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow 3\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)+2n\sum tg\frac{X}{2}\geq 2(n+1)\sqrt{3}\text{, (2)}.$$

In ΔXYZ ***following relationship holds:***

$$\sum tg\frac{X}{2}=\frac{4R+r}{p}\text{ și }\prod tg\frac{X}{2}=\frac{r}{p}\text{, we get }\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p}\text{, and}$$

inequality (2) becomes:

$$3\cdot\frac{2(2R-r)}{p}+2n\cdot\frac{4R+r}{p}\ge 2(n+1)\sqrt{3}\Leftrightarrow 3(2R-r)+n(4R+r)\ge(n+1)\cdot p\sqrt{3},$$

True from Doucet inequality $4R+r\ge p\sqrt{3}$.

We must show that:

$$3(2R-r)+n(4R+r)\ge(n+1)\cdot(4R+r)\Leftrightarrow R\ge 2r, \textbf{ (Euler inequality).}$$

Equality if only if triangle is equilateral.

Note:

1). For $n=1$ ***we get Problem JP.286 proposed by Daniel Sitaru in RMM, Number 20, Spring Edition 2021.***

2). For $n=0$ ***we get inequality:***

2) If $a,b,c>0$ ***such that*** $ab+bc+ca=3$***. Prove:***

$$\frac{ab(a^2+b^2)}{a+b}+\frac{bc(b^2+c^2)}{b+c}+\frac{ca(c^2+a^2)}{c+a}\ge 3.$$

Proposed by Marin Chirciu-Romania

Solution:

Applying inequality $x^2+y^2\ge\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\ge 0$, ***equality for*** $x=y$, ***we get:***

$$M_s=\sum\frac{bc(b^2+c^2)}{b+c}\ge\sum\frac{bc\cdot\frac{(b+c)^2}{2}}{b+c}=\sum\frac{bc(b+c)}{2}\overset{(1)}{\ge}3=M_d,$$

where (1) $\Leftrightarrow\sum bc(b+c)\ge 6$.

From $ab+bc+ca=3$***, where*** $a,b,c>0$, ***we can let trigonometric substitution:***

$$a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}, \textbf{ because:}$$

$$ab+bc+ca=3\Leftrightarrow\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}+\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}+\frac{c}{\sqrt{3}}\cdot\frac{a}{\sqrt{3}}=1\Leftrightarrow$$

$$\Leftrightarrow tg\frac{X}{2}tg\frac{Y}{2}+tg\frac{Y}{2}tg\frac{Z}{2}+tg\frac{Z}{2}tg\frac{X}{2}=1, \textbf{(true in any } \Delta XYZ \textbf{).}$$

With $a=\sqrt{3}tg\frac{X}{2},b=\sqrt{3}tg\frac{Y}{2},c=\sqrt{3}tg\frac{Z}{2}$, ***inequality*** $\sum bc(b+c)\ge 6$ ***becomes:***

$$\sum 3tg\frac{Y}{2}tg\frac{Z}{2}\left(\sqrt{3}tg\frac{Y}{2}+\sqrt{3}tg\frac{Z}{2}\right)\ge 6\Leftrightarrow\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)\ge\frac{2\sqrt{3}}{3}, \textbf{ (2)}.$$

In ΔXYZ ***following relationship holds:***

$\sum tg\frac{X}{2}=\frac{4R+r}{p}$ ***și*** $\prod tg\frac{X}{2}=\frac{r}{p}$***, we get*** $\sum tg\frac{Y}{2}tg\frac{Z}{2}\left(tg\frac{Y}{2}+tg\frac{Z}{2}\right)=\frac{2(2R-r)}{p}$ ***,***

and inequality (2) becomes:

$\frac{2(2R-r)}{p}\geq\frac{2\sqrt{3}}{3}\Leftrightarrow 3(2R-r)\geq p\sqrt{3}$***, true from Doucet inequality*** $4R+r\geq p\sqrt{3}$***.***

We must show that:

$3(2R-r)\geq 4R+r\Leftrightarrow R\geq 2r$***, (Euler inequality).***

Equality if only if triangle is equilateral.

Note by Editor:
Many thanks to Florică Anastase and Marin Chirciu-Romania for typed solutions.