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DANIEL SITARU

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Proposed by

Daniel Sitaru – Romania,Thanasis Gakopoulos-Larisa- Greece

Adil Abdullayev-Baku-Azerbaijan,Bogdan Fuștei-Romania

Seyran Ibrahimov-Maasilli-Azerbaijan,George Apostolopoulos-Messolonghi-Greece, Mustafa Tarek-Cairo-Egypt, Florentin Vișescu – Romania, Rahim Shahbazov-Baku-Azerbaijan, Radu Diaconu-Romania,Avishek Mitra-West Bengal-India,Marin Chirciu – Romania

D.M. Bătinețu – Giurgiu– Romania ,Neculai Stanciu – Romania

Emil Popa-Romania, Seyran Ibrahimov-Maasilli-Azerbaijan

Tran Hong-Dong Thap-Vietnam,Marian Ursărescu – Romania,Mokhtar Khassani-Mostaganem-Algerie, Jalil Hajimir-Toronto-Canada, Rovsen Pirguliyev-Sumgait-Azerbaijan,

Ionuț Florin Voinea-Romania

Solutions by

Daniel Sitaru – Romania,Thanasis Gakopoulos-Larisa- Greece

Soumava Chakraborty-Kolkata-India,Marian Ursărescu-Romania

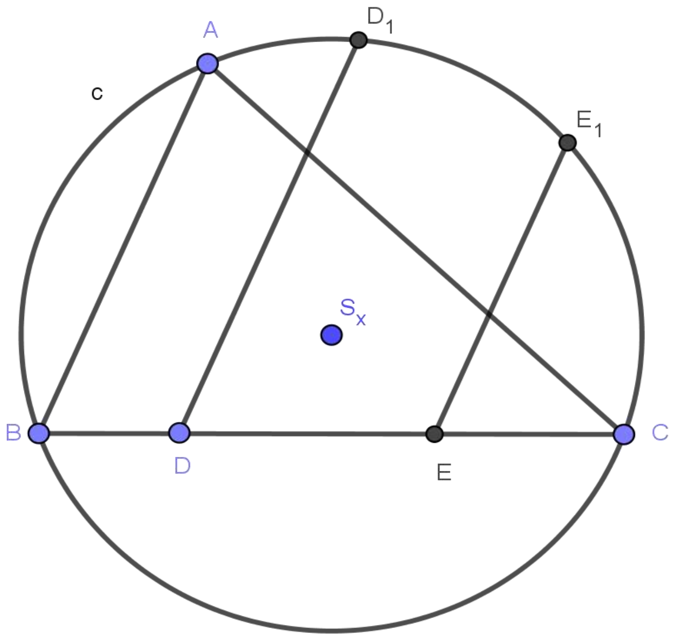
Mokhtar Khassani-Mostaganem-Algerie,Bogdan Fuștei-Romania

George Apostolopoulos-Messolonghi-Greece,Avishek Mitra-West Bengal-India,Soumava Chakraborty-Kolkata-India,Șerban George Florin-Romania

Boris Colakovic-Belgrade-Serbie,Adrian Popa – Romania

Ravi Prakash-New Delhi-India,Rahim Shahbazov-Baku-Azerbaijan,Jalil Hajimir-Toronto-Canada,Sanong Huayrerai-Nakon Pathom-Thailand,Myagmarsuren Yadamsuren-Darkhan-Mongolia,Remus Florin Stanca-Romania,Marin Chirciu-Romania,Rovsen Pirguliyev-Sumgait-Azerbaijan,George Apostolopoulos-Messolonghi-Greece

1301.

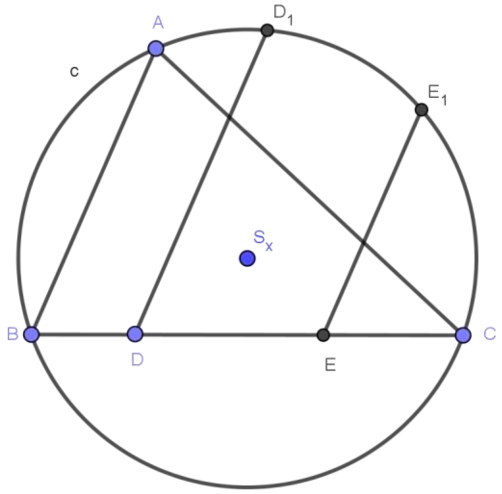


$$AB = 5, AC = 7, BD = 2, DE = 3, EC = 3, AB \parallel DD_1 \parallel EE_1$$

$$S_x = ?$$

Proposed by Thanasis Gakopoulos-Larisa- Greece

Solution by proposer



PLAGIOGONAL system: $BC \equiv Bx, BA \equiv By$

$$AC: f(x)=\frac{1}{2}\left(\sqrt{-3x^2+22x+25}-x+5\right)$$

$$[DEE_1D_1]=\left(\sin\frac{\pi}{3}\right)\cdot\int_2^5 f(x)\to$$

$$S_x=\frac{9\sqrt{3}}{8}+\sqrt{5}+\frac{5}{8}\sqrt{19}+\frac{49}{6}\left[\sin^{-1}\left(\frac{2}{7}\right)+\sin^{-1}\left(\frac{5}{14}\right)\right]=12.258$$

1302. In ΔABC the following relationship holds:

$$\frac{w_aw_bw_c}{h_ah_bh_c}\cdot\frac{(a+b)(b+c)(c+a)}{8abc}=\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\frac{w_aw_bw_c}{h_ah_bh_c}\cdot\frac{(a+b)(b+c)(c+a)}{8abc}=\frac{1}{8}\prod_{cyc}\frac{(b+c)w_a}{h_a\cdot a}=$$

$$=\frac{1}{8}\prod_{cyc}\frac{(b+c)w_a}{\frac{2S}{a}\cdot a}=\frac{1}{64}\prod_{cyc}(b+c)w_a==\frac{1}{64S^3}\prod_{cyc}(b+c)\cdot\frac{2bc}{b+c}cos\frac{A}{2}=$$

$$=\frac{1}{8S^3}(abc)^2\prod_{cyc}cos\frac{A}{2}=\frac{1}{8S^3}(4RS)^2\cdot\frac{s}{4R}=\frac{16R^2S^2s}{32RS^2rs}=\frac{R}{2r}$$

1303. In ΔABC the following relationship holds:

$$\frac{1}{r}\sum_{cyc}h_a=\left(\sum_{cyc}\frac{h_bh_c}{a^2}\right)\left(\sum_{cyc}\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}}\right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}}=\sum\frac{(b+c)\left(\frac{rs}{s-a}-\frac{rs}{s}\right)}{2bc\cos\frac{A}{2}}\sqrt{\frac{\left(\frac{2rs}{a}\right)}{\left(\frac{rs}{s-a}\right)}}=$$

$$=\sum\frac{(b+c)\left(\frac{rs(s-(s-a))}{s(s-a)}\right)}{2bc\cos\frac{A}{2}}\sqrt{\frac{s(s-a)}{bc}\left(\frac{2abc}{sa^2}\right)}$$

$$=\sum\frac{2ra(s+s-a)\cos\frac{A}{2}}{2(s-a)abc\cos\frac{A}{2}}\sqrt{\frac{2Rrs}{s}}=\sqrt{2Rr}\left(\frac{r}{4Rrs}\right)\sum\frac{a(s+s-a)}{s-a}=$$

$$=\frac{\sqrt{2Rr}}{4Rs}\left(s\sum\frac{a-s+s}{s-a}+\sum a\right)=\frac{\sqrt{2Rr}}{4Rs}\left(s(-3)+\frac{(4Rr+r^2)s^2}{sr^2}+2s\right)$$

$$=\frac{\sqrt{2Rr}}{4R}\left(-1+\frac{4R+r}{r}\right)=\frac{\sqrt{2Rr}}{4R}\left(\frac{4R}{r}\right)=\sqrt{\frac{2R}{r}}\Rightarrow\left(\sum\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}}\right)^2=\frac{2R}{r}$$

$$\Rightarrow\left(\sum\frac{h_bh_c}{a^2}\right)\left(\sum\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}}\right)^2=\frac{2R}{r}\sum\left(\frac{ca.ab}{4R^2a^2}\right)=\frac{\sum bc}{2Rr}=\frac{1}{r}\sum\left(\frac{bc}{2R}\right)=\frac{1}{r}\sum h_a\ \text{(Proved)}$$

1304. In ΔABC the following relationship holds:

$$h_a=\frac{m_a}{\sin A}\sqrt{\frac{2\cdot\sum_{cyc}\sin A\cdot\prod_{cyc}(\sin A+\sin B-\sin C)}{3+\cos 2A-2\cos 2B-2\cos 2C}}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Daniel Sitaru-Romania

$$2\cdot\sum_{cyc}\sin A\cdot\prod_{cyc}(\sin A+\sin B-\sin C)=$$

$$=\frac{1}{R}\sum_{cyc}2R\sin A\cdot\frac{1}{8R^3}\prod_{cyc}(a+b-c)=\frac{1}{R^4}\cdot 2s\cdot\prod_{cyc}(s-a)=\frac{2S^2}{R^4}$$

$$3+\cos 2A-2\cos 2B-2\cos 2C=3+1-2\sin^2 A-2+4\sin^2 B-2+4\sin^2 C=$$

$$=-2\sin^2 A+4\sin^2 B+4\sin^2 C=\frac{1}{2R^2}(2b^2+2c^2-a^2)=\frac{2m_a^2}{R^2}$$

$$\frac{m_a}{\sin A}\sqrt{\frac{2\cdot\sum_{cyc}\sin A\cdot\prod_{cyc}(\sin A+\sin B-\sin C)}{3+\cos 2A-2\cos 2B-2\cos 2C}}=$$

$$=\frac{m_a}{\sin A}\sqrt{\frac{\frac{2S^2}{R^4}}{\frac{2m_a^2}{R^2}}}=\frac{m_a}{\sin A}\cdot\frac{S}{m_a\cdot R}=\frac{2S}{2R\sin A}=\frac{a\cdot h_a}{a}=h_a$$

1305. **In ΔABC the following relationship holds:**

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\le 4\left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}\le\frac{R^2}{r^2}$ *(1)*

Because $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}, s=a+b+c=1$

$$\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}=1+\left(\frac{4R+r}{s^2}\right)^2\quad(2)$$

From (1)+(2) we must show: $1+\frac{(4R+r)^2}{s^2}\le\frac{R^2}{r^2}$ *(3)*

But $2s^2\ge 27Rr$ *(Cosniță and Turtoiu inequality) (4)*

From (3)+(4) we must show: $1+\frac{2(4R+r)^2}{27Rr}\le\frac{R^2}{r^2}\Leftrightarrow$

$$\Leftrightarrow\frac{27Rr+2(4R+r)^2}{27Rr}\le\frac{R^2}{r^2}\Leftrightarrow 27Rr^2+2(4R+r)^2\le\frac{27R^3}{r}\Leftrightarrow$$

$$27Rr^2+2r(4R+r)^2\le 27R^3\quad(5)$$

From Euler: $27Rr^2\le\frac{27R^3}{4}$ *(6)*

$$2r(4R+r)^2\le R\cdot\frac{81R^2}{4}=\frac{81R^3}{4}\quad(7)$$

From (6)+(7)$\Rightarrow 27Rr^2+2r(4R+r)^2\le\frac{27R^3}{4}+\frac{81R^3}{4}=\frac{108R^3}{4}=27R^3\Rightarrow$ *(5) it is true.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\sec^2\frac{A}{2}=\sum\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)}$$

$$=\frac{\sum bc(s^2-s(2s-a)+bc)}{r^2s^2}=\frac{-s^2\sum ab+(\sum ab)^2-2abc(2s)+3sabc}{r^2s^2}$$

$$=\frac{(s^2+4Rr+r^2)(4Rr+r^2)-4Rrs^2}{r^2s^2}=\frac{s^2r^2+r^2(4R+r)^2}{r^2s^2}=$$

$$=1+\frac{(4R+r)^2}{s^2}\le 4\left(\frac{R}{2r}\right)^2 \Leftrightarrow \frac{R^2-r^2}{r^2}\ge\frac{(4R+r)^2}{s^2}\Leftrightarrow s^2(R^2-r^2)\overset{(1)}{\ge} r^2(4R+r)^2$$

$$\because R^2-r^2=(R+2r)(R-2r)+3r^2\overset{Euler}{\ge}3r^2>0$$

$$\therefore \textit{LHS of (1)}\overset{Gerretsen}{\ge}(16Rr-5r^2)(R^2-r^2)\overset{?}{\ge} r^2(4R+r)^2$$

$$\Leftrightarrow 16t^3-21t^2-24t+4\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{16t^2+11(t-2)+22\}\overset{?}{\ge}0\rightarrow \textit{true}\because t\overset{Euler}{\ge}2$$

$$\Rightarrow \textit{(1)}\Rightarrow \textit{proposed inequality is true (Proved)}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\sum\sec^2\left(\frac{A}{2}\right)\le 4\left(\frac{R}{2r}\right)^2\Leftrightarrow\sum\left(1+\tan^2\left(\frac{A}{2}\right)\right)\le\left(\frac{R}{2}\right)^2\Leftrightarrow\left(\sum\tan\left(\frac{A}{2}\right)\right)^2+1\le\left(\frac{R}{r}\right)^2$$

$$\Rightarrow\left(\frac{4R+r}{s}\right)^2+1\le\left(\frac{R}{r}\right)^2\Rightarrow r^2(4R+r)^2\le(R^2-r^2)s^2$$

$$(R^2-r^2)s^2\overset{Gerretsen}{\ge}(R^2-r^2)(16Rr-5r^2)$$

Now, we will prove that: $(16Rr-5r^2)(R^2-r^2)\ge r^2(4R+r)^2\Leftrightarrow$

$$\Leftrightarrow r(16R^3-21R^2r-24Rr^2+4r^3)\ge 0\Rightarrow$$

$$\Rightarrow r\left(16R^2(R-2r)+11Rr(R-2r)-2r^2(R-2r)\right)\ge 0\overset{R\ge 2r}{\Rightarrow}$$

$$\Rightarrow 84r^3(R-2r)\ge 0\overset{true}{\rightarrow}\textit{(Euler } R\ge 2r\textit{)}\therefore\sum\sec^2\left(\frac{A}{2}\right)\le 4\left(\frac{R}{2r}\right)^2$$

1306. In ΔABC the following relationship holds:

$$\sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\le\sqrt{\frac{s}{r}}$$

Proposed by Mustafa Tarek-Cairo-Egypt

Solution 1 by George Apostolopoulos-Messolonghi-Greece

We have $\frac{a^2}{4F}=\frac{(2R\sin A)^2}{4\left(\frac{1}{2}bc\sin A\right)}=\frac{2R^2\sin A}{bc}=\frac{2R^2\sin A}{(2R\sin B)(2R\sin C)}=\frac{\sin A}{2\sin B\cdot\sin C}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos(B-C)-\cos(B+C)}\geq$

$\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1+\cos A}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1+\left(2\cos^2\frac{A}{2}-1\right)}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\tan\frac{A}{2}$. *So,* $\tan\frac{A}{2}\leq\frac{a^2}{4R}$. *Similarly,* $\tan\frac{B}{2}\leq\frac{b^2}{4F}$ *and*

$$\tan\frac{C}{2}\leq\frac{c^2}{4F}.\ \textit{So,}\ \sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\leq\frac{a+b+c}{2\sqrt{F}}=\frac{s}{\sqrt{rs}}=\sqrt{\frac{s}{r}}$$

Equality holds if and only if the triangle ABC is an equilateral triangle.

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow\sum\sqrt{\tan\frac{B}{2}}\leq\sqrt{\frac{s}{r}}\Rightarrow\sum\sqrt{\frac{r_b}{s}}\leq\sqrt{\frac{s}{r}}\Rightarrow\sum\sqrt{r_b}\leq\frac{s}{\sqrt{r}}\Rightarrow\left(\sum\sqrt{r_b}\right)^2\leq\frac{s^2}{r}$$

$$\Rightarrow\left(\sum\sqrt{r_b}\right)^2\leq\left(\sum\frac{1}{r_a}\right)\left(\sum r_ar_b\right)$$

$$\Leftrightarrow\left(\sum\frac{1}{\sqrt{r_a}}\cdot\sqrt{r_ar_b}\right)^2\overset{CBS}{\leq}\left\{\sum\left(\frac{1}{\sqrt{r_a}}\right)^2\right\}\left\{\sum\left(\sqrt{r_ar_b}\right)^2\right\}\ \textit{(* true)}$$

$$\Leftrightarrow\sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\leq\sqrt{\frac{s}{r}}\ \textit{(proved)}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\therefore\tan\left(\frac{A}{2}\right)=\frac{(a+b-c)(a+c-b)}{4S}=\frac{a^2-(b-c)^2}{4S}\leq\frac{a^2}{4S}\to\tan\left(\frac{B}{2}\right)\leq\frac{b^2}{4S},$$

$$\tan\left(\frac{C}{2}\right)\leq\frac{c^2}{4S}\therefore\sum\sqrt{\tan\left(\frac{A}{2}\right)}\leq\frac{a+b+c}{2\sqrt{S}}=\frac{s}{\sqrt{rs}}=\sqrt{\frac{s}{r}}$$

Solution 4 by Bogdan Fuștei-Romania

We know that: $\frac{r_a}{s}=\tan\frac{A}{2}$ *(and the analogs)*

$r_ar=(s-b)(s-c)$ *(and the analogs)*

The above inequality becomes: $\sqrt{\frac{r_a}{s}}+\sqrt{\frac{r_b}{s}}+\sqrt{\frac{r_c}{s}}\leq\sqrt{\frac{s}{r}}$

$$\Rightarrow\sqrt{r_ar}+\sqrt{r_br}+\sqrt{r_cr}\leq\sqrt{s\cdot s}=\sqrt{s^2}=s$$

$$\sqrt{(s-b)(s-c)}\leq\frac{s-b+s-c}{2}=\frac{2s-b-c}{2}=\frac{a}{2}\ \textit{(and the analogs)}$$

(The inequality between the geometric means and the arithmetic means)

Summing $\sqrt{r_a r} \le \frac{a}{2}$ *(and the analogs)* $\Rightarrow$ *we obtain the above inequality.*

1307. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\prod_{cyc}\left(1-\sqrt{\frac{2r}{r_b+r_c}}\right) \ge \frac{n_a n_b n_c}{2\sqrt{2}s^3}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Firstly, } r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\cos\frac{A}{2}\left(\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}\right)}{\prod\cos\frac{A}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} \overset{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$\textit{Now, } \frac{2r}{r_b+r_c} < 1 \overset{by\ (1)}{=} \frac{2r}{4R\cos^2\frac{A}{2}} < 1 \Leftrightarrow \frac{r}{R} < 2\cos^2\frac{A}{2}$$

$$\Leftrightarrow \sum\cos A - 1 < 1+\cos A \Leftrightarrow \cos B+\cos C < 2 \rightarrow \textit{true} \because \cos B, \cos C < 1$$

$$\therefore \frac{2r}{r_b+r_c} < 1 \Rightarrow \sqrt{\frac{2r}{r_b+r_c}} < 1 \Rightarrow 1-\sqrt{\frac{2r}{r_b+r_c}} \overset{(a)}{>} 0$$

$$\textit{Similarly, } 1-\sqrt{\frac{2r}{r_c+r_a}} \overset{(b)}{>} 0 \textit{ and, } 1-\sqrt{\frac{2r}{r_a+r_b}} \overset{(c)}{>} 0$$

$$\textit{Now, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc)$$

$$\Rightarrow s(b^2+c^2)-2sbc = an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2$$

$$\Rightarrow an_a^2 = as^2+s(2bc\cos A-2bc) = as^2-4sbc\sin^2\frac{A}{2}$$

$$= as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2-\frac{4\Delta^2}{s-a} = as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2-2ah_ar_a \Rightarrow n_a^2 \overset{(2)}{=} s^2-2h_ar_a$$

$$\textit{Now, } 1-\sqrt{\frac{2r}{r_b+r_c}} \ge \frac{n_a}{\sqrt{2}s} \Leftrightarrow 1+\frac{2r}{r_b+r_c}-2\sqrt{\frac{2r}{r_b+r_c}} \ge \frac{n_a^2}{2s^2}$$

$$\overset{by\ (1),(2)}{\Leftrightarrow} 1+\frac{2r}{4R\cos^2\frac{A}{2}}-2\sqrt{\frac{2r}{4R\cos^2\frac{A}{2}}} \ge \frac{s^2-2h_ar_a}{2s^2}$$

$$\Leftrightarrow 2s^2+\frac{rs^2}{R\cos^2\frac{A}{2}}-4s^2\sqrt{\frac{2r}{4R\cos^2\frac{A}{2}}}\geq s^2-2\left(\frac{2rs}{a}\right)\left(\frac{rs}{s-a}\right)$$

$$\Leftrightarrow s^2+\frac{4Rrs\cdot rs^2}{Rsa(s-a)}+\frac{4r^2s^2}{a(s-a)}-2s^2\sqrt{\frac{8r}{4R}}\sec\frac{A}{2}\geq 0$$

$$\Leftrightarrow s^2\left(1+\frac{8r^2}{a(s-a)}-2\sqrt{\frac{2r}{R}}\sec\frac{A}{2}\right)\geq 0\Leftrightarrow s^2\left(1+\frac{8sbcr^2}{abcs(s-a)}-2\sqrt{\frac{2r}{R}}\sec\frac{A}{2}\right)\geq 0$$

$$\Leftrightarrow s^2\left(1+\left(\frac{8sr^2}{4Rrs}\right)\left(\frac{bc}{s(s-a)}\right)-2\sqrt{\frac{2r}{R}}\sec\frac{A}{2}\right)\geq 0$$

$$\Leftrightarrow s^2\left(1+\frac{2r}{R}\sec^2\frac{A}{2}-2\sqrt{\frac{2r}{R}}\sec\frac{A}{2}\right)\geq 0\Leftrightarrow s^2\left(1-\sqrt{\frac{2r}{R}}\sec\frac{A}{2}\right)^2\geq 0$$

$$\rightarrow true\ \therefore 1-\sqrt{\frac{2r}{r_b+r_c}}\overset{(i)}{\geq}\frac{n_a}{\sqrt{2s}}.\ Similarly,\ 1-\sqrt{\frac{2r}{r_c+r_a}}\overset{(ii)}{\geq}\frac{n_b}{\sqrt{2s}}\ and,\ 1-\sqrt{\frac{2r}{r_a+r_b}}\overset{(iii)}{\geq}\frac{n_c}{\sqrt{2s}}$$

$$(a),(b),(c)\Rightarrow (i).(ii).(iii)\Rightarrow \prod\left(1-\sqrt{\frac{2r}{r_b+r_c}}\right)\geq\prod\left(\frac{n_a}{\sqrt{2s}}\right)=\frac{n_an_bn_c}{2\sqrt{2}s^3}\ \ (Done)$$

1308. In ΔABC the following relationship holds:

$$\frac{\sqrt{m_ar_a}}{w_a}+\frac{\sqrt{m_br_b}}{w_b}+\frac{\sqrt{m_cr_c}}{w_c}\leq 1+\frac{R}{r}$$

$$\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\leq\sqrt{\frac{\sin A+\sin B+\sin C}{\sin 2A+\sin 2B+\sin 2C}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\because m_a\leq\frac{R}{2r}\cdot h_a=\frac{R}{2r}\cdot\frac{2rs}{a}=\frac{Rs}{a}$$

$$\therefore m_ar_a\leq\frac{Rs}{a}\cdot\frac{rs}{s-a}=\frac{Rs}{a}\cdot\frac{rs}{s(s-a)}\cdot\frac{sbc}{bc}=\frac{Rrs^3}{4Rrs}\cdot\frac{bc}{s(s-a)}=\frac{s^2}{4\cos^2\frac{A}{2}}$$

$$\Rightarrow \sqrt{m_a r_a} \le \frac{s}{2\cos\frac{A}{2}} \Rightarrow \frac{\sqrt{m_a r_a}}{w_a} \le \frac{s}{2\cos\frac{A}{2}} \cdot \frac{b+c}{2bc\cos\frac{A}{2}}$$

$$= \frac{s(b+c)}{4bc\cdot\frac{s(s-a)}{b}} = \frac{b+c}{4(s-a)} \therefore \frac{\sqrt{m_a r_a}}{w_a} \overset{(1)}{\le} \frac{b+c}{4(s-a)}. \textit{ Similarly, } \frac{\sqrt{m_a r_b}}{w_b} \overset{(2)}{\le} \frac{c+a}{4(s-b)} \textit{ and } \frac{\sqrt{m_c r_c}}{w_c} \overset{(3)}{\le} \frac{a+b}{4(s-c)}$$

$$(1)+(2)+(3) \Rightarrow \sum \frac{\sqrt{m_a r_a}}{w_a}$$

$$\le \frac{1}{4}\sum \frac{s+s-a}{s-a} = \frac{1}{4}\left(\frac{s}{r^2 s}\sum (s-b)(s-c)+3\right)$$

$$\frac{1}{4}\left(\frac{4Rr+r^2}{r^2}+3\right) = 1+\frac{R}{r} \therefore \sum \frac{\sqrt{m_a r_a}}{w_a} \le 1+\frac{R}{r}$$

$$\textit{Now, } \sum m_a \overset{Ioscu}{\ge} \sum \frac{b+c}{2}\cos\frac{A}{2} \overset{Bogdan\ Fustei}{\ge} \sum \sqrt{8Rr}\cos^2\frac{A}{2} = \sqrt{2Rr}\sum(1+\cos A) =$$

$$= \sqrt{2Rr}\left(3+1+\frac{r}{R}\right) = \sqrt{2Rr}\left(\frac{4R+r}{R}\right) = \sqrt{\frac{2r}{R}}\left(\sum r_a\right) \Rightarrow \frac{\sum r_a}{\sum m_a} \overset{(i)}{\le} \sqrt{\frac{R}{2r}}$$

$$\textit{Now, } \sqrt{\frac{\sum \sin A}{\sum \sin 2A}} = \sqrt{\frac{\frac{s}{R}}{4\left(\frac{abc}{8R^3}\right)}} = \sqrt{\frac{s}{R}\cdot\frac{8R^3}{16Rrs}} = \sqrt{\frac{R}{2r}} \overset{by\ (i)}{\ge} \frac{\sum r_a}{\sum m_a} \Rightarrow \frac{\sum r_a}{\sum m_a} \le \sqrt{\frac{R}{2r}} \quad \textit{(Proved)}$$

1309. In acute ΔABC the following relationship holds:

$$8(1-\sin A)(1-\sin B)(1-\sin C)+15\sqrt{3} \le 26$$

Proposed by Florentin Vișescu – Romania

Solution by Marian Ursărescu – Romania

$$\textit{We must show: } 8(1-\sin A)(1-\sin B)(1-\sin C) \le 26-15\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow (1-\sin A)(1-\sin B)(1-\sin C) \le \left(\frac{2-\sqrt{3}}{2}\right)^3 \Leftrightarrow$$

$$\ln(1-\sin A)+\ln(1-\sin B)+\ln(1-\sin C) \le 3\ln\left(\frac{2-\sqrt{3}}{2}\right) \quad (1)$$

$$A,B,C \in \left(0,\frac{\pi}{2}\right) \Rightarrow \sin A, \sin B, \sin C \in (0,1)$$

$$\textit{Let } f(x) = \ln(1-\sin x);\ f:\left(0,\frac{\pi}{2}\right) \to \mathbb{R}$$

$$f'(x) = \frac{\cos x}{\sin x - 1}, f''(x) = \frac{-\sin x(\sin x-1)-\cos x\cdot\cos x}{(\sin x-1)^2} =$$

$$=\frac{-\sin^2 x+\sin x-\cos^2 x}{(\cos x-1)^2}=\frac{\sin x-1}{(\cos x-1)^2}<0\Rightarrow \textit{from Jensen}$$

$$\Rightarrow\frac{f(A)+f(B)+f(C)}{3}\le f\left(\frac{A+B+C}{3}\right)\Leftrightarrow$$

$$\frac{\ln(1-\sin A)+\ln(1-\sin B)+\ln(1-\sin C)}{3}\le\ln\left(1-\sin\frac{\pi}{3}\right)\Leftrightarrow$$

$$\ln(1-\sin A)+\ln(1-\sin B)+\ln(1-\sin C)\le 3\ln\left(\frac{2-\sqrt{3}}{2}\right)\Rightarrow \textit{(1) it is true.}$$

1310. In ΔABC the following relationship holds:

$$w_b w_a^3+w_c w_b^3+w_a w_c^3\le\frac{243R^4}{16}$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$w_b w_a^3+w_c w_b^3+w_a w_c^3\overset{(1)}{\le}\frac{243R^4}{16}$$

$$\textit{LHS of (1)}=w_a w_b w_c\left(\frac{w_a^2}{w_c}+\frac{w_b^2}{w_a}+\frac{w_c^2}{w_b}\right)=\prod\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\left[\frac{w_a^2}{w_c}+\frac{w_b^2}{w_a}+\frac{w_c^2}{w_b}\right]$$

$$\le\frac{8\cdot 16R^2r^2s^2\left(\frac{s}{4R}\right)}{\prod(b+c)}\left(\frac{s(s-a)}{h_c}+\frac{s(s-b)}{h_a}+\frac{s(s-c)}{h_b}\right)$$

$$\left(\because w_a^2\le s(s-a)\textit{ and analogs and }\frac{1}{w_c}\le\frac{1}{h_c}\textit{ and analogs}\right)$$

$$=\frac{32Rr^2s^3}{2abc+\sum ab(2s-c)}\cdot\left[\frac{s(s-a)c+s(s-b)a+s(s-c)b}{2rs}\right]$$

$$=\frac{16Rrs^3}{2s(s^2+4Rr+r^2)-4Rrs}\cdot\left[s(2s)-(s^2+4Rr+r^2)\right]$$

$$=\frac{8Rrs^2(s^2-4Rr-r^2)}{s^2+2Rr+r^2}=\frac{4Rrs^2(\sum a^2)}{s^2+2Rr+r^2}\overset{Leibnitz}{\le}\frac{4Rrs^2(9R^2)}{s^2+2Rr+r^2}\therefore \textit{LHS of (1)}\overset{(i)}{\le}\frac{36R^3rs^2}{s^2+2Rr+r^2}$$

$$\textit{(i)}\Rightarrow \textit{it suffices to prove: }\frac{36R^3rs^2}{s^2+2Rr+r^2}\le\frac{243R^4}{16}\Leftrightarrow 27R(s^2+2Rr+r^2)\ge 64rs^2$$

$$\Leftrightarrow(27R-54r)s^2+27R(2Rr+r^2)\overset{(2)}{\ge}10rs^2$$

$$\textit{Now, LHS of (2)}\underset{(a)}{\overset{Gerreten}{\ge}}(27R-54r)(16Rr-5r^2)+27R(2R+r^2)\textit{ and RHS of (2)}$$

$$\overset{Gerreten}{\underset{(b)}{\leq}} 10r(4R^2+4Rr+3r^2)$$

(a), (b) ⇒ in order to prove (2), it suffices to prove:

$$(27R-54r)(16R-5r)+27R(2R+r)-10(4R^2+4Rr+3r^2)\geq 0$$

$$\Leftrightarrow 223R^2-506Rr+120r^2\geq 0 \Leftrightarrow (R-2r)(223R-60r)\geq 0 \to true \because R \overset{Euler}{\geq} 2r$$

⇒ (2) ⇒ (1) is true (Proved)

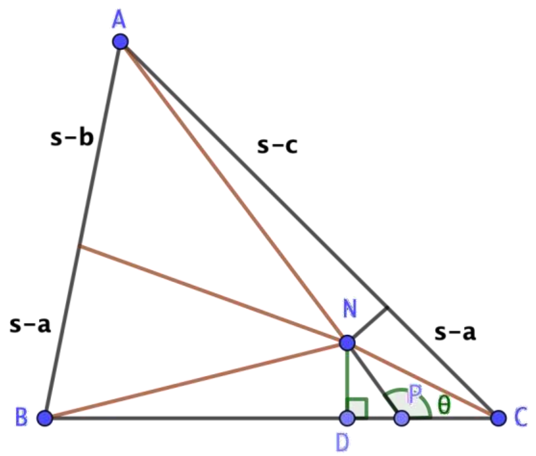
1311. In $\Delta ABC, N$ – Nagel's point, $ND\perp BC, NE\perp AC, NF\perp AB, D\in(BC)$, $E\in(CA), F\in(AB)$. Prove that:

$$\frac{r_a}{ND}+\frac{r_b}{NE}+\frac{r_c}{NF}\geq\left(\frac{3R}{2r}\right)^2\geq 9$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{ND}+\frac{r_b}{NE}+\frac{r_c}{NF}\overset{(i)}{\geq}\left(\frac{3R}{2r}\right)^2\geq 9$$



$$\text{Van Aubel's theorem} \Rightarrow \frac{AN}{n_a-AN}=\frac{s-c}{s-a}+\frac{s-b}{s-a}=\frac{a}{s-a}$$

$$\Rightarrow\frac{n_a-AN}{AN}=\frac{s-a}{a}\Rightarrow\frac{n_a}{AN}=\frac{s-a+a}{a}=\frac{s}{a}\Rightarrow\frac{AN}{n_a}\overset{(1)}{=}\frac{a}{s}$$

$$\text{Sine – rule on } \Delta APC \Rightarrow \frac{b}{\sin\theta}=\frac{n_a}{\sin C}\Rightarrow \sin(180^\circ-\theta)=\frac{bc}{2Rn_a}$$

$$\Rightarrow\frac{ND}{NP}=\frac{bc}{2Rn_a}\ (\text{using } \Delta NDP)$$

$$\Rightarrow \frac{ND}{n_a - AN} = \frac{bc}{2Rn_a} \Rightarrow \frac{n_a - AN}{n_a} = \frac{2R \cdot ND \cdot a}{abc} \Rightarrow 1 - \frac{AN}{n_a} = \frac{2R \cdot ND \cdot a}{4Rrs}$$

$$\overset{by\ (1)}{\Rightarrow} 1 - \frac{a}{s} = \frac{a \cdot ND}{2rs} \Rightarrow \frac{s-a}{s} = \frac{a \cdot ND}{2rs} \Rightarrow ND \overset{(a)}{=} 2r\left(\frac{s-a}{a}\right)$$

$$\text{Similarly, } NE \overset{(b)}{=} 2r\left(\frac{s-b}{b}\right) \text{ and } NF \overset{(c)}{=} 2r\left(\frac{s-c}{c}\right)$$

$$(a), (b), (c) \Rightarrow \text{LHS of } (i) = \sum \frac{r_a a}{2r(s-a)} = \frac{1}{2r}\sum \frac{r_a(a-s+s)}{s-a} = \frac{1}{2r}\sum\left(-r_a + \frac{r_a}{r} \cdot \frac{rs}{s-a}\right)$$

$$= \frac{1}{2r}\sum\left(-r_a + \frac{1}{r}r_a^2\right) = \frac{(4R+r)^2 - 2s^2}{2r^2} - \frac{4R+r}{2r}\left(\because \sum r_a^2 = (4R+r)^2 - 2s^2\right)$$

$$= \frac{(4R+r)^2 - 2s^2 - r(4R+r)}{2r^2} = \frac{8R^2 + 2Rr - s^2}{r^2} \geq \frac{9R^2}{4r^2}$$

$$\Leftrightarrow 32R^2 + 8Rr - 4s^2 \geq 9R^2 \Leftrightarrow 4s^2 \overset{(ii)}{\leq} 23R^2 + 8Rr$$

$$\text{Now, LHS of (ii)} \overset{Gerretsen}{\leq} 16R^2 + 16Rr + 12r^2 \overset{?}{\leq} 23R^2 + 8Rr$$

$$\Leftrightarrow 7R^2 - 8Rr - 12r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(7R+6r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \Rightarrow (ii) \Rightarrow (i) \text{ is true and } \because R \overset{Euler}{\geq} 2r \therefore \left(\frac{3R}{2r}\right)^2 \geq 9 \text{ and thus the proposed chain}$$

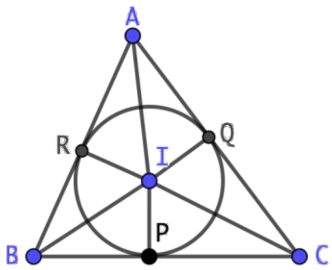
of inequalities is true (Proved)

1312. If in $\Delta ABC, I$ – incenter then:

$$[AIB] \cdot [AIC] + [BIC] \cdot [BIA] + [CIA] \cdot [CIB] \leq r^2(R+r)^2$$

Proposed by Marian Ursărescu – Romania

Solution by Avishek Mitra-West Bengal-India



$$\text{In } \Delta ABC \Rightarrow IP = IQ = IR = r$$

$$AB = a, BC = b, AC = c, [AIB] = [BIA] = \frac{1}{2}r_a, [BIC] = [CIB] = \frac{1}{2}r_b$$

$$[AIC]=[CIA]=\frac{1}{2}r_c$$

$$\therefore\ \Omega=[AIB]\cdot[AIC]+[BIC]\cdot[BIA]+[CIA]\cdot[CIB]$$

$$=\frac{1}{4}r^2(ab+bc+ca)=\frac{1}{4}r^2(s^2+r^2+4Rr)$$

$$\textit{Need to show}\Rightarrow\tfrac{1}{4}r^2(s^2+r^2+4Rr)\le r^2(R+r)^2$$

$$\Rightarrow\frac{s^2}{4}+\frac{r^2}{4}+Rr\le R^2+2Rr+r^2\Rightarrow\frac{s^2}{4}\le R^2+Rr+\frac{3r^2}{4}$$

$$\Rightarrow s^2\le 4R^2+4Rr+3r^2\ \ (\text{* True Gerretsen's Inequality})$$

1313. In ΔABC the following relationship holds:

$$m_a\ge\frac{1}{2\sqrt{2}}\left((b+c)\cos\frac{A}{2}+|b-c|\sin\frac{A}{2}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a\overset{(1)}{\ge}\frac{1}{2\sqrt{2}}\left((b+c)cos\frac{A}{2}+|b-c|sin\frac{A}{2}\right)$$

Upon squaring both sides, (1)

$$\Leftrightarrow 8m_a^2\ge(b+c)^2\left(cos\frac{A}{2}\right)^2+(b-c)^2\left(\sin\frac{A}{2}\right)^2+2(b+c)|b-c|\cos\frac{A}{2}\sin\frac{A}{2}$$

$$\Leftrightarrow 8m_a^2\ge(b-c)^2\left(\left(cos\frac{A}{2}\right)^2+\left(sin\frac{A}{2}\right)^2\right)+4bc\left(cos\frac{A}{2}\right)^2+\left(\frac{a}{2R}\right)(b+c)|b-c|\Leftrightarrow$$

$$\Leftrightarrow 8m_a^2\ge(b-c)^2+\frac{4bcs(s-a)}{bc}+\left(\frac{a}{2R}\right)(b+c)|b-c|$$

$$\Leftrightarrow 8m_a^2\ge(b-c)^2+(b+c+a)(b+c-a)+\left(\frac{a}{2R}\right)(b+c)|b-c|\Leftrightarrow 8m_a^2$$

$$\ge(b-c)^2+(b+c)^2-a^2+\left(\frac{a}{2R}\right)(b+c)|b-c|$$

$$\Leftrightarrow 8m_a^2\ge 4m_a^2+\left(\frac{a}{2R}\right)(b+c)|b-c|\Leftrightarrow 8Rm_a^2\ge a(b+c)|b-c|\Leftrightarrow$$

$$\left(\frac{2abc}{4\Delta}\right)4m_a^2\ge a(b+c)|b-c|\Leftrightarrow 4a^2b^2c^2(2b^2+2c^2-a^2)^2\ge$$

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)a^2(b^2-c^2)^2$$

$$\Leftrightarrow 4b^2c^2(2b^2+2c^2-a^2)^2 \geq \left(2\sum a^2b^2-\sum a^4\right)(b^2-c^2)^2$$

$$\Leftrightarrow a^4(b^2+c^2)^2-2a^2(b^6+c^6)-14a^2b^2c^2(b^2+c^2)+(b^2+c^2)^4+8b^2c^2(b^2+c^2)^2+16b^4c^4 \geq 0 \text{ (expanding and re-arranging)}$$

$$\Leftrightarrow \{a^4(b^2+c^2)^2+16b^4c^4-8a^2b^2c^2(b^2+c^2)\}-$$

$$-6a^2b^2c^2(b^2+c^2)+(b^2+c^2)^4+8b^2c^2(b^2+c^2)^2-$$

$$-2a^2(b^2+c^2)(b^4+c^4-b^2c^2) \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2)-4b^2c^2\}^2-6a^2b^2c^2(b^2+c^2)+(b^2+c^2)^4+8b^2c^2(b^2+c^2)^2-$$

$$-2a^2(b^2+c^2)\{(b^2+c^2)^2-3b^2c^2\} \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2)-4b^2c^2\}^2+(b^2+c^2)^4+8b^2c^2(b^2+c^2)^2-2a^2(b^2+c^2)^3 \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2)-4b^2c^2\}^2+(b^2+c^2)^4-2(b^2+c^2)^2\{a^2(b^2+c^2)-4b^2c^2\} \geq 0 \Leftrightarrow [\{a^2(b^2+c^2)-4b^2c^2\}-(b^2+c^2)^2]^2 \geq 0 \to \textit{true}$$

$\Rightarrow$ *(1) is true (Proved)*

1314. In ΔABC the following relationship holds:

$$\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+\frac{8r}{R} \geq 7$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y, s-c=z$$

$$\therefore 3s-2s=s=\sum x \Rightarrow a=y+z, b=z+x, c=x+y$$

$$\text{Now, proposed inequality} \Leftrightarrow \sum\left(\frac{s-b}{s-a}\right)^2+8\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right) \overset{\text{via above transformation}}{\Leftrightarrow}$$

$$\sum\frac{y^2}{x^2}+\frac{32s(s-a)(s-b)(s-c)}{s\prod(x+y)} \geq 7 \Leftrightarrow \sum\frac{y^2}{x^2}+\frac{32xyz}{\prod(x+y)} \geq 7$$

$$\Leftrightarrow \sum\frac{y^2}{x^2}+3+\frac{32xyz}{\prod(x+y)} \geq 10 \Leftrightarrow \sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)} \overset{(1)}{\geq} 10$$

$$\sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)} =$$

$$=\sum\frac{y^2+x^2}{x^2}+\frac{16xyz}{\prod(x+y)}+\frac{16xyz}{\prod(x+y)}\overset{A-G}{\geq}5\sqrt[5]{\frac{2^8(xyz)^2}{\prod x^2}\frac{\prod(x^2+y^2)}{\prod(x+y)^2}}\geq$$

$$\geq 5\sqrt[5]{2^8\frac{\prod\left(\frac{1}{2}(x+y)^2\right)}{\prod(x+y)^2}}=5\sqrt[5]{2^5}=10$$

$$\Rightarrow (1)\ is\ true\Rightarrow\ proposed\ inequality\ is\ true\ (Proved)$$

1315. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$R\geq\frac{r}{\sum_{cyc}h_a}\left(5R-r+\sum_{cyc}n_a\right)\geq\frac{r}{\sum_{cyc}h_a}\left(\sum_{cyc}(r_a+n_a)\right)\geq 2r$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$R\overset{(m)}{\geq}\frac{r}{\sum h_a}\left(5R-r+\sum n_a\right)\overset{(n)}{\geq}\frac{r}{\sum h_a}\left(\sum(r_a+n_a)\right)\overset{(p)}{\geq}2r$$

$$(m)\Leftrightarrow\sum h_a\geq r\left(5-\frac{r}{R}+\frac{\sum n_a}{R}\right)\Leftrightarrow\sum\left(\frac{2rs}{a}\right)\geq r\left(6-\left(1+\frac{r}{R}\right)+\frac{\sum n_a}{R}\right)$$

$$\Leftrightarrow\sum\left(\frac{a+b+c}{a}\right)\geq 6-\sum\cos A+\frac{\sum n_a}{R}$$

$$\Leftrightarrow\sum\left(1+\frac{b+c}{a}\right)\geq 6-\sum\left(1-2\sin^2\frac{A}{2}\right)+\frac{\sum n_a}{R}$$

$$\Leftrightarrow 3+\sum\frac{b+c}{a}\geq 3+2\sum\sin^2\frac{A}{2}+\frac{\sum n_a}{R}$$

$$\Leftrightarrow\sum\left(\frac{b}{c}+\frac{c}{b}\right)\geq\sum\left[\left(\frac{2(s-b)(s-c)}{bc}\right)+\frac{n_a}{R}\right]$$

$$\Leftrightarrow\sum\left[\frac{b^2+c^2}{bc}-\frac{2(s-b)(s-c)}{bc}\right]\geq\sum\frac{n_a}{R}$$

$$\Leftrightarrow\sum\left[\frac{2(b^2+c^2)-4(s-b)(s-c)}{2bc}\right]\geq\sum\frac{n_a}{R}$$

$$\Leftrightarrow\sum\left[\frac{2(b^2+c^2)-(a+b-c)(c+a-b)}{2bc}\right]\geq\sum\frac{n_a}{R}$$

$$\Leftrightarrow \sum\left[\frac{(2(b^2+c^2)-a^2)+(b-c)^2}{2bc}\right] \geq \sum \frac{n_a}{R} \Leftrightarrow \sum\left[\frac{4m_a^2+(b-c)^2}{2bc}\right] \overset{(i)}{\geq} \sum \frac{n_a}{R}$$

Now, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) \overset{(1)}{=} an_a^2+a(s-b)(s-c)$ *and,*

$$b^2(s-b)+c^2(s-c) \overset{(2)}{=} ag_a^2+a(s-b)(s-c)$$

$$(1)+(2) \Rightarrow (b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2) \Rightarrow 4m_a^2+(b-c)^2 \overset{(3)}{=} 2(n_a^2+g_a^2)$$

$$(3) \Rightarrow (i) \Leftrightarrow \sum\left(\frac{n_a^2+g_a^2}{bc}\right) \overset{(ii)}{\geq} \sum \frac{n_a}{R}$$

$$\text{Now, } \frac{bcn_a}{R}=2h_an_a \leq 2g_an_a \leq n_a^2+g_a^2 \Rightarrow \frac{n_a^2+g_a^2}{bc} \overset{(a)}{\geq} \frac{n_a}{R}$$

$$\text{Similarly, } \frac{n_b^2+g_b^2}{ca} \overset{(b)}{\geq} \frac{n_b}{R} \text{ and, } \frac{n_c^2+g_c^2}{ab} \overset{(c)}{\geq} \frac{n_c}{R}$$

(a)+(b)+(c)⇒(ii)⇒(i)⇒ (m) is true

$$\text{Now, } \frac{r}{\sum h_a}\left(5R-r+\sum n_a\right)=\frac{r}{\sum h_a}\left(4R+r+(R-2r)+\sum n_a\right)$$

$$\overset{Euler}{\geq} \frac{r}{\sum h_a}\left(4R+r+\sum n_a\right)=\frac{r}{\sum h_a}\left(\sum r_a+\sum n_a\right)=\frac{r}{\sum h_a}\left(\sum(r_a+n_a)\right) \Rightarrow (n) \text{ is true}$$

$$\text{Again, } \frac{r}{\sum h_a}\left(\sum(r_a+n_a)\right)=\frac{r}{\sum h_a}\left(\sum r_a+\sum n_a\right) \geq \frac{r}{\sum h_a}\left(\sum h_a+\sum h_a\right)=2r$$

⇒ (p) is true.

1316. In ΔABC the following relationship holds:

$$\frac{64}{81r^2} \leq \frac{a^2+b^2+c^2}{3S^2}+\frac{1}{(m_a+m_b+m_c)^2}+\sum_{cyc}\frac{1}{(m_a+m_b+m_c)^2} \leq \frac{R^6}{81r^8}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{64}{81r^2} \overset{(1)}{\lesssim} \frac{\sum a^2}{3S^2}+\frac{1}{(\sum m_a)^2}+\sum\frac{1}{(m_b+m_c-m_a)^2} \overset{(2)}{\lesssim} \frac{R^6}{81r^8}$$

Firstly, for simplicity, let $\Delta = S(= rs)$

Secondly, let ΔPQR *have sides* $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$. *Then, its medians =*

$\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ *respectively and its area* $= \frac{\Delta}{3}$

$$Now, \frac{64}{81}\left(\frac{\sum m_a}{\Delta}\right)^2 \overset{(i)}{\leq} \frac{4\sum {m_a}^2}{3\Delta^2} + \frac{4}{(\sum a)^2} + \sum \frac{4}{(b+c-a)^2}$$

$$\Leftrightarrow 64\left(\sum m_a\right)^2 \leq 108\left(\sum m_a^2\right) + 81r^2 + 81\sum\left(\frac{\Delta}{s-a}\right)^2 \Leftrightarrow$$

$$64\left(\sum m_a\right)^2 \leq 81\sum a^2 + 81r^2 + 81\sum r_a^2$$

$$\Leftrightarrow 64\left(\sum m_a\right)^2 \leq 162(s^2 - 4Rr - r^2) + 81r^2 + 81(4R + r)^2 - 162s^2$$

$$\Leftrightarrow 64\left(\sum m_a\right)^2 \overset{(ii)}{\leq} 81(4R + r)^2 + 81r^2 - 162(4Rr + r^2)$$

$$Now, 64\left(\sum m_a\right)^2 \leq 64(4R + r)^2 \overset{?}{\leq} 81(4R + r)^2 + 81r^2 - 162(4Rr + r^2)$$

$$\Leftrightarrow 17(4R + r)^2 + 81r^2 - 162(4Rr + r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(17R + 2r) \overset{?}{\geq} 0 \to true \because R \overset{Euler}{\geq} 2r \Rightarrow (ii) \Rightarrow (i) \text{ is true}$$

$$Now, applying\ (i)\ on\ \Delta PQR, we\ get: \frac{64}{81}\left(\frac{\frac{\sum a}{2}}{\frac{\Delta}{3}}\right)^2 \leq$$

$$\leq \frac{\left(\frac{4\sum a^2}{4}\right)}{\left(\frac{3\Delta^2}{9}\right)} + \frac{4}{\frac{4(\sum m_a)^2}{9}} + \sum \frac{4}{\frac{4(m_b + m_c - m_a)^2}{9}}$$

$$\Leftrightarrow \frac{64}{81}\left(\frac{9s^2}{r^2s^2}\right) \leq \frac{3\sum a^2}{\Delta^2} + \frac{9}{(\sum m_a)^2} + \sum \frac{9}{(m_b + m_c - m_a)^2} \Rightarrow \frac{64}{81r^2}$$

$$\leq \frac{\sum a^2}{3S^2} + \frac{1}{(\sum m_a)^2} + \sum \frac{1}{(m_b + m_c - m_a)^2} \Rightarrow \boxed{(1)\ is\ true}$$

$$Now, \frac{m_a m_b m_c}{\Delta} \overset{(iii)}{\leq} \frac{1}{\Delta^6}\left(\frac{abc}{8}\right)^3\left(\frac{\sum a}{2}\right)^4 \Leftrightarrow \prod m_a \leq \frac{s^4\left(\frac{4R\Delta}{8}\right)^3}{\Delta^5} = \frac{R^3s^4}{8r^2s^2} \Leftrightarrow$$

$$\prod m_a \overset{(iv)}{\leq} \frac{R^3s^2}{8r^2}$$

$$\because m_a \le \frac{Rh_a}{2r} = \frac{Rs}{a} \text{ etc } \therefore \prod m_a \le \frac{R^3s^3}{abc} = \frac{R^2s^2}{4r} \overset{?}{\leq} \frac{R^3s^2}{8r^2} \Leftrightarrow R \overset{?}{\geq} 2r \to true(Euler) \Rightarrow$$

$(iv) \Rightarrow (iii)$ *is true. Now, applying* (iii) *on* ΔPQR, *we get* :

$$\frac{\left(\frac{abc}{8}\right)}{\left(\frac{\Delta}{3}\right)} \le \frac{3^6}{\Delta^6}\left(\frac{\frac{8m_am_bm_c}{27}}{8}\right)^3\left(\frac{\left(\frac{2\sum m_a}{3}\right)}{2}\right)^4 \Rightarrow \frac{\left(\frac{4Rrs}{8}\right)}{\left(\frac{rs}{3}\right)} \le \frac{(\prod m_a)^6(\sum m_a)^4}{3^7\Delta^6}$$

$$\Rightarrow R \le \frac{2(\prod m_a)^3(\sum m_a)^4}{3^8\Delta^6} \Rightarrow 16R^2 \le \frac{64(\prod m_a)^6(\sum m_a)^8}{9^8\Delta^{12}}$$

$$\Rightarrow 2(s^2 - 4Rr - r^2) + r^2 + (4R + r)^2 - 2s^2 \le \frac{64(\prod m_a)^6(\sum m_a)^8}{9^8\Delta^{12}}$$

$$\Rightarrow \sum a^2 + r^2 + \sum r_a^2 \le \frac{64(\prod m_a)^6(\sum m_a)^8}{9^8\Delta^{12}} \Rightarrow$$

$$\Rightarrow \frac{\sum a^2}{\Delta^2} + \frac{1}{s^2} + \left(\frac{\Delta^2}{\Delta^2}\right)\sum\frac{4}{(b + c - a)^2} \le \frac{64(\prod m_a)^6(\sum m_a)^8}{9^8\Delta^{14}}$$

$$\Rightarrow \frac{\frac{4}{3}\sum {m_a}^2}{\Delta^2} + \frac{4}{(\sum a)^2} + \sum\frac{4}{(b + c - a)^2} \overset{(v)}{\leq} \frac{64(\prod m_a)^6(\sum m_a)^8}{9^8\Delta^{14}}$$

Now, applying (v) *on* ΔPQR, *we get* :

$$\frac{\frac{4}{12}\sum a^2}{\frac{\Delta^2}{9}} + \frac{4}{\frac{4(\sum m_a)^2}{9}} + \sum\frac{4}{\frac{4(m_b + m_c - m_a)^2}{9}} \le \frac{64\left(\frac{abc}{8}\right)^6\left(\frac{\sum a}{2}\right)^8}{9^8\left(\frac{\Delta}{3}\right)^{14}}$$

$$\Rightarrow \frac{\sum a^2}{3\Delta^2} + \frac{1}{(\sum m_a)^2} + \sum\frac{1}{(m_b + m_c - m_a)^2} \le \frac{64s^8}{81\Delta^{14}}\left(\frac{4R\Delta}{8}\right)^6 = \frac{R^6}{81r^8}$$

$$\Rightarrow \frac{\sum a^2}{3S^2} + \frac{1}{(\sum m_a)^2} + \sum\frac{1}{(m_b + m_c - m_a)^2} \le \frac{R^6}{81r^8}$$

$$\Rightarrow \boxed{(2) \text{ is true}} \text{ (Proved)}$$

1317. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} m_a\right)\left(\sum_{cyc}\sqrt{\frac{s^2 - r_br_c}{h_bh_c}}\right) \ge 3\sqrt{2}(r_a + r_b + r_c)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{b+c}{2}\ge\sqrt{2r(r_b+r_c)}\Leftrightarrow\left(\frac{b+c}{2}\right)^2\ge 2r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=\frac{2(s-a)(s-b)(s-c)(2s-b-c)}{(s-b)(s-c)}=a(b+c-a)\Leftrightarrow$$

$$(b+c-a+a)^2\ge 4a(b+c-a)$$

$$\to true\because (x+a)^2\ge 4xa, where\ x=b+c-a\Rightarrow\frac{b+c}{2}\overset{(1)}{\ge}\sqrt{2r(r_b+r_c)}$$

$$Now, r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\cos\frac{A}{2}\sin\left(\frac{B+C}{2}\right)}{\prod\cos\frac{A}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(2)}{=}4R\cos^2\frac{A}{2}, (1),(2)\Rightarrow\frac{b+c}{2}\overset{(3)}{\ge}\sqrt{8Rr}\cos\frac{A}{2}\ and\ analogs$$

$$Now,\sum m_a\overset{Ioscu}{\ge}\sum\frac{b+c}{2}\cos\frac{A}{2}\overset{by\ (3) and\ its\ analogs}{\ge}\sqrt{2Rr}\sum\left(2\cos^2\frac{A}{2}\right)=\sqrt{2Rr}\sum(1+\cos A)=$$

$$=\sqrt{2Rr}\left(3+1+\frac{r}{R}\right)\Rightarrow\sum m_a\ge\frac{\sqrt{2Rr}}{R}(4R+r)\Rightarrow\sum m_a\overset{(4)}{\ge}\sqrt{\frac{2r}{R}}\left(\sum r_a\right)$$

$$Again,\sum\sqrt{\frac{s^2-r_br_c}{h_bh_c}}=\sum\sqrt{\frac{s^2-\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}}{\frac{4r^2s^2}{bc}}}=\sum\sqrt{\frac{sabc}{4r^2s^2}}=\sum\sqrt{\frac{4Rrs^2}{4r^2s^2}}=$$

$$=3\sqrt{\frac{R}{r}}\Rightarrow\sum\sqrt{\frac{s^2-r_br_c}{h_bh_c}}\overset{(5)}{\ge}3\sqrt{\frac{R}{r}}$$

$$(4)\ and\ (5)\Rightarrow\left(\sum m_a\right)\left(\sum\sqrt{\frac{s^2-r_br_c}{h_bh_c}}\right)\ge 3\sqrt{2}\left(\sum r_a\right)\ (Proved)$$

1318. In acute ΔABC, H – orthocenter, the following relationship holds:

$$(A^2+B^2+C^2)\left(\frac{a^5}{AH}+\frac{b^5}{BH}+\frac{c^5}{CH}\right)\ge\frac{32\pi^2s^5}{243R}$$

Proposed by Radu Diaconu-Romania

Solution by Șerban George Florin-Romania

$$H^2=(A+B+C)^2\overset{C.B.S.}{\le}3(A^2+B^2+C^2)\Rightarrow\sum A^2\ge\frac{H^2}{3}$$

$$\sum\frac{a^5}{AH}\overset{Holder}{\ge}\frac{(a+b+c)^5}{3^{5-2}\sum AH}=\frac{2^5s^5}{27\sum AH}$$

$$\Rightarrow\left(\sum A^2\right)\cdot\left(\sum\frac{a^5}{AH}\right)\ge\frac{H^2}{3}\cdot\frac{32s^5}{27\cdot\sum AH}=\frac{32s^5H^2}{81\sum AH}$$

$$\sum AH=\sum 2R\cos A=2R\sum\cos A=2R\left(1+\frac{r}{R}\right)\le 2R\left(1+\frac{1}{2}\right)=3R$$

$$\Rightarrow\sum AH\le 3R\Rightarrow\left(\sum A^2\right)\left(\sum\frac{a^5}{AH}\right)\ge\frac{321^5H^2}{81\sum AH}\ge\frac{32s^5H^2}{81\cdot 3R}=\frac{32s^5H^2}{243R}\ \textit{(true)}$$

1319. In ΔABC the following relationship holds:

$$(w_a+w_b+w_c)\left(\frac{A}{b+c}+\frac{B}{c+a}+\frac{C}{a+b}\right)\ge\frac{27\pi r}{4s}$$

Proposed by Radu Diaconu – Romania

Solution 1 by Șerban George Florin – Romania

$$WLOG:\ \ A\le B\le C\Rightarrow a\le b\le c\Rightarrow b+c\ge a+c\ge a+b$$

$$\Rightarrow\frac{1}{b+c}\le\frac{1}{a+c}\le\frac{1}{a+b}.\ \textit{Applying Cebyshev's inequality}$$

$$\sum\frac{A}{b+c}\ge\frac{\left(\sum A\right)\left(\sum\frac{1}{b+c}\right)}{3}=\frac{\pi\cdot\sum\frac{1}{b+c}}{3}$$

$$\sum\frac{1}{b+c}\ge\sum\frac{1^2}{b+c}\overset{Bergstrom}{\ge}\frac{(1+1+1)^2}{\sum(b+c)}=\frac{9}{4s}\Rightarrow\sum\frac{A}{b+c}\ge\frac{\pi\cdot\frac{9}{4s}}{3}=\frac{3\pi}{4s}$$

$$\textit{Applying the following inequality: } w_a+w_b+w_c\ge 9r$$

$$\Rightarrow\left(\sum w_a\right)\left(\sum\frac{A}{b+c}\right)\ge 9r\cdot\frac{3\pi}{4s}=\frac{27\pi r}{4s}\ \textit{true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{A}{b+c}=\sum\frac{A^2}{A(b+c)}\overset{Bergstrom}{\ge}\frac{\left(\sum A\right)^2}{\sum A(b+c)}\therefore\sum\frac{A}{b+c}\overset{(1)}{\ge}\frac{\pi^2}{\sum A(b+c)}$$

$$Now, WLOG, we\ may\ assume$$

$$a\ge b\ge c\therefore A\ge B\ge C\ and\ (b+c)\le(c+a)\le(a+b)$$

$$\therefore \text{via Chebyshev and using } (1), \text{we get}: \sum \frac{A}{b+c} \geq \frac{\pi^2}{\frac{1}{3}(\sum A)(\sum(b+c))} = \frac{3\pi^2}{4s\pi} = \frac{3\pi}{4s}$$

$$\therefore \sum w_a\left(\sum \frac{A}{b+c}\right) \geq \sum w_a\left(\frac{3\pi}{4s}\right) \overset{?}{\geq} \frac{27\pi r}{4s} \Leftrightarrow \sum w_a \underset{(2)}{\overset{?}{\geq}} 9r$$

$$\text{Now}, \sum w_a \geq \sum h_a \overset{?}{\geq} 9r \Leftrightarrow \frac{\sum ab}{2R} \overset{?}{\geq} 9r \Leftrightarrow s^2 + 4Rr + r^2 \overset{?}{\geq} 18Rr \Leftrightarrow s^2 \underset{(3)}{\overset{?}{\geq}} 14Rr - r^2$$

$$\text{Now}, s^2 \overset{Gerretsen}{\geq} 16Rr - 5r^2 = 14Rr - r^2 + 2r(R-2r) \overset{Euler}{\geq} 14Rr - r^2$$

$$\Rightarrow (3) \Rightarrow (2) \Rightarrow \text{proposed inequality is true (Proved)}$$

1320. In ΔABC the following relationship holds:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{4r}{R} \geq 5$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y, s-c=z$$

$$\therefore 3s - \sum a = 3s - 2s = s = \sum x \Rightarrow a = y+z, b = z+x, c = x+y$$

$$\text{Now}, \sum \frac{r_a}{r_b} + \frac{4r}{R} = \sum \frac{s-b}{s-a} + 4\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right) = \sum \frac{y}{x} + \frac{16s(s-a)(s-b)(s-c)}{sabc}$$

$$= \frac{\sum x^2 y}{xyz} + \frac{16xyz}{\prod(x+y)} = \frac{(\prod(x+y))(\sum x^2 y) + 16x^2y^2z^2}{xyz \cdot \prod(x+y)} \geq 5$$

$$\Leftrightarrow \left(\sum x^2 y\right) \cdot \prod(x+y) + 16x^2y^2z^2 \geq 5xyz \cdot \prod(x+y)$$

$$\Leftrightarrow \sum x^4y^2 + \sum x^3y^3 + xyz\left(\sum x^3\right) + 9x^2y^2z^2 \overset{(1)}{\geq} 3xyz\left(\sum x^2y + \sum xy^2\right)$$

$$\text{Now}, xyz\left(\sum x^3\right) + 3x^2y^2z^2 \underset{(i)}{\overset{Schur}{\geq}} xyz\left(\sum x^2y + \sum xy^2\right)$$

$$\text{Also}, \sum x^3y^3 + 3x^2y^2z^2 \underset{(ii)}{\overset{Schur}{\geq}} xyz\left(\sum x^2y + \sum xy^2\right)$$

$$\text{Again, as}, 4+2 = 3+3 \text{ and } 4 > 3, \therefore (4,2) \succ (3,3)$$

$$\Rightarrow \sum x^4y^2 \overset{Muirhead}{\geq} \sum x^3y^3 \Rightarrow \sum x^4y^2 + 3x^2y^2z^2$$

$$\geq \sum x^3y^3 + 3x^2y^2z^2 \overset{Schur}{\geq} xyz\left(\sum x^2y + \sum xy^2\right)$$

$$\therefore \sum x^4y^2 + 3x^2y^2z^2 \overset{(iii)}{\geq} xyz\left(\sum x^2y + \sum xy^2\right)$$

(i)+(ii)+(iii) $\Rightarrow$ *(1) is true* $\therefore \sum \frac{r_a}{r_b} + \frac{4r}{R} \geq 5$ *(Proved)*

1321. In ΔABC the following relationship holds:

$$\frac{m_a\sqrt{h_a}}{w_a} + \frac{m_b\sqrt{h_b}}{w_b} + \frac{m_c\sqrt{h_c}}{w_c} \geq s\sqrt{\frac{2}{R}}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{m_a\sqrt{h_a}}{w_a} = \sum \frac{m_a\sqrt{\frac{bc}{2R}}(b+c)}{2bc\sqrt{\frac{s(s-a)}{bc}}} = \frac{1}{2\sqrt{2R}}\sum\left(\frac{m_a}{\sqrt{s(s-a)}}\cdot(b+c)\right)$$

$$\geq \frac{1}{2\sqrt{2R}}\sum(b+c) \ \ (\because m_a \geq \sqrt{s(s-a)} \text{ and analogs})$$

$$= \frac{4s}{2\sqrt{2R}} = s\sqrt{\frac{2}{R}} \ \ \text{(Proved)}$$

1322. In $\Delta ABC, n_a$ – Nagel's cevian then the following relationship holds:

$$2 + \frac{a^2+b^2+c^2}{4r^2} \geq \frac{4R}{r} + \sum_{cyc} \frac{n_an_b}{h_ah_b}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc)$$

$$\Rightarrow s(b^2+c^2) - 2sbc = an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)}{bc} = as^2 - \frac{4s(s-b)(s-c)(s-a)}{s-a} = as^2 - \frac{4r^2s^2}{s-a}$$

$$\Rightarrow a^2n_a^2 \overset{(a)}{=} a^2s^2 - 4r^2s^2\left(\frac{a}{s-a}\right)$$

Similarly, $b^2n_b^2 \overset{(b)}{=} b^2s^2 - 4r^2s^2\left(\frac{b}{s-b}\right)$ *and,* $c^2n_c^2 \overset{(c)}{=} c^2s^2 - 4r^2s^2\left(\frac{c}{s-c}\right)$

Now, $\sum\frac{n_an_b}{h_ah_b} = \sum\frac{an_a\cdot bn_b}{4r^2s^2} = \frac{\sum an_a\cdot bn_b}{4r^2s^2} \le \frac{\sum a^2n_a^2}{4r^2s^2}\ (\because xy+yz+zx \le \sum x^2)$

$$\overset{by\ (a)+(b)+(c)}{=} \left(\frac{1}{4r^2s^2}\right)\sum\left(a^2s^2 - 4r^2s^2\left(\frac{a}{s-a}\right)\right) = \frac{s^2\sum a^2}{4r^2s^2} - \sum\left(\frac{a-s+s}{s-a}\right)$$

$$= \frac{\sum a^2}{4r^2} - \left(-3 + \frac{s\sum(s-b)(s-c)}{r^2s}\right) = \frac{\sum a^2}{4r^2} + 3 - \frac{\sum(s^2 - s(b+c) + bc)}{r^2}$$

$$= \frac{\sum a^2}{4r^2} + 3 - \frac{3s^2 - 4s^2 + s^2 + 4Rr + r^2}{r^2} = \frac{\sum a^2}{4r^2} + 3 - \frac{4R+r}{r}$$

$$= \frac{\sum a^2}{4r^2} + 2 - \frac{4R}{r} \Rightarrow 2 + \frac{\sum a^2}{4r^2} \ge \frac{4R}{r} + \sum\frac{n_an_b}{h_ah_b}\ \ (Proved)$$

1323. In $\Delta ABC, I -$ incenter, n_a – Nagel's cevian the following relationship holds:

$$\frac{n_a+r_a}{AI} + \frac{n_b+r_b}{BI} + \frac{n_c+r_c}{CI} \le \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)\left(1 + \frac{4R}{r}\right)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc)$$

$$\Rightarrow s(b^2+c^2) - 2sbc = an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_an_a \Rightarrow n_a^2 = s^2 - 2h_an_a$$

$$\Rightarrow n_a^2 = s^2 - \frac{4rs^2\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}\left(\because n_a = s\tan\frac{A}{2}\ and\ a = 4R\tan\frac{A}{2}\cos^2\frac{A}{2}\right)$$

$$\Rightarrow n_a^2 \overset{(1)}{=} s^2 - \frac{rs^2}{R\cos^2\frac{A}{2}}. \text{ Now, } \frac{n_a+r_a}{AI} + \sqrt{\frac{r}{R}}\left(\frac{r_a}{r}\right)$$

$$= \frac{1}{r}\left(n_a \sin\frac{A}{2} + n_a \sin\frac{A}{2} + \sqrt{\frac{r}{R}}n_a\right)\left(\because AI = \frac{r}{\sin\frac{A}{2}}\right)$$

$$\overset{CBS}{\leq} \frac{\sqrt{3}}{r}\sqrt{n_a^2\sin^2\frac{A}{2} + r_a^2\sin^2\frac{A}{2} + \frac{r}{R}r_a^2}$$

$$\overset{by\ (1)}{=} \frac{\sqrt{3}}{r}\sqrt{s^2\sin^2\frac{A}{2} - \frac{rs^2}{R}\left(\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}\right) + n_a^2\sin^2\frac{A}{2} + \frac{r}{R}n_a^2}$$

$$= \frac{\sqrt{3}}{r}\sqrt{\sin^2\frac{A}{2}(s^2 + r_a^2) - \frac{r}{R}\left(s\tan\frac{A}{2}\right)^2 + \frac{r}{R}r_a^2}$$

$$= \frac{\sqrt{3}}{r}\sqrt{\sin^2\frac{A}{2}\left(s^2 + s^2\tan^2\frac{A}{2}\right) - \frac{r}{R}r_a^2 + \frac{r}{R}r_a^2}$$

$$= \frac{\sqrt{3}}{r}\sqrt{s^2\left(\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}\right)} = \frac{\sqrt{3}}{r}\left(s\tan\frac{A}{2}\right) = \frac{\sqrt{3}}{r}r_a$$

$$\therefore \frac{n_a + r_a}{AI} + \sqrt{\frac{r}{R}}\frac{r_a}{r} \leq \frac{\sqrt{3}}{r}n_a \Rightarrow \frac{n_a + r_a}{AI} \overset{(a)}{\leq} \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)r_a\left(\frac{1}{r}\right)$$

Similarly, $\frac{n_b+r_b}{BI} \overset{(b)}{\leq} \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)r_b\left(\frac{1}{r}\right)$ *and* $\frac{n_c+r_c}{CI} \overset{(c)}{\leq} \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)r_c \cdot \left(\frac{1}{r}\right)$

(a)+(b)+(c) $\Rightarrow \sum\frac{n_a+r_a}{AI} \leq \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)\left(\frac{\sum n_a}{r}\right) = \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)\left(\frac{4R+r}{r}\right)$

$$= \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)\left(1 + \frac{4R}{r}\right) \quad \textit{(Proved)}$$

1324. In ΔABC the following relationship holds:

$$\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \leq 2\sqrt{3\left(\frac{R}{2r}\right)^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

$$\text{We must show: } \left(\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\right)^2 \le \frac{3}{2}\left(\frac{R}{r}\right)^3 \quad (1)$$

$$\text{From Cauchy's inequality: } \left(\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\right)^2 \le 3\left(\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\right) \quad (2)$$

$$\text{From (1)+(2): We must show: } \frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2} \le \frac{1}{2}\left(\frac{R}{r}\right)^3 \quad (3)$$

$$\text{But in any } \Delta ABC \text{ we have: } m_a \ge \sqrt{s(s-a)};\ s=\frac{a+b+c}{2}$$

$$\Rightarrow m_a^2 \ge s(s-a)\ (4)$$

$$\text{From (3)+(4) we must show: } \frac{1}{s}\left(\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}\right) \le \frac{1}{2}\left(\frac{R}{r}\right)^3 \quad (5)$$

$$\text{But } \frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}=\frac{4s(R-r)}{r} \quad (6)$$

$$\text{From (5)+(6) we must show: } 4\left(\frac{R}{r}-1\right) \le \frac{1}{2}\left(\frac{R}{r}\right)^3.\ \text{Let } \frac{R}{r}=x, x\ge 2 \quad \text{(Euler)}$$

$$\text{We must show: } \frac{1}{2}x^3 \ge 4(x-1) \Leftrightarrow x^3-8x+8 \ge 0 \Leftrightarrow$$

$$(x-2)(x^2+2x-4)\ge 0, \text{ true because } x\ge 2.$$

Solution 2 by Șerban George Florin-Romania

$$a\le b\le c \Rightarrow m_a \ge m_b \ge m_c. \text{ Applying Chebyshev's inequality}$$

$$\Rightarrow 3\sum_{cyc}\frac{a}{m_a} \le \sum_{cyc} a\cdot\sum_{cyc}\frac{1}{m_a} \Rightarrow \sum\frac{a}{m_a} \le \frac{2s}{3}\cdot\sum\frac{1}{m_a}$$

$$\text{We prove that } \sum\frac{1}{m_a} \le \frac{1}{n},\ \left(\sum_{cyc}\frac{1}{m_a}\right)^2 = \left(\sum\frac{2}{(b+c)\cos\frac{A}{2}}\right)^2 \overset{AM-GM}{\le}$$

$$\left(\sum\frac{2}{2bc\sqrt{\frac{s(s-a)}{bc}}}\right)^2 = \frac{1}{s}\left(\sum\frac{1}{s-a}\right)^2 \overset{CBS}{\le} \frac{3}{s}\sum\frac{1}{s-a} = \frac{3}{s}\cdot\frac{4R+r}{rs} \le$$

$$\le \frac{1}{r^2} \Rightarrow 3(4R+r)r \le s^2 \Rightarrow s^2 \ge 12Rr+3r^2, \text{ applying Gerretsen's inequality}$$

$$s^2 \ge 16Rr-5r^2 \ge 12Rr+3r^2 \Rightarrow 4Rr \ge 8r^2$$

$$\Rightarrow R \ge 2r, \text{ true (Euler's inequality)} \Rightarrow \sum\frac{1}{m_a} \le \frac{1}{r}$$

$$\Rightarrow \sum \frac{a}{m_a} \le \frac{2s}{3} \sum \frac{1}{m_a} \le \frac{2s}{3} \cdot \frac{1}{r} = \frac{2s}{3r} \le 2\sqrt{3\left(\frac{R}{2r}\right)^3}$$

$$\frac{s}{3r} \le \sqrt{3\left(\frac{R}{2r}\right)^3} \Rightarrow \frac{s^2}{9r^2} \le 3 \cdot \frac{R^3}{8r^3}, 8s^2r \le 27R^3$$

$s^2 \le \frac{27R^3}{8r}$. *Applying Mitrinovic's inequality* $s \le \frac{3\sqrt{3}R}{2}$

$\Rightarrow s^2 \le \frac{27R^2}{4} \le \frac{27R^3}{8r} \Rightarrow 2R \le R$, *true, Euler's inequality.*

Solution 3 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \left(\sum \frac{a}{m_a}\right)^2 \overset{CBS}{\le} \left(\sum a^2\right)\left(\sum \frac{1}{m_a^2}\right)$$

$$\Rightarrow \Omega^2 \overset{Leibnitz}{\le} 9R^2\left(\sum \frac{1}{m_a^2}\right) \overset{m_a \ge \sqrt{s(s-a)}}{\le} 9R^2\left(\sum \frac{1}{s(s-a)}\right)$$

$$\Rightarrow \Omega^2 \le 9R^2 \cdot \frac{1}{s}\left(\frac{\sum r_a}{\Delta}\right) = \frac{9R^2(4R+r)}{s^2r}$$

$\Leftrightarrow$ *Given* $\Omega = \sum \frac{a}{m_a} \le 2\sqrt{3\left(\frac{R}{2r}\right)^3} \Rightarrow \left(\sum \frac{a}{m_a}\right)^2 \le \frac{12R^3}{8r^3}$

$\Leftrightarrow$ *Need to show* $\frac{9R^2(4R+r)}{s^2r} \le \frac{12R^3}{8r^3} \Rightarrow \frac{4R+r}{s^2} \le \frac{R}{6r^2}$

$$\Leftrightarrow \frac{4R+r}{s^2} \overset{Euler}{\le} \frac{4R+\frac{R}{2}}{s^2} = \frac{9R}{2s^2}$$

$\Leftrightarrow \frac{9R}{2s^2} \le \frac{R}{6r^2} \Rightarrow s^2 \ge 27r^2 \Rightarrow s \overset{Mitrinovic}{\ge} 3\sqrt{3}r$ *(* true)*

$\Leftrightarrow \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \le 2\sqrt{3\left(\frac{R}{2r}\right)^3}$ *(Proved)*

1325. If in $\Delta ABC, R_a, R_b, R_c$ —are circumradii of $BIC, \Delta CIA, \Delta AIB, I$ —incenter then the following relationship holds:

$$\sum_{cyc} (h_a - 2r)\sqrt{\frac{R_a}{AI}} \le (R+r)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\angle BIC = \pi - \left(\frac{B+C}{2}\right) = \pi - \left(\frac{\pi - A}{2}\right) = \frac{\pi}{2} + \frac{A}{2}$$

$$\textit{Using sine rule on } \Delta\, BIC, 2R_a \sin\left(\frac{\pi}{2} + \frac{A}{2}\right) = 4R\sin\frac{A}{2}\cos\frac{A}{2} \Rightarrow R_a \overset{(a)}{=} 2R\sin\frac{A}{2}$$

$$\textit{Similarly}, R_b \overset{(b)}{=} 2R\sin\frac{B}{2} \text{ and } R_c \overset{(c)}{=} 2R\sin\frac{C}{2}$$

$$\textit{Also}, b + c - a = 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\sin\frac{A}{2}\cos\frac{A}{2} =$$

$$4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Rightarrow$$

$$\Rightarrow s - a \overset{(i)}{=} 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\textit{Similarly}, s - b \overset{(ii)}{=} 4R\cos\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2} \text{ and } s - c \overset{(iii)}{=} 4R\cos\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}$$

$$\textit{Using } (a),(b),(c), \sum(h_a - 2r)\sqrt{\frac{R_a}{AI}} = \sum\left(\frac{2rs}{a} - 2r\right)\sqrt{\frac{2R\sin^2\frac{A}{2}}{r}} =$$

$$= 2r\sqrt{\frac{2R}{r}}\sum\left(\frac{s-a}{a}\sin\frac{A}{2}\right) \overset{\text{by (i),(ii),(iii)}}{=} 2r\sqrt{\frac{2R}{r}}\sum\left(\frac{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\sin\frac{A}{2}\right)$$

$$= 2r\sqrt{\frac{2R}{r}}\left(\frac{r}{4R}\right)\sum \operatorname{cosec}\frac{A}{2} = \sqrt{\frac{2R}{r}}\left(\frac{r^2}{2R}\right)\sum\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} =$$

$$= \sqrt{\frac{2R}{rs}}\left(\frac{r}{2R}\right)\sum\sqrt{bc(s-a)} \overset{CBS}{\le} \sqrt{\frac{2R}{rs}}\left(\frac{r}{2R}\right)\sqrt{\sum ab}\sqrt{\sum(s-a)} =$$

$$= \sqrt{\frac{2R}{r}}\left(\frac{r}{2R}\right)\sqrt{s^2 + 4Rr + r^2}$$

$$\overset{Gerretsen}{\le} \sqrt{\frac{r}{2R}}\sqrt{4R^2 + 8Rr + 4r^2} = 2(R + r)\sqrt{\frac{r}{2R}} = (r + R)\sqrt{\frac{2r}{R}} \text{ (Proved)}$$

1326. In ΔABC the following relationship holds:

$$\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+\frac{2nr}{R}\ge n+3, n\le 4$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$We\ shall\ first\ prove:\sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}\overset{(a)}{\ge}7$$

$Proof: Let\ s-a=x, s-b=y, s-c=z$

$$\therefore 3s-2s=s=\sum x\Rightarrow a=y+z, b=z+x, c=x+y$$

$$Now, (a)\Leftrightarrow\sum\left(\frac{s-b}{s-a}\right)^2+8\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right)\ge 7$$

$$\overset{\text{via above transformation}}{\Leftrightarrow}\sum\frac{y^2}{x^2}+\frac{32s(s-a)(s-b)(s-c)}{s\prod(x+y)}\ge 7\Leftrightarrow\sum\frac{y^2}{x^2}+\frac{32xyz}{\prod(x+y)}\ge 7$$

$$\Leftrightarrow\sum\frac{y^2}{x^2}+3+\frac{32xyz}{\prod(x+y)}\ge 10\Leftrightarrow\sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}\overset{(1)}{\ge}10$$

$$Now,\sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}=\sum\frac{y^2+x^2}{x^2}+\frac{16xyz}{\prod(x+y)}+\frac{16xyz}{\prod(x+y)}\overset{A-G}{\ge}$$

$$\overset{A-G}{\ge}5\sqrt[5]{\frac{2^8(xyz)^2}{\prod x^2}\frac{\prod(x^2+y^2)}{\prod(x+y)^2}}\ge 5\sqrt[5]{2^8\frac{\prod\left(\frac{1}{2}(x+y)^2\right)}{\prod(x+y)^2}}=5\sqrt[5]{2^5}=10$$

$$\Rightarrow(1)\Rightarrow(a)\ is\ true$$

$$Now,\sum\frac{r_a^2}{r_b^2}+\frac{2nr}{R}=\sum\frac{r_a^2}{r_b^2}+\frac{(2n-8+8)r}{R}=$$

$$=\sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}+\frac{(2n-8)r}{R}\overset{by\ (a)}{\ge}7+\frac{(2n-8)r}{R}\overset{?}{\ge}n+3\Leftrightarrow$$

$$\Leftrightarrow\frac{(2n-8)r}{R}\overset{?}{\ge}n-4\Leftrightarrow(n-4)\left(\frac{2r}{R}-1\right)\overset{?}{\ge}0$$

$$\rightarrow true\ \because(n-4),\left(\frac{2r}{R}-1\right)\le 0\left(\because\frac{2r}{R}\overset{Euler}{\le}1\right)\Rightarrow(n-4)\left(\frac{2r}{R}-1\right)\ge 0$$

$$\therefore \sum \frac{r_a^2}{r_b^2} + \frac{2nr}{R} \ge n + 3 \ \forall \ n \le 4 \ (Proved)$$

1327. **In ΔABC the following relationship holds:**

$$8 \le \frac{(b^3+c^3)(c^3+a^3)(a^3+b^3)}{a^3b^3c^3} \le \frac{1}{8}\left(\frac{R}{r}\right)^6$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\prod(b^3+c^3)}{a^3b^3c^3} = \frac{\prod(b+c)\cdot\prod(b^2+c^2-bc)}{a^3b^3c^3}$$

$$\overset{G\le A}{\le} \frac{2s(s^2+2Rr+r^2)}{4Rrs\cdot 16R^2r^2s^2}\cdot\frac{(2\sum a^3 - 2\sum ab)^3}{27}$$

$$\overset{Gerretsen}{\le} \frac{(4R^2+6Rr+4r^2)}{2Rr}\cdot\frac{(4(s^2-4Rr-r^2)-s^2-4Rr-r^2)^3}{27\cdot 16R^2r^2s^2}$$

$$= \frac{(2R^2+3Rr+2r^2)}{Rr}\cdot\frac{(3s^2-20Rr-5r^2)^3}{27\cdot 16R^2r^2s^2}$$

$$\overset{Gerretsen}{\le} \frac{(2R^2+3Rr+2r^2)}{Rr}\cdot\frac{(12R^2-8Rr+4r^2)^3}{27\cdot 16R^2r^2(16Rr-5r^2)}$$

$$= \frac{4(2R^2+3Rr+2r^2)(3R^2-2Rr+r^2)^3}{27R^3r^4(16R-5r)} \overset{?}{\le} \frac{1}{8}\left(\frac{R}{r}\right)^6$$

$$\Leftrightarrow 27R^9(16R-5r) \overset{?}{\ge} 32(2R^2+3Rr+2r^2)(3R^2-2Rr+r^2)^3r^2$$

$$\Leftrightarrow 432t^{10} - 135t^9 - 1728t^8 + 864t^7 - 576t^6 + 224t^5 -$$

$$-1152t^4 + 1184t^3 - 832t^2 + 288t - 64 \overset{?}{\ge} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left[\begin{matrix}(t-2)(432t^8+1593t^7+2916t^6+6156t^5\\ +12384t^4+25136t^3+49856t^2+100064t+\\ +200000)+4000032\end{matrix}\right] \overset{?}{\ge} 0$$

$$\to true \because t \overset{Euler}{\ge} 2 \Rightarrow \frac{\prod(b^3+c^3)}{a^3b^3c^3} \le \frac{1}{8}\left(\frac{R}{r}\right)^6$$

$$Also, \frac{\prod(b^3+c^3)}{a^3b^3c^3} \overset{Cesaro}{\ge} 8 \ \ (Hence\ proved)$$

Solution 2 by Boris Colakovic-Belgrade-Serbie

$$a^3+b^3\ge ab(a+b)\ge 2ab\sqrt{ab}$$

$$b^3+c^3\ge bc(b+c)\ge 2bc\sqrt{bc}$$

$$c^3+a^3\ge ca(c+a)\ge 2ca\sqrt{ca}$$

$$(a^3+b^3)(b^3+c^3)(c^3+a^3)\ge 8a^2b^2c^2\cdot abc=8a^3b^3c^3$$

$$\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{a^3b^3c^3}\ge 8$$

$$\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{a^3b^3c^3}\le\frac{8}{27}\left(\frac{a^3+b^3+c^3}{abc}\right)^3=\frac{64}{27}\cdot\frac{1}{64}\left(\frac{s^3-3r^2s-6Rrs}{Rrs}\right)^3=$$

$$=\frac{1}{27}\left(\frac{s^2-3r^2-6Rr}{Rr}\right)^3\overset{Gerretsen}{\le}\frac{1}{27}\left(\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{Rr}\right)^3=\frac{8}{27}\left(\frac{2R-r}{r}\right)^3=$$

Now, is $\frac{8}{27}\left(\frac{2Rr}{r}\right)^3\le\frac{1}{8}\left(\frac{R}{r}\right)^6\Leftrightarrow 3R^2-8Rr+4r^2\ge 0\Leftrightarrow (R-2r)(3R-2r)\ge 0$

$\Rightarrow R\ge 2r$ *(Euler)*

1328. In ΔABC the following relationship holds:

$$2\left(\frac{m_a^5}{m_b^3}+\frac{m_b^5}{m_c^3}+\frac{m_c^5}{m_a^3}\right)\ge s^2+3r^2+12Rr$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Marian Ursărescu-Romania

First, we want to show: $\frac{x^5}{y^3}+\frac{y^5}{z^3}+\frac{z^5}{x^3}\ge x^2+y^2+z^2\Leftrightarrow$

$$x^8z^3+y^8x^3+z^8y^3\ge x^3y^3z^3(x^2+y^2+z^2)\quad (1)$$

From inequality of weighted means we have:

$$\frac{275}{539}x^8z^3+\frac{165}{539}y^8x^3+\frac{99}{539}z^8y^3\ge\sqrt[539]{(x^8z^3)^{275}\cdot(y^8x^3)^{165}\cdot(z^8y^3)^{99}}=x^5y^3z^3\quad (2)$$

By permutation we have:

$$\frac{275}{539}y^8x^3+\frac{165}{539}z^8y^3+\frac{99}{539}x^8z^3\ge x^3y^5z^3\quad (3)$$

$$\frac{275}{539}z^8y^3+\frac{165}{539}x^8z^3+\frac{99}{539}y^8x^3\ge x^3y^3z^5\quad (4)$$

From (2)+(3)+(4) by summing $\Rightarrow$

$$x^8z^3+y^8z^3+z^8x^3\ge x^3y^3z^3(x^2+y^2+z^2)$$

Now, in our case $x=m_a, y=m_b, z=m_c\Rightarrow$ *from (1)* $\Rightarrow$

$$2\left(\frac{m_a^5}{m_b^3}+\frac{m_b^5}{m_c^3}+\frac{m_c^5}{m_a^3}\right)\geq 2\left(m_a^2+m_b^2+m_c^2\right)\ \ (5)$$

But $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ *(6)*

From (5)+(6)⇒

$$2\left(\frac{m_a^5}{m_b^3}+\frac{m_b^5}{m_c^3}+\frac{m_c^5}{m_a^3}\right)\geq \frac{3}{2}(a^2+b^2+c^2)\ \ (7)$$

But $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ *(8)*

From (7)+(8)⇒ $2\left(\frac{m_a^5}{m_b^3}+\frac{m_b^5}{m_c^3}+\frac{m_c^5}{m_a^3}\right)\geq 3(s^2-r^2-4Rr)\Rightarrow$ *we must show:*

$$3s^2-3r^2-12Rr\geq s^2+3r^2+12Rr\Leftrightarrow$$

$2s^2\geq 24Rr+6r^2\Leftrightarrow s^2\geq 12Rr+3r^2$*, inequality which it is true, because it is Carlitz inequality.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$2\left(\frac{m_a^5}{m_b^3}+\frac{m_b^5}{m_c^3}+\frac{m_c^5}{m_a^3}\right)\overset{(1)}{\geq} s^2+3r^2+12Rr$$

$$s^2\overset{Gerretsen}{\geq}16Rr-5r^2\overset{?}{\geq}12Rr+3r^2$$

$$\Leftrightarrow 4Rr\overset{?}{\geq}8r^2\Leftrightarrow R\overset{?}{\geq}2r\rightarrow \text{true (Euler)}$$

$$\therefore 12Rr+3r^2\leq s^2\Rightarrow s^2+3r^2+12Rr\overset{(i)}{\leq}2s^2$$

(i)⇒ in order to prove (1), it suffices to prove: $\sum\frac{m_a^5}{m_b^3}\overset{(2)}{\geq}s^2$

We shall now prove: $\sum\frac{a^5}{b^3}\overset{(ii)}{\geq}\frac{4}{3}\sum m_a^2$

$$(ii)\Leftrightarrow\sum\frac{a^5}{b^3}\overset{(iii)}{\geq}\frac{4}{3}\cdot\frac{3}{4}\sum a^2=\sum a^2$$

$$\text{Now, }\sum\frac{a^5}{b^3}=\sum\frac{a^6}{ab^3}\overset{Bergstrom}{\geq}\frac{(\sum a^3)^2}{\sum ab^3}\overset{?}{\geq}\sum a^2$$

$$\Leftrightarrow\sum a^6+\sum a^3b^3\underset{(iv)}{\overset{(1)}{\geq}}\sum ab^5+abc\left(\sum a^2b\right)$$

$$\because 6+0=1+5 \text{ and } 6>1, \therefore (6,0)\succ(1,5)$$

$$\therefore\sum a^6\underset{(a)}{\overset{Muirhead}{\geq}}\sum ab^5$$

$$\textit{Also, } \because x^3+y^3+z^3 \ge \sum xy^2$$

$$\therefore \sum a^3b^3 \ge \sum ab(b^2c^2) = \sum ab^3c^2 = abc\left(\sum b^2c\right) \Rightarrow \sum a^3b^3 \overset{(b)}{\ge} abc\left(\sum a^2b\right)$$

(a)+(b)⇒(iv)⇒(iii) ⇒(ii) is true.

Applying (ii) on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ *whose medians will of course be*

$$\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \textit{ we get, } \sum \frac{\left(\frac{2}{3}m_a\right)^5}{\left(\frac{2}{3}m_b\right)^3} \ge \frac{4}{3}\sum\left(\frac{a}{2}\right)^2 \Rightarrow \frac{4}{9}\sum\frac{m_a^5}{m_b^3} \ge \frac{1}{3}\sum a^2$$

$$\Rightarrow \sum \frac{m_a^5}{m_b^3} \ge \frac{1}{4}\cdot 3\sum a^2 \ge \frac{1}{4}\left(\sum a\right)^2 = \frac{4s^2}{4} = s^2 \Rightarrow (2) \Rightarrow (1) \textit{ is true (Proved)}$$

1329. In $\Delta ABC, H$ – orthocenter the following relationship holds:

$$AH\cdot CH^3 + BH\cdot AH^3 + CH\cdot BH^3 \le \frac{16}{3}(4R^2-13r^2)^2$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$AH = 2R|\cos A| \textit{ and analogs. Now, } AH\cdot CH^3 = 16R^4|\cos A||\cos C|^3$$

$$= 16R^4|\cos A||\cos C|\cos^2 C \overset{(i)}{\le} \frac{16R^4}{2}(\cos^2 A+\cos^2 C)\cos^2 C$$

$$= 8R^4(\cos^2 A\cos^2 C + \cos^4 C)$$

$$\textit{Similarly, } BH\cdot AH^3 \overset{(ii)}{\le} 8R^4(\cos^2 B\cos^2 A+\cos^4 A) \textit{ and}$$

$$CH\cdot BH^3 \overset{(iii)}{\le} 8R^4(\cos^2 C\cos^2 B+\cos^4 B)$$

$$\textit{(i)+(ii)+(iii)} \Rightarrow LHS \overset{(1)}{\le} 8R^4\left(\sum\cos^2 B\cos^2 C+\sum\cos^4 A\right)$$

$$\textit{Now, } \sum\cos^2 C\cos^2 B = \sum(1-\sin^2 B)(1-\sin^2 C) =$$

$$= \sum(1-\sin^2 B-\sin^2 C+\sin^2 B\sin^2 C) = 3-2\sum\frac{a^2}{4R^2}+\frac{1}{16R^4}\sum b^2c^2$$

$$\overset{(a)}{=} 3-\frac{\sum a^2}{2R^2}+\frac{\sum b^2c^2}{16R^4}$$

$$\textit{Again, } \sum\cos^4 A = \sum(1-\sin^2 A)^2 = \sum(1+\sin^4 A-2\sin^2 A)$$

$$\overset{(b)}{=} 3+\frac{\sum a^4}{16R^4}-\frac{\sum a^2}{2R^2}$$

$$\text{(a), (b), (1)}\Rightarrow LHS \overset{(m)}{\leq} 8R^4\left(6+\frac{\sum a^4+\sum a^2b^2}{16R^4}-\frac{\sum a^2}{R^2}\right)$$

$$=48R^4+\frac{\sum a^4+\sum a^2b^2}{2}-8R^2\sum a^2 \overset{?}{\leq} \frac{16}{3}(4R^2-13r^2)^2$$

$$\Leftrightarrow 288R^4-48R^2\sum a^2+3\left(\sum a^4+\sum a^2b^2\right) \underset{(2)}{\overset{?}{\leq}} 32(4R^2-13r^2)^2$$

$$\text{Now, } \sum a^4+\sum a^2b^2 = \left(\sum a^2\right)^2-\sum a^2b^2$$

$$=4(s^2-4Rr-r^2)^2-\{(s^2+4Rr+r^2)^2-2abc(2s)\}$$

$$=4(s^2-4Rr-r^2)^2-(s^2+4Rr+r^2)^2-16Rrs^2$$

$$=3s^4-s^2(24Rr+10r^2)+3r^2(4R+r)^2$$

$$\underset{(iv)}{\overset{Gerretsen}{\leq}} 3s^2(4R^2+4Rr+3r^2)-s^2(24Rr+10r^2)+3r^2(4R+r)^2$$

$$=s^2(12R^2-12Rr-r^2)+3r^2(4R+r)^2$$

$$\text{(iv)}\Rightarrow LHS \text{ of (2)} \leq 288R^4-96R^2(s^2-4Rr-r^2)+$$

$$+s^2(36R^2-36Rr-3r^2)+9r^2(4R+r)^2 \overset{?}{\leq} 32(4R^2-13r^2)^2$$

$$\Leftrightarrow 288R^4+96R^2(4Rr+r^2)+9r^2(4R+r)^2$$

$$\underset{(3)}{\overset{?}{\leq}} 32(4R^2-13r^2)^2+s^2(60R^2+36Rr+3r^2)$$

$$\text{Now, RHS of (3)} \geq 32(4R^2-13r^2)^2+(16Rr-5r^2)(60R^2+36Rr+3r^2)$$

$$\overset{?}{\geq} 288R^4+96R^2(4Rr+r^2)+9r^2(4R+r)^2$$

$$\Leftrightarrow 56t^4+144t^3-823t^2-51t+1346 \overset{?}{\geq} 0 \left(t=\frac{R}{t}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(56t^2+368t+425)+117\} \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow \text{(3)}\Rightarrow \text{(2) is true and} \therefore \text{(m)} \Rightarrow LHS \leq \frac{16}{3}(4R^2-13r^2)^2 \quad \text{(Proved)}$$

1330. In ΔABC the following relationship holds:

$$\left(\sum r_ar_b\right)\left(\sum (r_a+r_b)^2(r_a+r_c)^2\right) \geq \left(\prod (r_a+r_b)^2\right)\left(\sum \cos^2\left(\frac{A}{2}\right)\right)$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum r_a r_b\right)\left(\sum (r_a+r_b)^2(r_a+r_c)^2\right) \overset{(1)}{\geq} \left(\prod (r_a+r_b)^2\right)\left(\sum \cos^2\left(\frac{A}{2}\right)\right)$$

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}} = \left(\frac{s}{\frac{s}{4R}}\right)\cos^2\frac{A}{2} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(a)}{=} 4R\cos^2\frac{A}{2} \textit{ and analogs}$$

$$\textit{Now, LHS of (1)} = s^2\cdot\sum(r_a^2+\sum r_a r_b)^2 = s^2\sum\left(s^2\tan^2\frac{A}{2}+s^2\right)^2 \overset{(i)}{=} s^6\sum\sec^4\frac{A}{2}$$

Using (a) and its analogs, RHS of (1)

$$= \prod\left(16R^2\cos^4\frac{A}{2}\right)\sum\cos^2\frac{A}{2}$$

$$= 16^3R^6\left(\frac{s}{4R}\right)^4\left(\sum\cos^2\frac{A}{2}\right) \overset{(ii)}{=} 16R^2s^4\left(\sum\cos^2\frac{A}{2}\right)$$

$$\textit{(i), (ii)} \Rightarrow \textit{(1)} \Leftrightarrow \left(\frac{s}{4R}\right)^2\sum\sec^4\frac{A}{2} \geq \sum\cos^2\frac{A}{2}$$

$$\Leftrightarrow \left(\prod\cos^2\frac{A}{2}\right)\sum\sec^4\frac{A}{2} \geq \sum\cos^2\frac{A}{2} \Leftrightarrow \sum\sec^4\frac{A}{2} \geq \sum\sec^2\frac{B}{2}\sec^2\frac{C}{2}$$

$$\rightarrow \textit{true} \because x^2+y^2+z^2 \geq xy+yz+zx, \textit{ where } x=\sec^2\frac{A}{2}, y=\sec^2\frac{B}{2}, z=\sec^2\frac{C}{2}$$

1331. A TERESHIN TYPE INEQUALITY BY BOGDAN FUȘTEI

In ΔABC the following relaionship holds:

$$m_a \geq \frac{n_a^2+g_a^2+2rr_a}{4R}, n_a - \text{Nagel's cevian}, g_a - \text{Gergonne's cevian}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc)$$

$$\Rightarrow s(b^2+c^2)-2sbc = an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2$$

$$\Rightarrow an_a^2 = as^2+s(2bc\cos A-2bc) = as^2-4sbc\sin^2\frac{A}{2}$$

$$= as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2-\frac{4\Delta^2}{s-a} = as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_a r_a \Rightarrow n_a^2 \overset{(1)}{=} s^2 - 2h_a r_a$$

Again, Stewart's theorem $\Rightarrow b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - (b^3+c^3) = ag_a^2 + a(s^2 - s(2s-a) + bc) = ag_a^2 + a(-s^2 + as + bc)$$

$$\Rightarrow ag_a^2 = as^2 + \frac{(b^2+c^2)(\sum a) - 2(b^3+c^3) - a^2(\sum a) - 2abc}{2}$$

$$= as^2 + \frac{ab^2 + ac^2 + b^3 + bc^2 + b^2c + c^3 - 2(b^3+c^3) - a^3 - a^2b - a^2c - 2abc}{2}$$

$$= as^2 + \frac{a(b-c)^2 - (a^3+b^3+c^3) + b^2c + bc^2 - a^2b - a^2c}{2}$$

$$= as^2 + \frac{a(b-c)^2 - a^2(\sum a) - (b+c)(b^2 - bc + c^2) + bc(b+c)}{2}$$

$$= as^2 + \frac{a(b-c)^2 - 2sa^2 - (2s-a)(b-c)^2}{2}$$

$$= as^2 + \frac{2a(b-c)^2 - 2sa^2 - 2s(b^2+c^2-2bc)}{2} = as^2 + a(b-c)^2 - s\sum a^2 + 2sbc$$

$$\Rightarrow g_a^2 \overset{(2)}{=} (b-c)^2 + s^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a}$$

$$(1)+(2) \Rightarrow n_a^2 + g_a^2 + 2r_b r_c$$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - 2\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) + 2\frac{\Delta^2}{(s-b)(s-c)}$$

$$= 2s^2 + (b-c)^2 - \frac{s\sum a^2}{a} + \frac{2sbc}{a} - \frac{4s(s-a)(s-b)(s-c)}{a(s-a)} + \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{\sum a^2 + 4(s-b)(s-c) - 2bc}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left\{\frac{a^2 - (b-c)^2 + a^2 + (b^2+c^2-2bc)}{a}\right\}$$

$$= (b-c)^2 + 2s(s-a) + 2s^2 - s\left(\frac{2a^2}{a}\right) = (b-c)^2 + 4s(s-a)$$

$$= (b-c)^2 + (b+c+a)(b+c-a)$$

$$= (b-c)^2 + (b+c)^2 - a^2 = 2b^2 + 2c^2 - a^2 = 4m_a^2$$

$$\therefore n_a^2 + g_a^2 + 2r_b r_c = 4m_a^2 \Rightarrow n_a^2 + g_a^2 + 2rr_a = 4m_a^2 - 2r_b r_c + 2rr_a$$

$$= 2b^2 + 2c^2 - a^2 - \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)} + \frac{2s(s-a)(s-b)(s-c)}{s(s-a)}$$

$$= b^2 + c^2 + (b^2 + c^2 - a^2) - 2s(s-a) + 2(s-b)(s-c)$$

$$= (b-c)^2 + 2bc + 2bc\cos A + 2\left(s^2 - s(2s-a) + bc - s(s-a)\right)$$

$$= (b-c)^2 + 2bc \cdot \frac{2s(s-a)}{bc} + 2(-2s^2 + 2as + bc)$$

$$= (b-c)^2 + 4s(s-a) - 4s(s-a) + 2bc = b^2 + c^2$$

$$\therefore n_a^2 + g_a^2 + 2rr_a = b^2 + c^2 \Rightarrow \frac{n_a^2 + g_a^2 + 2rr_a}{4R} = \frac{b^2+c^2}{4R} \overset{Tereshin}{\leq} m_a \text{ (proved)}$$

1332. In $\Delta ABC, n_a -$ Nagel's cevian, the following relationship holds:

$$n_a + n_b + n_c + r_a + r_b + r_c \leq \left(\sqrt{\frac{6R}{r}} - \sqrt{2}\right) \sum_{cyc} \sqrt{h_a r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{h_a r_a} = \sqrt{\frac{2rs}{abc} \cdot \frac{bcrs^2}{s(s-a)}} = \sqrt{\frac{2r^2 s}{4Rrs}\left(s^2 \sec^2\frac{A}{2}\right)} = \sqrt{\frac{r}{2R}}\, s \sec\frac{A}{2}$$

$$\text{and analogs} \Rightarrow \sum \sqrt{h_a r_a} = s\sqrt{\frac{r}{2R}} \sum \sec\frac{A}{2}$$

$$\Rightarrow \sqrt{\frac{6R}{r}} \sum \sqrt{h_a r_a} = \left(s\sqrt{\frac{6R}{r}}\sqrt{\frac{r}{2R}}\right)\sum \sec\frac{A}{2} \overset{(1)}{=} \sqrt{3}s \sum \sec\frac{A}{2}$$

$$\text{Now, Stewarts theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc)$$

$$\Rightarrow s(b^2 + c^2) - 2sbc = an_a + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2$$

$$\Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= as^2 - 2ah_a r_a \Rightarrow n_a^2 \overset{(2)}{=} s^2 - 2h_a r_a$$

Now, $n_a+\sqrt{2h_ar_a}+r_a\overset{CBS}{\le}\sqrt{3}\sqrt{n_a^2+2h_ar_a+r_a^2}\overset{by\,(1)}{=}\sqrt{3}\sqrt{s^2-2h_ar_a+2h_ar_a+r_a^2}$

$$=\sqrt{3}\sqrt{s^2+s^2\tan^2\frac{A}{2}}=\sqrt{3}s\sec\frac{A}{2}\Rightarrow n_a+r_a\overset{(a)}{=}\sqrt{3}s\sec\frac{A}{2}-\sqrt{2h_ar_a}$$

Similarly, $n_b+r_b\overset{(b)}{\le}\sqrt{3}s\sec\frac{B}{2}-\sqrt{2h_br_b}$ *and,* $n_c+r_c\overset{(c)}{\le}\sqrt{3}s\sec\frac{C}{2}-\sqrt{2h_cr_c}$

$$(a)+(b)+(c)\Rightarrow\sum n_a+\sum r_a\overset{(i)}{\le}\sqrt{3}s\sum\sec\frac{A}{2}-\sqrt{2}\sum\sqrt{h_ar_a}$$

$$\overset{by\,(1)}{=}\sqrt{\frac{6R}{r}}\sum\sqrt{h_ar_a}-\sqrt{2}\sum\sqrt{h_ar_a}=\left(\sqrt{\frac{6R}{r}}-\sqrt{2}\right)\sum\sqrt{h_ar_a}\quad \textit{(Proved)}$$

1333. In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{2R^2}\le\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}\le\frac{\sqrt{3}}{8r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Marian Ursărescu-Romania

$$\frac{r_a}{a^3}+\frac{r_b}{b^3}+\frac{r_c}{c^3}\ge 3\sqrt[3]{\frac{r_ar_br_c}{(abc)^3}}\quad (1)$$

$$\textit{But } r_ar_br_c=s^2r \textit{ and } abc=4sRr\ (2),\ s=\frac{a+b+c}{2}$$

$$\textit{From (1)+(2) we must show: } 3\sqrt[3]{\frac{s^2r}{64s^3R^3r^3}}\ge\frac{\sqrt{3}}{2R^2}\Leftrightarrow 27\frac{1}{64sR^3r^2}\ge\frac{3\sqrt{3}}{8R^6}\Leftrightarrow$$

$$\Leftrightarrow 3\sqrt{3}R^3\ge 8sr^2,\ \textit{true because } R^2\ge 4r^2 \textit{ and } 3\sqrt{3}R\ge 2s$$

$$\textit{Now, } r\cdot r_a=\frac{S}{s}\cdot\frac{S}{s-a}=\frac{s(s-a)(s-b)(s-c)}{s(s-a)}=(s-b)(s-c)\le\frac{a^2}{4}$$

$$\Rightarrow r_a\le\frac{a^2}{4r}\Rightarrow\frac{r_a}{a^3}\le\frac{1}{4ar}\Rightarrow \textit{we must show:}$$

$$\frac{1}{4r}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\frac{\sqrt{3}}{8r^2}\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2r},\ \textit{true, because it is Steining inequality.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{Firstly, } \left(\sum ab\right)^2\ge 24Rrs^2\Leftrightarrow(s^2+4Rr+r^2)^2\ge 24Rrs^2$$

$$\Leftrightarrow s^4+(4Rr+r^2)^2+2(4Rr+r^2)s^2\ge 24Rrs^2$$

$$\Leftrightarrow s^4+(4Rr+r^2)^2\overset{(i)}{\ge}s^2(16Rr-2r^2)$$

$$\text{Now, LHS of (i)} \overset{\text{Gerretsen}}{\geq} s^2(16Rr-5r^2)+(4Rr+r^2)^2 \overset{?}{\geq} s^2(16Rr-2r^2)$$

$$\Leftrightarrow (4Rr+r^2)^2 \overset{?}{\geq} 3r^2s^2 \Leftrightarrow 4R+r \overset{(i)}{\geq} s\sqrt{3} \to \text{true (Trucht)} \Rightarrow \text{(i) is true.}$$

$$\therefore \left(\sum ab\right)^2 \overset{(1)}{\geq} 24Rrs^2$$

$$\text{Secondly, } \left(\sum ab\right)^2 \leq 12R^2s^2 \Leftrightarrow s^4+(4Rr+r^2)^2+2(4Rr+r^2)s^2 \leq 12R^2s^2$$

$$\Leftrightarrow s^4+(4Rr+r^2)^2 \overset{(ii)}{\leq} s^2(12R^2-8Rr-2r^2)$$

$$\text{Now, LHS of (ii)} \overset{\text{Gerretsen}}{\leq} s^2(4R^2+4Rr+3r^2)+(4Rr+r^2)^2$$

$$\overset{?}{\leq} s^2(12R^2-8Rr-2r^2) \Leftrightarrow s^2(8R^2-12Rr-5r^2) \underset{(iii)}{\overset{?}{\geq}} (4R+r^2)^2$$

$$\text{Again, LHS of (iii)} \overset{\text{Gerretsen}}{\geq} (16Rr-5r^2)(8R^2-12Rr-5r^2) \overset{?}{\geq} (4Rr+r^2)^2$$

$$\Leftrightarrow 32t^3-62t^2-7t+6 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(32t^2+2(t-2)+1) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii) is true} \therefore \left(\sum ab\right)^2 \overset{(2)}{\leq} 12R^2s^2$$

$$\text{Now, } \sum \frac{r_a}{a^3} = \sum \left[\frac{\left(\frac{1}{a}\right)^3}{\left(\frac{1}{r_a}\right)}\right] \overset{\text{Holder}}{\geq} \frac{\left(\sum \frac{1}{a}\right)^3}{3\left(\sum \frac{1}{r_a}\right)} = \frac{r}{3}\left(\frac{\sum ab}{4Rrs}\right)^3 \overset{\text{by (1)}}{\geq} \frac{24Rr^2s^2\left(\sum ab\right)}{192R^3r^3s^3}$$

$$\overset{\substack{\text{Ionescu}\\ \text{Weitzenbock}}}{\geq} \frac{96\sqrt{3}Rr^3s^3}{192R^3r^3s^3} = \frac{\sqrt{3}}{2R^2} \therefore \frac{\sqrt{3}}{2R^2} \leq \sum \frac{r_a}{a^3}$$

$$\text{Now, } a^3 = \left((s-b)+(s-c)\right)^3 = (s-b)^3+(s-c)^3+3(s-b)(s-c)a$$

$$\geq (s-b)(s-c)a+3(s-b)(s-c)a = 4a(s-b)(s-c)$$

$$= 4a\left(\frac{(s-a)(s-b)(s-c)}{s-a}\right) = \frac{4ar^2s}{s-a} = 4ar\left(\frac{rs}{s-a}\right) = 4arr_a \Rightarrow a^3 \overset{(a)}{\geq} 4arr_a$$

$$\text{Similarly, } b^3 \overset{(b)}{\geq} 4brr_b, c^3 \overset{(c)}{\geq} 4crr_c$$

$$\text{(a), (b), (c)} \Rightarrow \sum \frac{r_a}{a^3} \leq \sum \frac{1}{4ar} = \frac{1}{4r}\left(\frac{\sum ab}{4Rrs}\right) = \frac{\sum ab}{16Rr^2s} \overset{?}{\leq} \frac{\sqrt{3}}{8r^2}$$

$$\Leftrightarrow 2\sqrt{3}Rs \overset{?}{\geq} \sum ab \Leftrightarrow 12R^2s^2 \overset{?}{\geq} \left(\sum ab\right)^2 \to \text{true by (2)} \therefore \sum \frac{r_a}{a^3} \leq \frac{\sqrt{3}}{8r^2} \text{ (Done)}$$

1334. In ΔABC the following relationship holds:

$$4\left(\sum_{cyc}\frac{r_a}{a}\right)\left(\sum_{cyc}\frac{r_a^2}{r_b+r_c}\right)\geq 9s$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Șerban George Florin-Romania

$$\left(\sum\frac{r_a}{a}\right)\cdot\left(\sum\frac{r_a^2}{r_b+r_c}\right)\geq\frac{9s}{4}$$

$$\left(\sum\frac{r_a}{a}\right)\cdot\left(\sum\frac{r_a^2}{r_b+r_c}\right)\overset{Bergstrom}{\geq}\left(\sum\frac{r_a}{a}\right)\cdot\frac{(r_a+r_b+r_c)^2}{\sum(r_b+r_c)}=\left(\sum\frac{r_a}{a}\right)\cdot\frac{(r_a+r_b+r_c)^2}{2(r_a+r_b+r_c)}$$

$$=\left(\sum\frac{r_a}{a}\right)\cdot\frac{r_a+r_b+r_c}{2}=\sum\frac{S}{a(s-a)}\cdot\frac{S}{2}\cdot\sum\frac{1}{s-a}=\frac{S^2}{2}\cdot\sum\frac{1}{a(s-a)}\cdot\sum\frac{1}{s-a}$$

$$=\frac{S^2}{2}\cdot\frac{s^2+(4R+r)^2}{4Rrs^2}\cdot\frac{4R+r}{rs}=\frac{S^2[s^2+(4R+r)^2]\cdot(4R+r)}{8Rr^2s^3}=$$

$$=\frac{[s^2+(4R+r)^2]\cdot(4R+r)\cdot r^2}{8Rr^2s}=\frac{[s^2+(4R+r)^2]\cdot(4R+r)}{8Rs}\geq\frac{9s}{4}$$

$$\Rightarrow[s^2+(4R+r)^2]\cdot(4R+r)\geq\frac{72Rs^2}{4}=18Rs^2$$

$$s^2(4R+r)+(4R+r)^3\geq 18Rs^2, s^2(18R-4R-r)\leq(4R+r)^3$$

$$s^2(14R-r)\leq(4R+r)^3; R\geq 2r\ (Euler)\Rightarrow 14R\geq 28r$$

$$\Rightarrow 14R-r\geq 28r-r=27r>0\Rightarrow s^2\leq\frac{(4R+r)^3}{14R-r}$$

Applying Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$

$$s^2\leq 4R^2+4Rr+3r^2\leq\frac{(4R+r)^3}{14R-r}$$

$$\Rightarrow(14R-r)(4R^2+4Rr+3r^2)\leq(4R+r)^3|:r^3,\frac{R}{r}=t\geq 2\ (Euler)$$

$$\Rightarrow(14t-1)(4t^2+4t+3)\leq(4t+1)^3$$

$$56t^3+56t^2+42t-4t^2-4t-3\leq 64t^3+3\cdot 16t^2\cdot 1+3\cdot 4t\cdot 1^2+1$$

$$56t^3+52t^2+38t-3\leq 64t^3+48t^2+12t+1$$

$$8t^3-4t^2-26t+4\geq 0|:2\ 4t^3-2t^2-13t+2\geq 0$$

$$4t^3-8t^2+6t^2-12t-t+2\geq 0$$

$$4t^2(t-2)+6t(t-2)-(t-2)\geq 0,(t-2)(4t^2+6t-1)\geq 0$$

$$t-2\geq 0 \text{ because } t\geq 2 \text{ (Euler)}$$

$$4t^2+6t-1\geq 4\cdot 2^2+6\cdot 2-1=16+12-1=27>0\Rightarrow \text{true.}$$

Solution 2 by Marian Ursărescu-Romania

(another approach). From Cauchy inequality we have:

$$\sum\frac{r_a}{a}\cdot\sum\frac{a}{r_a}\geq 9 \quad (1)$$

$$\text{But } \sum\frac{a}{r_a}=\frac{2(4R+r)}{s} \quad (2)$$

$$\text{From (1)+(2)}\Rightarrow\sum\frac{r_a}{a}\cdot\frac{2(4R+r)}{s}\geq 9\Rightarrow\sum\frac{r_a}{a}\geq\frac{9s}{2(4R+r)} \quad (3)$$

$$\text{From (3) we must show: } 4\cdot\frac{9s}{2(4R+r)}\cdot\left(\sum\frac{r_a^2}{r_b+r_c}\right)\geq 9s\Leftrightarrow$$

$$\sum\frac{r_a^2}{r_b+r_c}\geq\frac{4R+r}{2} \quad (4)$$

From Bergström we have:

$$\sum\frac{r_a^2}{r_b+r_c}\geq\frac{(r_a+r_b+r_c)^2}{2(r_a+r_b+r_c)}=\frac{r_a+r_b+r_c}{2} \quad (5)$$

$$\text{But } r_a+r_b+r_c=4R+r \quad (6)$$

$$\text{From (5)+(6)}\Rightarrow\sum\frac{r_a^2}{r_b+r_c}\geq\frac{4R+r}{2}\Rightarrow \text{(4) it is true.}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\prod\cos\frac{A}{2}}=\frac{s}{\frac{s}{4R}}\cos^2\frac{A}{2}\overset{(1)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Using (1) and analogs, } 4\left(\sum\frac{r_a}{a}\right)\left(\sum\frac{r_a^2}{r_b+r_c}\right)$$

$$=4\left(\sum\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}\right)\left(\sum\frac{s^2\tan^2\frac{A}{2}}{4R\cos^2\frac{A}{2}}\right)$$

$$=\frac{s^3}{4R^2}\left(\sum\sec^2\frac{A}{2}\right)\left(\sum\tan^2\frac{A}{2}\sec^2\frac{A}{2}\right)$$

$$\overset{Chebyshev}{\geq}\frac{s^3}{4R^2}\left(\sum\sec^2\frac{A}{2}\right)\cdot\frac{1}{3}\left(\sum\tan^2\frac{A}{2}\right)\left(\sum\sec^2\frac{A}{2}\right)$$

$$(\because \text{ if we assume } a\geq b\geq c \text{ then}$$

$\tan^2\frac{A}{2}\geq\tan^2\frac{B}{2}\geq\tan^2\frac{C}{2}$ *and* $\sec^2\frac{A}{2}\geq\sec^2\frac{B}{2}\geq\sec^2\frac{C}{2}$)

$$\overset{Jensen}{\geq}\frac{16s^3}{12R^2}\left(\sum\tan^2\frac{A}{2}\right)$$

$$\left(\because f(x)=\sec^2\frac{x}{2}\text{ is convex }\forall x\in(0,\pi)\Rightarrow\sum\sec^2\frac{A}{2}\overset{Jensen}{\geq}3\sec^2\frac{\pi}{6}=4\right)$$

$$=\frac{4s}{3R^2}\sum r_a^2=\frac{4s}{3R^2}((4R+r)^2-2s^2)\overset{?}{\geq}9s$$

$$\Leftrightarrow 4(4R+r)^2-27R^2\underset{(2)}{\overset{?}{\geq}}8s^2$$

$$\text{Now, RHS of (2)}\overset{Gerretsen}{\leq}8(4R^2+4Rr+3r^2)\overset{?}{\leq}$$

$$4(4R+r)^2-27R^2\Leftrightarrow R^2\overset{?}{\geq}4r^2\to\text{true (Euler)}$$

$$\Rightarrow\text{(2) is true}\Rightarrow 4\left(\sum\frac{r_a}{a}\right)\left(\sum\frac{r_a^2}{r_b+r_c}\right)\geq 9s$$

(proved)

1335. FUȘTEI'S REFINEMENT FOR EULER'S INEQUALITY

In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$R\geq r\left(1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\right)\geq 2r$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$R\overset{(1)}{\geq}r\left(1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\right)\overset{(2)}{\geq}2r$$

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)$$

$$\Rightarrow s(b^2+c^2)-2sbc=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$= as^2 - \frac{4s\,bc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4r^2S^2}{s-a} \Rightarrow an_a^2 \overset{(a)}{=} a^2s^2 - 4r^2S^2\left(\frac{a}{s-a}\right)$$

Similarly, $b^2n_b^2 \overset{(b)}{=} b^2s^2 - 4r^2S^2\left(\frac{b}{s-b}\right)$ *and* $c^2n_c^2 \overset{(c)}{=} c^2s^2 - 4r^2S^2\left(\frac{c}{s-c}\right)$

Now, (1) $\Leftrightarrow \frac{R-r}{r} \ge \sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}} \Leftrightarrow \sqrt[3]{\frac{n_a^2n_b^2n_c^2}{h_a^2h_b^2h_c^2}} \le \left(\frac{R-r}{r}\right)^2$

$$\Leftrightarrow \sqrt[3]{\frac{(a^2n_a^2)(b^2n_b^2)(c^2n_c^2)}{(4r^2S^2)^3}} \le \left(\frac{R-r}{r}\right)^2 \Leftrightarrow \frac{1}{4s^2}\sqrt[3]{\prod(a^2n_a^2)} \overset{(i)}{\le} (R-r)^2$$

$GM \le AM \Rightarrow$ *LHS of (i)* $\le \frac{1}{4s^2}\left(\frac{\sum a^2n_a^2}{3}\right) = \frac{\sum\left[a^2s^2 - 4r^2S^2\left(\frac{a}{s-a}\right)\right]}{12s^2}$ *(using (a), (b), (c))*

$$= \frac{2(s^2-4Rr-r^2) - 4r^2\sum\left(\frac{a-s+s}{s-a}\right)}{12}$$

$$= \frac{s^2-4Rr-r^2-2r^2\left(-3+\frac{s}{r^2s}\sum(s-b)(s-c)\right)}{6}$$

$$= \frac{s^2-4Rr-r^2-2r^2\left(-3+\frac{4Rr+r^2}{r^2}\right)}{6} = \frac{s^2-4Rr-r^2-2r(4R-2r)}{6}$$

$$= \frac{s^2-12Rr+3r^2}{6} \Rightarrow \textit{LHS of (i)} \overset{(ii)}{\le} \frac{s^2-12Rr+3r^2}{6}$$

(ii)⇒ in order to prove (i), it suffices to prove:

$$s^2 - 12Rr + 3r^2 \le 6(R^2 - 2Rr + r^2) \Leftrightarrow s^2 \le 6R^2 + 3r^2$$

$$\Leftrightarrow (s^2 - 4R^2 - 4Rr - 3r^2) - 2R(R-2r) \le 0$$

$$\rightarrow \textit{true} \because s^2 - 4R^2 - 4Rr - 3r^2 \overset{Gerretsen}{\le} 0 \textit{ and } R - 2r \overset{Euler}{\ge} 0$$

$$\Rightarrow (i) \Rightarrow (1) \textit{ is true} \Rightarrow R \ge r\left(1 + \sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\right)$$

Also, ∵ perpedincular distance from a vertex to opposite side is least among all line segments from that vertex to opposite side, ∴ $n_a \ge h_a$ *etc*

$$\Rightarrow \sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}} \ge 1 \Rightarrow r\left(1 + \sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\right) \ge 2r \quad \textit{(Proved)}$$

1336. In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\le\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution 1 by Marian Ursărescu-Romania

First, we prove: $\cos A\cos B\cos C\le\frac{1}{2}\left(\frac{r}{R}\right)^2$ *(1)*

Because $\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}, s=\frac{a+b+c}{2}\Rightarrow$

We must show: $\frac{s^2-(2R+r)^2}{4R^2}\le\frac{r^2}{2R^2}\Leftrightarrow$

$s^2-4R^2-4Rr-r^2\le 2r^2\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$*, which it is true, because it is Gerretsen inequality. From (1) we must show:*

$$4\left(\frac{r}{R}\right)^2\le\left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2\Leftrightarrow\frac{ab+bc+ac}{a^2+b^2+c^2}\ge\frac{2r}{R}\quad (2)$$

But $ab+ab+ac=s^2+r^2+4Rr$ *(3)*

and $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ *(4)*

From (2)+(3)+(4) we must show:

$$\frac{s^2+r^2+4Rr}{2(s^2-r^2-4Rr)}\ge\frac{2r}{R}\Leftrightarrow R(s^2+r^2+4Rr)\ge 4r(s^2-r^2-4Rr)\quad (5)$$

Now, using Gerretsen's inequality:

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2\quad (6)$$

From (5)+(6), we must show:

$$R(20Rr-4r^2)\ge 4r(4R^2+2r^2)\Leftrightarrow R(5Rr-r^2)\ge r(4R^2+2r^2)$$

$$\Leftrightarrow 5R^2r-Rr^2\ge 4R^2r+2r^3\Leftrightarrow R^2r\ge Rr^2+2r^3\Leftrightarrow$$

$$\Leftrightarrow R^2\ge Rr+2r^2\Leftrightarrow (R-2r)(R+r)\ge 0\text{, true (Euler).}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sum ab}{\sum a^2}\right)^2\ge\frac{2r}{R}\Leftrightarrow R(s^2+4Rr+r^2)^2\ge 8r(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow R\left(s^4+(4Rr+r^2)^2+2s^2(4Rr+r^2)\right)\ge 8r\left(s^4+(4Rr+r^2)^2-2s^2(4Rr+r^2)\right)$$

$$\Leftrightarrow (R-2r)s^4+2(R+8r)(4Rr+r^2)s^2+(R-8r)(4Rr+r^2)^2\overset{(1)}{\ge} 6rs^4$$

$$\text{Now, LHS of (1)} \underset{(i)}{\overset{Gerretsen}{\geq}} (R-2r)(16Rr-5r^2)s^2+2(R+8r)(4Rr+s^2)s^2+$$

$$+(R-8r)(4Rr+r^2)^2$$

$$\text{and, RHS of (1)} \underset{(ii)}{\overset{Gerretsen}{\leq}} 6r(4R^2+4Rr+3r^2)s^2$$

(i), (ii) ⇒ in order to prove (1), it suffices to prove:

$$s^2\left((R-2r)(16Rr-5r^2)+2(R+8r)(4Rr+r^2)-6r(4R^2+4Rr+3r^2)\right)+$$

$$+(R-8r)(4Rr+r^2)^2 \geq 0 \Leftrightarrow s^2(5R+8r)+(R-8r)(4R+r)^2 \overset{(2)}{\geq} 0$$

$$\text{Now, LHS of (2)} \overset{Gerretsen}{\geq} (16Rr-5r^2)(5R+8r)+(R-8r)(4R+r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 2t^3-5t^2+5t-6 \overset{?}{\geq} 0\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)(2t+3)+9\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (2) \Rightarrow (1) \text{ is true} \Rightarrow \left(\frac{\sum ab}{\sum a^2}\right)^2 \overset{(a)}{\geq} \frac{2r}{R} \overset{?}{\geq} 8\prod \cos A$$

$$\Leftrightarrow \frac{2r}{R}-\frac{2(s^2-(2R+r)^2)}{R^2} \overset{?}{\geq} 0 \Leftrightarrow \frac{Rr-s^2+(2R+r)^2}{R^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow 4R^2+5Rr+r^2 \underset{(3)}{\overset{?}{\geq}} s^2$$

$$\text{Now, } \overset{Gerretsen}{\leq} 4R^2+4Rr+3r^2 \overset{?}{\leq} 4R^2+5Rr+r^2 \Leftrightarrow Rr \overset{?}{\geq} 2r^2 \rightarrow \text{true (Euler)}$$

$$\Rightarrow \text{(3) is true} \Rightarrow \frac{2r}{R} \overset{?}{\geq} 8\prod \cos A \therefore (a) \Rightarrow \left(\frac{\sum ab}{\sum a^2}\right)^2 \geq 8\prod \cos A \text{ (Proved)}$$

1337. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \leq \frac{1}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbie

$$\text{Substitutions: } \frac{1}{a+b}=x, \frac{1}{b+c}=y, \frac{1}{c+a}=z$$

$$WLOG\ x \geq y \geq z$$

$$\text{Given inequality becomes } (x+y-z)(y+z-x)(x+z-y) \leq xyz$$

$$(x-y)^2 \ge 0 \Leftrightarrow z^2 + (x-y)^2 \ge z^2 \Leftrightarrow z^2 \ge z^2 - (x-y)^2 = (x+z-y)(y+z-x).$$

Similarly $y^2 \ge (x+y-z)(y+z-x); x^2 \ge (x+y-z)(x+z-y)$

$$x^2y^2z^2 \ge (x+y-z)^2(y+z-x)^2(x+z-y)^2 \Leftrightarrow$$

$$\Leftrightarrow xyz \ge \left|\underbrace{(x+y-z)}_{>0}\underbrace{(y+z-x)}_{>0}\underbrace{(x+z-y)}_{>0}\right|$$

Triangle's rule

Sign "=" holds for $x = y = z$.

$$x+y-z = \frac{1}{a+b} + \frac{1}{b+c} - \frac{1}{c+a} \ge \frac{4}{a+2b+c} - \frac{1}{c+a} > \frac{4}{3(a+c)} - \frac{1}{c+a} = \frac{1}{3(a+c)} > 0$$

From triangle's rule $a+c > b \Rightarrow 2(a+c) > 2b \Rightarrow 3(a+c) > a+2b+c \Rightarrow$

$$\Rightarrow \frac{1}{a+2b+c} > \frac{1}{3(a+c)}$$

$$y+z-x = \frac{1}{b+c} + \frac{1}{c+a} - \frac{1}{a+b} \ge \frac{4}{a+b+2c} - \frac{1}{a+b} > \frac{4}{3(a+b)} - \frac{1}{a+b} = \frac{1}{3(a+b)} > 0$$

From triangle's rule: $a+b > c \Rightarrow 2(a+b) > 2c \Leftrightarrow 3(a+b) > a+b+2c \Rightarrow$

$$\Rightarrow \frac{1}{a+b+2c} > \frac{1}{3(a+b)}$$

$$x+z-y = \frac{1}{a+b} + \frac{1}{c+a} - \frac{1}{b+c} \ge \frac{4}{2a+b+c} - \frac{1}{b+c} > \frac{4}{3(b+c)} - \frac{1}{b+c} = \frac{1}{3(b+c)} > 0$$

From triangle's rule $b+c \Leftrightarrow 2(b+c) > 2a \Rightarrow 3(b+c) > 2a+b+c \Rightarrow$

$$\Rightarrow \frac{1}{2ab+b+c} > \frac{1}{3(b+c)}$$

Solution 2 by Ravi Prakash-New Delhi-India

As a, b, c are the sides of a triangle, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also sides of a triangle.

Let $x = \frac{1}{b+c}, y = \frac{1}{c+a}, z = \frac{1}{a+b}$. Then $x+y-z > 0$, etc.

$$LHS = \prod_{cyc}(x+y-z)$$

$$= \sqrt{(x+y-z)(y+z-x)}\sqrt{(x+y-z)(z+x-y)}$$

$$\sqrt{(y+z-x)(z+x-y)}$$

$$\le \left[\frac{1}{2}(x+y-z+y+z-x)\right]\left[\frac{1}{2}(x+y-z+z+x-y)\right]$$

$$\times\left[\frac{1}{2}(y+z-x+z+x-y)\right]=xyz=\left(\frac{1}{b+c}\right)\left(\frac{1}{c+a}\right)\left(\frac{1}{a+b}\right)$$

Equality when triangle is equilateral. Let's assume $a\geq b\geq c, 2s=a+b+c$

$$\Rightarrow 2s-a\leq 2s-b\leq 2s-c\Rightarrow \frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$$

It is sufficient to show that: $\frac{1}{b+c}<\frac{1}{c+a}+\frac{1}{a+b}$

$$\Leftrightarrow (c+a)(a+b)<(a+b)(b+c)+(b+c)(c+a)$$

$$\Leftrightarrow bc+a(a+b+c)<ac+b(a+b+c)+ab+c(a+b+c)$$

$$\Leftrightarrow bc<a(b+c)+(a+b+c)(b+c-a)$$

$$\Leftrightarrow bc<a(b+c)+(b+c)^2-a^2\Leftrightarrow 0<a(b+c-a)+b^2+c^2+bc$$

Which is true.

Solution 3 by Soumava Chakraborty-Kolkata-India

Let $b+c=x, c+a=y, a+b=z$

Then, the proposed inequality gets transformed into:

$$\frac{1}{xyz}-\left(\frac{1}{z}+\frac{1}{x}-\frac{1}{y}\right)\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)\left(\frac{1}{y}+\frac{1}{z}-\frac{1}{x}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\frac{x^2y^2z^2-(xy+yz-zx)(yz+zx-xy)(zx+xy-yz)}{(xyz)^3}\geq 0$$

$$\Leftrightarrow \sum x^3y^3+3x^2y^2z^2\geq xyz\left(\sum x^2y+\sum xy^2\right)\to true$$

$$\because \sum m^3+3mnp\overset{Schur}{\geq}\sum m^2n+\sum mn^2, where\ m=xy, n=yz, p=zx$$

$$\therefore \prod\left(\frac{1}{a+b}+\frac{1}{b+c}-\frac{1}{c+a}\right)\leq\frac{1}{(a+b)(b+c)(c+a)}\ (proved)$$

1338. If in $\Delta ABC, a+b+c=1$ then the following relationship holds:

$$\sum_{cyc}\left(\frac{\mu^2(A)}{9}+\frac{2\mu(A)}{3\tan\frac{A}{3}}+\frac{ab}{c}\right)>7$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Let\ f(x)=\frac{x^2}{9}+\frac{2x\cot\frac{x}{3}}{3}\ \forall x\in(0,\pi)$$

$$\therefore f'(x)=\frac{6\cot\frac{x}{3}-2x\left(\csc^2\frac{x}{3}-2\right)}{9}>0\Leftrightarrow 3\cot\frac{x}{3}\overset{(1)}{>}x\left(\cot^2\frac{x}{3}-1\right)$$

$$Now,\cot^2\frac{x}{3}-1\le 0\Leftrightarrow \cot\frac{x}{3}\le 1\Leftrightarrow \frac{x}{3}\ge\frac{\pi}{4}\Leftrightarrow x\ge\frac{3\pi}{4}$$

$$\boxed{\text{Case 1}}\ \boxed{x\ge\frac{3\pi}{4}}\ \therefore \cot^2\frac{x}{3}-1\le 0\Rightarrow x\left(\cot^2\frac{x}{3}-1\right)\le 0<3\cot\frac{x}{3}\left(\because\frac{\pi}{4}\le\frac{x}{3}<\frac{\pi}{3}\right)$$

$$\Rightarrow (1)\text{is true}\Rightarrow \boxed{f'(x)>0}$$

$$\boxed{\text{Case 2}}\ \boxed{x<\frac{3\pi}{4}}\ \therefore \frac{x}{3}<\frac{\pi}{4}\Rightarrow \cot\frac{x}{3}>1\Rightarrow \cot^2\frac{x}{3}-1>0$$

$$\therefore (1)\Leftrightarrow \frac{3\cot\frac{x}{3}}{\cot^2\frac{x}{3}-1}>x\Leftrightarrow \frac{3\cot\frac{x}{3}}{\cot^2\frac{x}{3}-1}-x\overset{(2)}{>}0$$

$$Let\ g(x)=\frac{3\cot\frac{x}{3}}{\cot^2\frac{x}{3}-1}-x\ \forall x\in\left(0,\frac{3\pi}{4}\right)$$

$$\therefore g'(x)=\frac{\csc^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}+\frac{2\cot^2\frac{x}{3}\csc^2\frac{x}{3}}{\left(1-\cot^2\frac{x}{3}\right)^2}-1=\frac{\csc^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}\left(1+\frac{2\cot^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}\right)-1$$

$$=\left(\frac{1+\cot^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}\right)\left(\frac{1+\cot^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}\right)-1=\left(\frac{1+\cot^2\frac{x}{3}}{1-\cot^2\frac{x}{3}}\right)^2-1=\left(\frac{1+\frac{\cos^2\frac{x}{3}}{\sin^2\frac{x}{3}}}{1-\frac{\cos^2\frac{x}{3}}{\sin^2\frac{x}{3}}}\right)^2-1$$

$$=\frac{\left(\cos^2\frac{x}{3}+\sin^2\frac{x}{3}\right)^2}{\left(\cos^2\frac{x}{3}-\sin^2\frac{x}{3}\right)^2}-1=\frac{1}{\cos^2\left(\frac{2x}{3}\right)}-1>0$$

$$\therefore g'(x)>0\Rightarrow g(x)is\uparrow on\left(0,\frac{3\pi}{4}\right)\Rightarrow g(x)>\lim_{x\to 0^+}g(x)=\lim_{x\to 0^+}\frac{3\cot\frac{x}{3}\sin^2\frac{x}{3}}{\sin^2\frac{x}{3}\cot^2\frac{x}{3}-\sin^2\frac{x}{3}}=$$

$$=\lim_{x\to0^+}\frac{\left(\frac{3}{2}\right)2\cos\frac{x}{3}\sin\frac{x}{3}}{\cos^2\frac{x}{3}-\sin^2\frac{x}{3}}=\left(\frac{3}{2}\right)\lim_{x\to0^+}\left(\frac{\sin\frac{2x}{3}}{\cos\frac{2x}{3}}\right)$$

$$=\left(\frac{3}{2}\right)\left[\lim_{\frac{2x}{3}\to0^+}\left(\frac{\sin\frac{2x}{3}}{\frac{2x}{3}}\right)\right]\left[\lim_{\frac{2x}{3}\to0^+}\left(\frac{2x}{3}\right)\right]\left[\lim_{\frac{2x}{3}\to0^+}\sec\frac{2x}{3}\right]=\left(\frac{3}{2}\right)(1)(0)(1)=0\Rightarrow g(x)=$$

$$=\frac{3\cot\frac{x}{3}}{\cot^2\frac{x}{3}-1}-x>0\ \forall x\in\left(0,\frac{3\pi}{4}\right)\Rightarrow(2)\Rightarrow(1)\text{is true}\Rightarrow\boxed{f'(x)>0}$$

Combining both the cases,

$$f'(x)>0\ \forall x\in(0,\pi)\Rightarrow f(x)\text{is}\uparrow on\ (0,\pi)\Rightarrow$$

$$\boxed{f(x)>}\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\left(\frac{x^2}{9}\right)+\lim_{x\to0^+}\left(\frac{2x\cot\frac{x}{3}}{3}\right)$$

$$=0+3\left(\frac{2}{3}\right)\left[\lim_{\frac{2x}{3}\to0^+}\left(\frac{\frac{x}{3}}{\sin\frac{x}{3}}\right)\right]\left[\lim_{\frac{2x}{3}\to0^+}\cos\frac{x}{3}\right]=2(1)(1)=\boxed{2}$$

$$\therefore f(x)=\boxed{\frac{x^2}{9}+\frac{2x\cot\frac{x}{3}}{3}\overset{(i)}{>}2\ \forall x\in(0,\pi)}$$

$$Via\ (i),\sum\left(\frac{1}{9}.\mu^2(A)+\frac{2}{3}.\frac{\mu(A)}{\tan\frac{A}{3}}+\frac{ab}{c}\right)>$$

$$>\sum\left(2+\frac{ab}{c}\right)=6+\frac{\sum a^2b^2}{abc}\geq6+\frac{abc\sum a}{abc}\overset{\because\sum a=1}{=}6+1=7\ (\text{Proved})$$

1339. In acute ΔABC the following relationship holds:

$$\left(\sum_{cyc}\sin^2 A\right)\left(\sum_{cyc}\left(\frac{\mu(A)}{\cos A}\right)^2\right)>\frac{8\pi^2}{3}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

For the sake of simplicity let us denote $\mu(A)$ *by* A, $\mu(B)$ *by* B *and* $\mu(C)$ *by* C.

$$(\sum\sin^2 A)\left(\sum\left(\frac{\mu(A)}{\cos A}\right)^2\right)\overset{\text{reverse CBS}}{\geq}\left(\sum\frac{A\sin A}{\cos A}\right)^2=(\sum A\tan A)^2\overset{?}{>}\frac{8\pi^2}{3}\Leftrightarrow\sum A\tan A\underset{(1)}{\overset{?}{>}}\frac{2\sqrt{2}\pi}{\sqrt{3}}$$

Let $f(x)=x\tan x\ \forall x\in\left(0,\frac{\pi}{2}\right)$. *Then* $f''(x)=2\sec^2 x(x\tan x+1)>0\ \because x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f(x)$

is convex.

$$\therefore\sum A\tan A\overset{\text{Jensen}}{\geq}3\left(\frac{A+B+C}{3}\right)\tan\left(\frac{A+B+C}{3}\right)=\frac{3\pi\sqrt{3}}{3}=\sqrt{3}\pi\overset{?}{>}\frac{2\sqrt{2}\pi}{\sqrt{3}}\Leftrightarrow 3\overset{?}{>}2\sqrt{2}\Leftrightarrow 9\overset{?}{>}8\rightarrow \textit{true}$$

$$\Rightarrow \textit{(1) is true}\ \therefore\ (\sum\sin^2 A)\left(\sum\left(\frac{\mu(A)}{\cos A}\right)^2\right)>\frac{8\pi^2}{3}\ \textit{(Proved)}$$

Solution 2 by Șerban George Florin-Romania

$$(\sum\sin^2 A)\cdot\left(\sum\left(\frac{A}{\cos A}\right)^2\right)\overset{CBS}{\geq}(\sum A\cdot\tan A)^2$$

$$\mu(A)\leq\mu(B)\leq\mu(C)\Rightarrow\tan A\leq\tan B\leq\tan C$$

$$\textit{Applying Chebyshev's inequality}\Rightarrow\sum A\tan A\geq\frac{1}{3}\sum A\sum\tan A=\frac{\pi}{3}\cdot\pi\tan A$$

$$(\sum\sin^2 A)\left(\sum\left(\frac{A}{\cos A}\right)^2\right)\geq(\sum A\tan A)^2\geq\frac{\pi^2}{9}(\pi\tan A)^2\geq\frac{8\pi^2}{3}$$

$$(\pi\tan A)^2\geq 24,\tan A\tan B\tan C\geq 2\sqrt{6}$$

$$\tan A\tan B\tan C=\frac{2rs}{s^2-(2R+r)^2}\geq 2\sqrt{6},s^2-(2R+r)^2\leq\frac{rA}{\sqrt{6}}$$

$$s^2\leq(2R+r)^2+\frac{s}{\sqrt{6}}$$

Applying Mitrinovic's inequality:

$$s\leq\frac{R\sqrt{3}}{2}\Rightarrow s^2\leq\frac{3R^2}{4}\leq 4R^2+4Rr+r^2+\frac{S}{\sqrt{6}}$$

$$\Rightarrow\frac{13}{4}R^2+4Rr+r^2+\frac{S}{\sqrt{6}}\geq 0,\ \textit{true.}$$

1340. If $m\geq 0$ then in ΔABC the following relationship holds:

$$\frac{a^2\sin^{2m}A}{(\sin B\sin C)^m}+\frac{b^2\sin^{2m}B}{(\sin C\sin A)^m}+\frac{c^2\sin^{2m}C}{(\sin A\sin B)^m}\geq 36r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Adrian Popa-Romania

$$\frac{a^2\sin^{2m}A}{(\sin B\sin C)^m}+\frac{b^2\sin^{2m}B}{(\sin C\sin A)^m}+\frac{c^2\sin^{2m}C}{(\sin A\sin B)^m}\ge 36r^2\quad (1)$$

$$\frac{a}{\sin A}=2R\Rightarrow \sin A=\frac{a}{2R}\Rightarrow \sin^{2m}A=\frac{a^{2m}}{2^{2m}\cdot R^{2m}}$$

$$(\sin B\cdot\sin C)^m=\left(\frac{b}{2R}\cdot\frac{c}{2R}\right)^m=\frac{b^mc^m}{2^{2m}\cdot R^{2m}}$$

$$(1)\Leftrightarrow \frac{a^2\cdot a^{2m}}{b^m\cdot c^m}+\frac{b^2\cdot b^{2m}}{a^m\cdot c^m}+\frac{c^2\cdot c^{2m}}{a^m\cdot b^m}\ge 36r^2$$

$$\frac{a^2\cdot a^{2m}}{b^m\cdot c^m}+\frac{b^2\cdot b^{2m}}{a^m\cdot c^m}+\frac{c^2\cdot c^{2m}}{a^m\cdot b^m}\ge 3\sqrt[3]{\frac{a^2b^2c^2\cdot a^{2m}\cdot b^{2m}\cdot c^{2m}}{a^{2m}\cdot b^{2m}\cdot c^{2m}}}=3\sqrt[3]{a^2b^2c^2}=$$

$$=3\sqrt[3]{16R^2S^2}=3\sqrt[3]{16R^2s^2r^2}\ge 3\sqrt[3]{16\cdot 4r^2\cdot s^2\cdot r^2}=3\sqrt[3]{64s^2r^4}=$$

$$=3\cdot 4\sqrt[3]{s^2r^4}\overset{?}{\ge}36r^2|:12$$

$$\sqrt[3]{s^2r^4}\overset{?}{\ge}3r^2\Leftrightarrow s^2r^4\ge 27r^6|:r^4$$

$$s^2\ge 27r^2\Leftrightarrow s\ge 3\sqrt{3}r\quad (A)$$

(Mitrinovic)

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \Omega=\sum\frac{a^2\cdot\sin^{2m}A}{(\sin B\cdot\sin C)^m}\overset{AM-GM}{\ge}3\left\{a^2b^2c^2\frac{(\sin A\cdot\sin B\cdot\sin C)^{2m}}{(\sin^2A\cdot\sin^2B\cdot\sin^2C)^m}\right\}^{\frac{1}{3}}$$

$$\Rightarrow \Omega\ge 3(abc)^{\frac{2}{3}}=3(4Rrs)^{\frac{2}{3}}$$

$$\Rightarrow \Omega\ge 3\left(4\cdot 2r\cdot 3\sqrt{3}r\right)^{\frac{2}{r}}\left[\because R\ge 2r, s\ge 3\sqrt{3}r\right]$$

$$\Rightarrow \Omega\ge 3\cdot 8^{\frac{2}{3}}\cdot\left(3^{\frac{3}{2}}\right)^{\frac{2}{3}}(r^3)^{\frac{2}{3}}\Leftrightarrow \Omega\ge 36r^2\quad \textit{(proved)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS=4R^2\sum\frac{(\sin^2A)^{m+1}}{(\sin B\sin C)^m}$$

$$\overset{Radon}{\ge}4R^2\frac{(\sum\sin^2A)^{m+1}}{(\sum\sin B\sin C)^m}\ge\frac{4R^2(\sum\sin^2A)^{m+1}}{(\sum\sin^2A)^m}$$

$$=4R^2\sum\sin^2A=\sum a^2\overset{Ionescu-Weitzenbock}{\ge}4\sqrt{3}rs$$

$$\overset{Mitrinovic}{\geq}\left(4\sqrt{3}r\right)\left(3\sqrt{3}r\right)=36r^2 \text{ (proved)}$$

1341. If in $\Delta ABC, \mu(B)=2\mu(A), \mu(C)=2\mu(B)$ then the following relationship holds:

$$h_a^2+h_b^2+h_c^2>\frac{7\sqrt{21}}{10}R^2$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\pi=A+2A+4A\Rightarrow A=\frac{\pi}{7}, B=\frac{2\pi}{7} \text{ and } C=\frac{4\pi}{7}$$

$$Now, \cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{3\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{5\pi}{7}}{2\sin\frac{\pi}{7}}=$$

$$\frac{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{2\pi}{7}+\sin\frac{6\pi}{7}-\sin\frac{4\pi}{7}}{2\sin\frac{\pi}{7}}$$

$$=\frac{\sin\left(\pi-\frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}}=\frac{\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}=\frac{1}{2}\therefore \boxed{\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\overset{(1)}{=}\frac{1}{2}}$$

$$\therefore \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\left(\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)\overset{\text{by }(1)}{=}-\frac{1}{2}$$

$$\therefore \boxed{\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\overset{(2)}{=}-\frac{1}{2}}$$

$$\text{Now, }\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)^2+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=$$

$$=3+2\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+2\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}+2\cos\frac{8\pi}{7}\cos\frac{2\pi}{7}$$

$$+2\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}+2\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}+2\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}=$$

$$=3+2\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}-\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)+2\left(\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}-\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}\right)$$

$$+2\left(\cos\frac{8\pi}{7}\cos\frac{2\pi}{7}-\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}\right)=3+2\left(\cos\frac{6\pi}{7}+\cos\frac{12\pi}{7}+\cos\frac{10\pi}{7}\right)=$$

$$=3-2\left(\cos\frac{\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}\right)\overset{\text{by }(1)}{\cong}3-1=2$$

$$\therefore\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)^2+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=2$$

$$\overset{\text{by }(2)}{\Rightarrow}\frac{1}{4}+\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=2$$

$$\Rightarrow\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right)^2=\frac{7}{4}\Rightarrow\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{\sqrt{7}}{2}\Rightarrow$$

$$\Rightarrow\boxed{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}\overset{(3)}{\cong}\frac{\sqrt{7}}{2}}$$

$$\textit{Again},\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)=$$

$$=\frac{\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\left(2\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}\right)}{8}=\frac{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{6\pi}{7}}{8}$$

$$=\frac{\left(\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{\pi}{7}\right)}{8}\Rightarrow\boxed{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\overset{(4)}{\cong}\frac{1}{8}}$$

$$\textit{Also},\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=$$

$$=\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right)$$

$$=1+\frac{1}{2}\left(2\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+2\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+2\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$$

$$=1+\frac{1}{2}\left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}$$

$$\overset{\text{by }(4)}{\Longleftrightarrow} 1+\frac{1}{2}\left(-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\frac{1}{8}$$

$$=\frac{7}{8}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}=\frac{7}{8}$$

$$\Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\sqrt{\frac{7}{64}}\ \therefore\ \boxed{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\overset{(5)}{=}\frac{\sqrt{7}}{8}}$$

$$\textit{Moreover,}\ \frac{1}{\sin\frac{2\pi}{7}}+\frac{1}{\sin\frac{3\pi}{7}}=\frac{\sin\frac{3\pi}{7}+\sin\frac{5\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}=\frac{2\sin\frac{4\pi}{7}\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{4\pi}{7}}=\frac{1}{\sin\frac{\pi}{7}}$$

$$\Rightarrow\left(\frac{1}{\sin\frac{2\pi}{7}}+\frac{1}{\sin\frac{3\pi}{7}}-\frac{1}{\sin\frac{\pi}{7}}\right)^2=0$$

$$\Rightarrow\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{3\pi}{7}}+\frac{1}{\sin^2\frac{\pi}{7}}+$$

$$+2\left(\frac{1}{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}-\frac{1}{\sin\frac{3\pi}{7}\sin\frac{\pi}{7}}-\frac{1}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}\right)=0$$

$$\Rightarrow\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{3\pi}{7}}+\frac{1}{\sin^2\frac{\pi}{7}}-\left(\frac{2}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}}\right)\left(\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}\right)=0$$

$$\overset{\text{by }(3)\text{and }(5)}{\Longleftrightarrow}\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{3\pi}{7}}+\frac{1}{\sin^2\frac{\pi}{7}}-\left(\frac{16}{\sqrt{7}}\right)\left(\frac{\sqrt{7}}{2}\right)=0$$

$$\Rightarrow\boxed{\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{3\pi}{7}}+\frac{1}{\sin^2\frac{\pi}{7}}\overset{(6)}{=}8}$$

$$\text{Now, } \sum h_a^2 = \sum\left(\frac{bc}{2R}\right)^2 = \sum\left(\frac{4R^2\sin B\sin C}{2R}\right)^2 = 4R^2\sum\sin^2 B\sin^2 C$$

$$= 4R^2\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2\left(\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{3\pi}{7}}+\frac{1}{\sin^2\frac{\pi}{7}}\right)$$

$$\overset{\text{by (5) and (6)}}{=} 4R^2\left(\frac{56}{64}\right) = \frac{7R^2}{2} \therefore \boxed{LHS = \frac{7R^2}{2}} > \frac{7\sqrt{21}R^2}{10} \ (\textit{Proved})$$

1342. In ΔABC the following relationship holds:

$$\left(\frac{4}{(b+c)^2}+\frac{9}{(c+a)^2}+\frac{1}{(a+b)^2}\right)\left(\frac{9}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{4}{(a+b)^2}\right) > 49\sum\frac{1}{(a+b)^2(b+c)^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{36}{(b+c)^4}+\frac{4}{(b+c)^2(c+a)^2}+\frac{16}{(a+b)^2(b+c)^2}+\frac{81}{(b+c)^2(c+a)^2}$$
$$+\frac{9}{(c+a)^4}+\frac{36}{(c+a)^2(a+b)^2}+\frac{9}{(a+b)^2(b+c)^2}$$
$$+\frac{1}{(c+a)^2(a+b)^2}+\frac{4}{(a+b)^4} > RHS =$$
$$=\frac{49}{(a+b)^2(b+c)^2}+\frac{49}{(b+c)^2(c+a)^2}+\frac{49}{(c+a)^2(a+b)^2}$$

$$\Leftrightarrow\left(\frac{36}{(b+c)^4}+\frac{4}{(a+b)^4}-\frac{24}{(a+b)^2(b+c)^2}\right)+\frac{36}{(b+c)^2(c+a)^2}+\frac{9}{(c+a)^4} > \frac{12}{(c+a)^2(a+b)^2}$$

$$\Leftrightarrow\left(\frac{6}{(b+c)^2}-\frac{2}{(a+b)^2}\right)^2+\left(\frac{3}{(c+a)^2}\right)\left(\frac{12}{(b+c)^2}+\frac{3}{(c+a)^2}-\frac{4}{(a+b)^2}\right)\overset{(1)}{\gtrsim} 0$$

$$(1)\Rightarrow \textit{it suffices to prove:}\ \frac{12}{(b+c)^2}+\frac{3}{(c+a)^2}-\frac{4}{(a+b)^2}\overset{(2)}{\gtrsim} 0$$

$$\textit{Let } s-a=x, s-b=y \textit{ and } s-c=z \therefore s=x+y+z$$

$$\Rightarrow a=y+z, b=z+x, c=x+y \textit{ and using this substitution,}$$

$$(2)\Leftrightarrow\frac{12}{(2x+y+z)^2}+\frac{3}{(2y+z+x)^2}-\frac{4}{(2z+x+y)^2}>0$$

$$\Leftrightarrow 12(2y+z+x)^2(2z+x+y)^2+3(2z+x+y)^2(2x+y+z)^2-$$

$$-4(2x+y+z)^2(2y+z+x)^2>0$$

$$\Leftrightarrow 8x^4+28x^3y+84x^3z+63x^2y^2+294x^2yz+203x^2z^2+82xy^3+420xy^2z$$

$$+518xyz^2+180xz^3+35y^4+210y^3z+383y^2z^2$$

$$+252yz^3+56z^4>0\to \boldsymbol{true}\Rightarrow(2)\Rightarrow(1)$$

$$\Rightarrow \boldsymbol{proposed\ inequality\ is\ true\ (Proved)}$$

1343. In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\sum_{cyc}(h_a-2r)\sqrt{\frac{R_a}{AI}}\le(r+R)\sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\angle BIC=\pi-\left(\frac{B+C}{2}\right)=\pi-\left(\frac{\pi-A}{2}\right)=\frac{\pi}{2}+\frac{A}{2}$$

Using sine rule on Δ BIC, $2R_a\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)=4R\sin\frac{A}{2}\cos\frac{A}{2}\Rightarrow R_a\overset{(a)}{\triangleq}2R\sin\frac{A}{2}$

Similarly, $R_b\overset{(b)}{\triangleq}2R\sin\frac{B}{2}$ and $R_c\overset{(c)}{\triangleq}2R\sin\frac{C}{2}$. *Also,*

$$b+c-a=4R\cos\frac{A}{2}\cos\frac{B-C}{2}-4R\sin\frac{A}{2}\cos\frac{A}{2}=$$

$$=4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)=8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow s-a\overset{(i)}{\triangleq}4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Similarly, $s-b\overset{(ii)}{\triangleq}4R\cos\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$ and $s-c\overset{(iii)}{\triangleq}4R\cos\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}$

Using (a), (b), (c):

$$\sum(h_a-2r)\sqrt{\frac{R_a}{AI}}=\sum\left(\frac{2rs}{a}-2r\right)\sqrt{\frac{2R\sin^2\frac{A}{2}}{r}}=$$

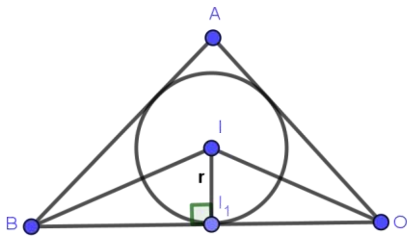
$$= 2r\sqrt{\frac{2R}{r}}\sum\left(\frac{s-a}{a}\sin\frac{A}{2}\right)\overset{\text{by (i),(ii),(iii)}}{\cong} 2r\sqrt{\frac{2R}{r}}\sum\left(\frac{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\sin\frac{A}{2}\right)$$

$$= 2r\sqrt{\frac{2R}{r}}\left(\frac{r}{4R}\right)\sum\operatorname{cosec}\frac{A}{2} = \sqrt{\frac{2R}{r}}\left(\frac{r^2}{2R}\right)\sum\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} =$$

$$= \sqrt{\frac{2R}{rs}}\left(\frac{r}{2R}\right)\sum\sqrt{bc(s-a)}\overset{\text{CBS}}{\leq}\sqrt{\frac{2R}{rs}}\left(\frac{r}{2R}\right)\sqrt{\sum ab}\sqrt{\sum(s-a)} = \sqrt{\frac{2R}{r}}\left(\frac{r}{2R}\right)\sqrt{s^2+4Rr+r^2}$$

$$\overset{\text{Gerretsen}}{\leq}\sqrt{\frac{r}{2R}}\sqrt{4R^2+8Rr+4r^2} = 2(R+r)\sqrt{\frac{r}{2R}} = (r+R)\sqrt{\frac{2r}{R}}\ \textit{(Proved)}$$

Solution 2 by Șerban George Florin-Romania



$$AI = \frac{r}{\sin\frac{A}{2}}, \sigma_{BIC} = \frac{a\cdot r}{2}$$

$$R_a = \frac{BI\cdot CI\cdot BC}{4\sigma_{BIC}} = \frac{\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}\cdot a}{2ar}$$

$$R_a = \frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2}}, \frac{R_a}{AI} = \frac{r}{2\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\frac{\sin\frac{A}{2}}{r} = \frac{\sin\frac{A}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\frac{R_a}{AI} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{2\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}} = \frac{a}{2(s-a)}, \sqrt{\frac{R_a}{AI}} = \frac{\sqrt{a}}{\sqrt{2}\sqrt{s-a}}$$

$$h_a - 2r = \frac{2S}{a} - \frac{2S}{s} = \frac{2S(s-a)}{as}, (h_a-2r)\sqrt{\frac{R_a}{AI}} = \frac{2S(s-a)}{as}\cdot\frac{\sqrt{a}}{\sqrt{2}\cdot\sqrt{s-a}} =$$

$$= \sqrt{2}\cdot r\sqrt{\frac{s-a}{a}} \Rightarrow \sum (h_a - 2r)\sqrt{\frac{R_a}{AI}} = \sqrt{2}R\cdot\sum\sqrt{\frac{s-a}{a}} \le (r+R)\sqrt{\frac{2r}{R}}$$

$$\Rightarrow \sum\sqrt{\frac{s-a}{a}} \le \frac{r+R}{R}\sqrt{\frac{r}{R}}, \left(\sum\sqrt{\frac{s-a}{a}}\right)^2 \overset{CBS}{\le} 3\cdot\sum\left(\sqrt{\frac{s-a}{a}}\right)^2 \le$$

$$\le \frac{(r+R)^2\cdot r}{r^2\cdot R}, 3\sum\frac{s-a}{a} \le \frac{R^2+2Rr+r^2}{Rr}$$

$$\Rightarrow 3\cdot\frac{s^2+r^2-8Rr}{4Rr} \le \frac{R^2+2Rr+r^2}{Rr}, 3s^2+3r^2-24Rr \le 4R^2+8Rr+4r^2$$

$$3s^2 \le 4R^2+32Rr+r^2 \Rightarrow s^2 \le \frac{4R^2+32Rr+r^2}{3}$$

Applying Mitrinovic's inequality: $s \le \frac{R\sqrt{3}}{2} \Rightarrow s^2 \le \frac{R\sqrt{3}}{2} \Rightarrow s^2 \le \frac{3R^2}{4}$

$$\Rightarrow s^2 \le \frac{3R^2}{4} \le \frac{4R^2+32Rr+r^2}{3} \Rightarrow 9R^2 \le 16R^2+128Rr+4r^2$$

$$\Rightarrow 7R^2+128Rr+4r^2 \ge 0, \textit{ true.}$$

1344. Let $\Delta A'B'C'$ be the orthic triangle of acute $\Delta ABC, H$ – orthocenter.

Prove that:

$$\sum_{cyc}\left(\frac{AH}{B'C'}\right)^n\cdot\sum_{cyc}\left(\frac{a^2}{r_br_c}\right)^m \ge \frac{1}{9}\left(\frac{2}{\sqrt{3}}\right)^n\cdot\left(\frac{4}{3}\right)^m, m,n \ge 2$$

Proposed by Radu Diaconu-Romania

Solution by Șerban George Florin-Romania

$$\frac{AH}{B'C'} = \frac{2R\cos A}{a\cos A} = \frac{2R}{2R\sin A} = \frac{1}{\sin A}$$

$$\sum_{cyc}\left(\frac{AH}{B'C'}\right)^n \overset{Holder}{\ge} \frac{\left(\sum\frac{AH}{B'C'}\right)^n}{3^{n-1}} = \frac{\left(\sum\frac{1}{\sin A}\right)^n}{3^{n-1}} \ge \frac{(2\sqrt{3})^n}{3^n\cdot 3^{-1}} = \left(\frac{2\sqrt{3}}{3}\right)^n\cdot 3 = \left(\frac{2}{\sqrt{3}}\right)^n\cdot 3$$

$$\sum_{cyc}\left(\frac{a^2}{r_br_c}\right)^m \overset{Holder}{\ge} \frac{\left(\sum\frac{a^2}{r_br_c}\right)^m}{3^{m-1}} \overset{Bergstrom}{\ge} \frac{\left[\frac{(a+b+c)^2}{\sum r_br_c}\right]^m}{3^{m-1}} = \frac{\left(\frac{4s^2}{s^2}\right)^m}{3^m}\cdot 3 = \left(\frac{4}{3}\right)^m\cdot 3$$

$$\Rightarrow \sum_{cyc}\left(\frac{AH}{B'C'}\right)^n \cdot \sum_{cyc}\left(\frac{a^2}{r_b r_c}\right)^m \ge \left(\frac{2}{\sqrt{3}}\right)^n \cdot 3 \cdot \left(\frac{4}{3}\right)^m \cdot 3 =$$

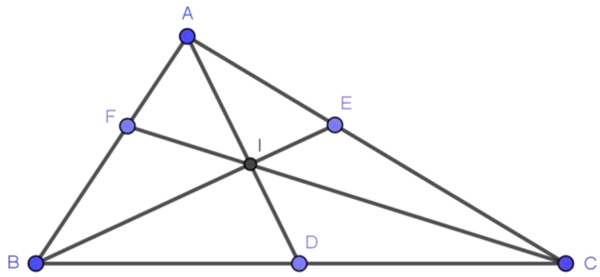
$$= 9 \cdot \left(\frac{2}{\sqrt{3}}\right)^n \cdot \left(\frac{4}{3}\right)^m > \frac{1}{9} \cdot \left(\frac{2}{\sqrt{3}}\right)^n \cdot \left(\frac{4}{3}\right)^m, \textit{true.}$$

1345. If in $\Delta ABC, AD, BE, CF$ – internal bisectors, I – incenter then the following relationship holds:

$$\left(\sum_{cyc} r_a^{2n}\right)\left(\sum_{cyc}\left(\frac{DI}{AI}\right)^m\right) \ge \frac{9 \cdot s^{2n}}{2^m \cdot 3^n}, m, n \ge 2$$

Proposed by Radu Diaconu-Romania

Solution by Marian Ursărescu-Romania



From Hölder's inequality $\Rightarrow$

$$(r_a^2)^n + (r_b^2)^n + (r_c^2)^n \ge \frac{(r_a^2+r_b^2+r_c^2)^n}{3^{n-1}} \quad (1)$$

$$\textit{But } r_a^2 + r_b^2 + r_c^2 \ge s^2 \ \textit{(Bokov's inequality)} \quad (2)$$

$$\textit{From (1)+(2)} \Rightarrow \sum r_a^{2n} \ge \frac{s^{2n}}{3^{n-1}} \quad (3)$$

$$\Delta ABD \Rightarrow \frac{DI}{AI} = \frac{BD}{AB} = \frac{BD}{c}$$

$$\textit{But } \Delta ABC: \frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD}{a} = \frac{c}{b+c} \Rightarrow BD = \frac{ac}{b+c} \Rightarrow$$

$$\frac{DI}{AI} = \frac{a}{b+c}. \textit{ Again Hölder's inequality} \Rightarrow$$

$$\sum\left(\frac{DI}{AI}\right)^m \ge \frac{\left(\sum\frac{DI}{AI}\right)^m}{3^{m-1}} = \frac{\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)^m}{3^{m-1}} \quad (4)$$

$$\textit{From Nesbitt inequality} \Rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} \quad (5)$$

$$\text{From (4)+(5)}\Rightarrow \sum\left(\frac{DI}{AI}\right)^m \geq \frac{1}{3^{m-1}}\left(\frac{3}{2}\right)^m = \frac{3}{2^m} \quad (6)$$

$$\text{From (3)+(6)}\Rightarrow \left(\sum r_a^{2n}\right)\cdot\sum\left(\frac{DI}{AI}\right)^m \geq 9\cdot\frac{s^{2n}}{3^n\cdot 2^m}$$

1346. In $\Delta ABC, O$ – circumcentre, I – incentre the following relationship holds:

$$(w_a-w_b)^2+(w_b-w_c)^2+(w_c-w_a)^2 \leq n\cdot OI^2, n\geq\frac{35}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show:

$$w_a^2+w_b^2+w_c^2-(w_aw_b+w_bw_c+w_aw_c)\leq\frac{n}{2}OI^2 \quad (1)$$

$$\text{But } OI^2 = R^2-2Rr \quad (2)$$

$$w_a\leq\sqrt{s(s-a)}, s=\frac{a+b+c}{2}\Rightarrow w_a^2+w_a^2+w_c^2\leq$$

$$s(s-a)+s(s-b)+s(s-c)=s^2 \quad (3)$$

$$w_aw_b+w_bw_c+w_aw_c\geq h_ah_b+h_bh_c+h_ah_c=\frac{2s^2r}{R} \quad (4)$$

From (1)+(2)+(3)+(4) we must show:

$$s^2-\frac{2s^2r}{R}\leq n(R^2-2Rr)\Leftrightarrow\frac{s^2(R-2r)}{R}\leq\frac{nR}{2}(R-2r)\Leftrightarrow$$

$$s^2\leq\frac{n}{2}R^2 \quad (5)$$

(From Euler $R\geq 2r$)

From Mitrinovic's inequality $s^2\leq\frac{27}{4}R^2\leq\frac{n}{2}R^2\Leftrightarrow 27\leq 2n$, true because

$2n\geq 35\Rightarrow$ (5) it is true.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum {w_a}^2=\sum\frac{4bcs(s-a)}{(b+c)^2}=\sum\frac{bc(b+c-a)(b+c+a)}{(b+c)^2}=\sum\frac{bc((b+c)^2-a^2)}{(b+c)^2}=$$

$$\sum ab-\sum\frac{a^2bc}{\left(4R\cos\frac{A}{2}\cos\frac{B-C}{2}\right)^2}\leq\sum ab-\sum\frac{a^2bc}{16R^2\frac{s(s-a)}{bc}}$$

$$\left(\because 0 \le \cos\frac{B-C}{2} \le 1 \text{ as } \frac{B-C}{2} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right) =$$

$$= s^2 + 4Rr + r^2 - \frac{16R^2r^2s^2}{16R^2s.r^2s}\sum(s-b)(s-c) =$$

$$= s^2 + 4Rr + r^2 - (4Rr + r^2) = s^2 \therefore \sum {w_a}^2 \overset{(1)}{\lesssim} s^2$$

$$\text{Now, } 2\sum w_a w_b = 2\left(\prod w_a\right)\sum\frac{1}{w_a} \overset{\text{Berstrom}}{\geq} 2\left(\prod w_a\right)\frac{9}{\sum w_a} = \frac{18\left(\prod w_a\right)}{\sum m_a} \ge \frac{18\left(\prod w_a\right)}{4R+r} =$$

$$= \left(\frac{18}{4R+r}\right)\prod\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right) = \left(\frac{18}{4R+r}\right)\frac{128R^2r^2s^2\left(\frac{s}{4R}\right)}{2s(s^2+2Rr+r^2)}$$

$$= \frac{288Rr^2s^2}{(4R+r)(s^2+2Rr+r^2)} \therefore 2\sum w_a w_b \overset{(2)}{\geq} \frac{288Rr^2s^2}{(4R+r)(s^2+2Rr+r^2)}$$

$$(1),(2) \Rightarrow \sum(w_a - w_b)^2 = 2\sum {w_a}^2 - 2\sum w_a w_b \le 2s^2 - \frac{288Rr^2s^2}{(4R+r)(s^2+2Rr+r^2)} =$$

$$= \frac{2s^2(4R+r)(s^2+2Rr+r^2) - 288Rr^2s^2}{(4R+r)(s^2+2Rr+r^2)}$$

$$= \frac{2(4R+r)s^4 + s^2\{(4Rr+2r^2)(4R+r) - 288Rr^2\}}{(4R+r)(s^2+2Rr+r^2)} \overset{\text{Gerretsen}}{\leq}$$

$$\le \frac{s^2\{2(4R+r)(4R^2+4Rr+3r^2) + (4Rr+2r^2)(4R+r) - 288Rr^2\}}{(4R+r)(s^2+2Rr+r^2)}$$

$$= \frac{(32R^3+56R^2r - 244Rr^2 + 8r^3)s^2}{(4R+r)(s^2+2Rr+r^2)} = \frac{4(R-2r)(8R^2+30Rr-r^2)s^2}{(4R+r)(s^2+2Rr+r^2)} \overset{?}{\leq} n.OI^2 =$$

$$= nR(R-2r) \Leftrightarrow nR(4R+r)(s^2+2Rr+r^2) \overset{?}{\geq} 4(8R^2+30Rr-r^2)s^2$$

$$\Leftrightarrow s^2(n.4R^2 + n.Rr - 32R^2 - 120Rr + 4r^2) + nR(4R+r)(2Rr+r^2) \underset{(a)}{\overset{?}{\geq}} 0$$

$$\because n \ge \frac{35}{2} \therefore \text{LHS of (a)} \ge$$

$$\ge s^2\left(70R^2 + \frac{35}{2}Rr - 32R^2 - 120Rr + 4r^2\right) + \left(\frac{35}{2}\right)R(4R+r)(2Rr+r^2) \overset{?}{>} 0$$

$$\Leftrightarrow s^2(R-2r)(76R-53r) + 35R(4R+r)(2Rr+r^2) \underset{(b)}{\overset{?}{>}} 98r^2s^2$$

$$\textit{Gerretsen} \Rightarrow \text{LHS of (b)} \overset{(i)}{\geq} (16Rr - 5r^2)(R - 2r)(76R - 53r) + 35R(4R + r)(2Rr + r^2)$$

$$\textit{and RHS of (b)} \overset{(ii)}{\leq} 98r^2(4R^2 + 4Rr + 3r^2)$$

(i), (ii) ⇒ in order to prove (b), it suffices to prove:

$$(16Rr - 5r^2)(R - 2r)(76R - 53r) + 35R(4R + r)(2Rr + r^2) >$$
$$> 98r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 748t^3 - 1921t^2 + 1182t - 412 > 0 \left(\text{where } t = \frac{R}{r}\right) \Leftrightarrow$$

$$\Leftrightarrow (t - 2)(748t(t - 2) + 1071t + 332) + 252 > 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

⇒ (b) ⇒ (a) ⇒ proposed inequality is true (Proved)

1347. In acute ΔABC the following relationship holds:

$$\left(\sum_{cyc} a^m \sin^m A\right)\left(\sum_{cyc}\left(\frac{a}{s-a}\right)^q\right) \geq \frac{9 \cdot 2^{m+q} \cdot s^{m+n}}{3^{m+n} \cdot R^n}, m, n, q \geq 2$$

Proposed by Radu Diaconu-Romania

Solution by Șerban George Florin-Romania

$$a \leq b \leq c \Rightarrow \sin A \leq \sin B \leq \sin C$$

$$a^m \leq b^m \leq c^m \Rightarrow \sin^n A \leq \sin^n B \leq \sin^n C$$

$$\sum_{cyc} a^m \sin^n A \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^m\right)\left(\sum \sin^n A\right) \overset{Holder}{\geq} \frac{1}{3}\frac{(\sum a)^m}{3^{m-1}} \cdot \frac{(\sum \sin A)^n}{3^{n-1}}$$

$$= \frac{(2s)^m \cdot \left(\frac{a+b+c}{2R}\right)^n}{3^{m+n-1}} = \frac{2^m \cdot s^m \cdot 2^n \cdot s^n}{2^n \cdot R^n \cdot 3^{m+n-1}} = \frac{2^m \cdot s^{m+n}}{R^n \cdot 3^{m+n-1}}$$

$$\left(\sum_{cyc}\left(\frac{a}{s-a}\right)^q\right) \overset{Holder}{\geq} \frac{\left(\sum \frac{a}{s-a}\right)^q}{3^{q-1}} = \frac{\left[\frac{2(2R-r)}{r}\right]^q}{3^{q-1}} \geq \frac{6^q}{3^{q-1}} = 3 \cdot 2^q$$

$$\textit{Because (Euler)} \frac{2(2R-r)}{r} \geq 6 \Rightarrow 4R - 2r \geq 6r \Rightarrow 4R \geq 8r, R \geq 2r$$

$$\Rightarrow \left(\sum_{cyc} a^m \sin^n A\right) \cdot \left(\sum \left(\frac{a}{s-a}\right)^q\right) \geq \frac{2^m \cdot s^{m+n}}{R^n \cdot 3^{m+n-1}} \cdot 3 \cdot 2^q = \frac{2^{m+q} \cdot s^{m+n} \cdot 9}{R^n \cdot 3^{m+n}}, \textit{ true.}$$

1348. If in $\Delta ABC, \mu(B) = 2\mu(A), \mu(C) = 2\mu(B)$ then the following relationship holds:

$$(a^2 + b^2 + c^2)(a^4 + b^4 + c^4) > 15a^2b^2c^2$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

$$\Delta ABC: \widehat{B} = 2\widehat{A};\ \widehat{C} = 2\widehat{B}$$

$$\widehat{A} + \widehat{B} + \widehat{C} = \pi \Rightarrow \widehat{A} + 2\widehat{A} + 4\widehat{A} = \pi$$

$$\widehat{A} = \frac{\pi}{7}; \widehat{B} = \frac{2\pi}{7};\ \widehat{C} = \frac{4\pi}{7}$$

$$(a^2 + b^2 + c^2)(a^4 + b^4 + c^4) > 15a^2b^2c^2$$

$$\frac{a}{\sin A} + \frac{b}{\sin B} + \frac{c}{\sin C} = 2R \Rightarrow a = 2R\sin\frac{\pi}{7}$$

$$b = 2R\sin\frac{2\pi}{7}$$

$$c = 2R\sin\frac{4\pi}{7}$$

$$(a^2 + b^2 + c^2)(a^4 + b^4 + c^4) \overset{Bergstrom}{\geq} (a^2 + b^2 + c^2)\left(\frac{(a^2 + b^2 + c^2)^2}{3}\right) =$$

$$= \frac{(a^2 + b^2 + c^2)^3}{3} = \frac{\left(4R^2\left(\sin\frac{2\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7}\right)\right)^3}{3} =$$

$$= \frac{64R^6 \cdot \left(\frac{7}{4}\right)^3}{3} = \frac{343R^6}{3}$$

$$15a^2b^2c^2 = 15 \cdot 4R^2\sin^2\frac{\pi}{7} \cdot 4R^2\sin^2\frac{2\pi}{7} \cdot 4R^2\sin^2\frac{4\pi}{7} =$$

$$= R^6 \cdot 15 \cdot 4^3\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2 = 15 \cdot 4^3 \cdot \frac{7}{64} = 105R^6$$

$$\frac{343}{3}R^6 \overset{?}{>} 105R^6 |: R^6 \Rightarrow 343 > 3 \cdot 105$$

$$343 > 315 \quad (True)$$

We must prove two relationships that we've used here:

$$\textit{1)}\ \sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{9}{4}$$

$$\textit{2)}\ \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}=\frac{\sqrt{7}}{8}$$

$$\textit{1)}\ \sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{1-\cos\frac{2\pi}{7}}{2}+\frac{1-\cos\frac{4\pi}{7}}{2}+\frac{1-\cos\frac{8\pi}{7}}{2}=$$

$$=\frac{3}{2}-\frac{\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}}{2}$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{3\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{5\pi}{7}}{2\sin\frac{\pi}{7}}=$$

$$=\frac{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{2\pi}{7}+\sin\frac{6\pi}{7}-\sin\frac{4\pi}{7}}{2\sin\frac{\pi}{7}}=\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}}=\frac{\sin\left(\pi-\frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}}=\frac{1}{2}$$

$$\Rightarrow\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\left(\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}\right)=-\frac{1}{2}\Rightarrow$$

$$\Rightarrow\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{3}{2}+\frac{1}{4}=\frac{7}{4}$$

$$\textit{2)}\ \left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)=$$

$$=\frac{\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\left(2\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}\right)}{8}=$$

$$=\frac{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{6\pi}{7}}{8}=\frac{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{\pi}{7}}{8}\Rightarrow$$

$$\Rightarrow\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}$$

Now:

$$\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right)=$$

$$=1+\frac{1}{2}\left(2\cos\frac{2\pi}{7}\cos\frac{4\pi}{2}+2\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+2\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}=$$

$$=1+\frac{1}{2}\left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\underbrace{\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}}_{\frac{1}{8}}$$

$$\left(2\sin^2\frac{\pi}{7}\right)\left(2\sin^2\frac{2\pi}{7}\right)\left(2\sin^2\frac{3\pi}{7}\right)=$$

$$=1+\frac{1}{2}\left(-\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)-$$

$$-\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}-\cos\frac{6\pi}{7}-\frac{1}{8}=$$

$$=\frac{7}{8}+\cos\frac{2\pi}{7}-\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}=\frac{7}{8}$$

$$\textit{So, } 2\sin^2\tfrac{\pi}{7}\cdot 2\sin^2\tfrac{2\pi}{7}\cdot 2\sin^2\tfrac{3\pi}{7}=\tfrac{7}{8}\Rightarrow$$

$$\Rightarrow \sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7}=\frac{7}{64}\Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8}$$

1349. In ΔABC the following relationship holds:

$$\max\{\mu^2(A),\mu^2(B),\mu^2(C)\}+\sum_{cyc}\left(\sin^2\frac{A}{2}+\frac{1}{2}\mu(A)\tan\frac{A}{2}\right)\ge\frac{5\pi^2}{18}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Șerban George Florin-Romania

$$\max\left(\mu^2(A),\mu^2(B),\mu^2(C)\right)\ge\mu^2(A),\max\left(\mu^2(A),\mu^2(B),\mu^2(C)\right)\ge\mu^2(B)$$

$$\max\left(\mu^2(A),\mu^2(B),\mu^2(C)\right)\ge\mu^2(C)\Rightarrow\max\left(\mu^2(A),\mu^2(B),\mu^2(C)\right)\ge$$

$$\ge\frac{\mu^2(A)+\mu^2(B)+\mu^2(C)}{3},(x+y+z)^2\overset{CBS}{\le}3(x^2+y^2+z^2)$$

$$\Rightarrow x^2+y^2+z^2\ge\frac{(x+y+z)^2}{3}$$

$$\max\left(\mu^2(A),\mu^2(B),\mu^2(C)\right)\ge\frac{1}{3}\sum\mu^2(A)\overset{CBS}{\ge}\frac{1}{9}\left(\sum\mu(A)\right)^2=\frac{\pi^2}{9}$$

$$\sum\sin^2\frac{A}{2}=\sum\frac{1-\cos A}{1}=\frac{3}{2}-\frac{1}{2}\sum\cos A=\frac{3}{2}-\frac{1}{2}\left(1+\frac{r}{R}\right)$$

$$\tfrac{r}{R}\le\tfrac{1}{2}\textit{(Euler)},\ 1+\tfrac{r}{R}\le\tfrac{3}{2},-\tfrac{1}{2}\left(1+\tfrac{r}{R}\right)\ge-\tfrac{3}{4}$$

$$\sum \sin^2\frac{A}{2} \geq \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$$

$$\mu(A) \leq \mu(B) \leq \mu(C) \Rightarrow \tan\frac{\widehat{A}}{2} \leq \tan\frac{\widehat{B}}{2} \leq \tan\frac{\widehat{C}}{2}$$

Applying Chebyshev's inequality:

$$\frac{1}{2}\sum \mu(A)\tan\frac{A}{2} \geq \frac{1}{2}\cdot\frac{1}{3}\left(\sum \mu(A)\right)\cdot\left(\sum \tan\frac{A}{2}\right) = \frac{\pi}{6}\cdot\frac{4R+r}{s} \geq$$

$\geq \frac{\pi}{6}\cdot\frac{s\sqrt{3}}{s} = \frac{\pi\sqrt{3}}{6}$, ***we've applied Doucet's inequality*** $s\sqrt{3} \leq 4R + r$

$$\Rightarrow \sum\left(\sin^2\frac{A}{2} + \frac{1}{2}\mu(A)\tan\frac{A}{2}\right) \geq \frac{3}{4} + \frac{\pi\sqrt{3}}{6}$$

$$\Rightarrow \max\left(\mu^2(A), \mu^2(B), \mu^2(C)\right) + \sum\left(\sin^2\frac{A}{2} + \frac{1}{2}\mu(A)\tan\frac{A}{2}\right) \geq$$

$$\geq \frac{\pi^2}{9} + \frac{3}{4} + \frac{\pi\sqrt{3}}{6} \geq \frac{5\pi^2}{18}, \frac{3}{4} + \frac{\pi\sqrt{3}}{6} \geq \frac{5\pi^2}{18} - \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

$$\frac{3}{4} + \frac{\pi\sqrt{3}}{6} \geq \frac{\pi^2}{6} \Rightarrow 9 + 2\pi\sqrt{3} \geq 2\pi^2$$

$$9 + 2\pi\sqrt{3} \backsimeq 9 + 2\cdot 3.14\cdot 1.73 \backsimeq 9 + 10.86 \backsimeq 19.86$$

$$2\pi^2 \backsimeq 2\cdot 3.14^2 \backsimeq 2\cdot 9.85 \approx 19.71$$

$$19.86 \geq 19.71 \textit{ true} \Rightarrow 9 + 2\pi\sqrt{3} \geq 2\pi^2 \textit{ true.}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{Let } f(x) = x - \sin x \;\forall\, x \in [0, \pi)$$

$$\therefore f'(x) = 1 - \cos x \geq 0 \Rightarrow f(x) \text{ is } \uparrow \text{ on } [0, \pi) \Rightarrow f(x) \geq f(0) = 0 \Rightarrow$$

$$\Rightarrow x \geq \sin x \;\forall\, x \in [0, \pi) \Rightarrow \forall\, x \in (0, \pi), x \overset{(1)}{>} \sin x$$

$$\textit{Let } g(x) = \sin^2\frac{x}{2} + \frac{1}{2}x\tan\frac{x}{2} \;\forall\, x \in (0, \pi)$$

$$\therefore g''(x) = \frac{1}{4}\left(x\sec^2\frac{x}{2}\tan\frac{x}{2} - 2\sin^2\frac{x}{2} + 2\sec^2\frac{x}{2} + 2\cos^2\frac{x}{2}\right)$$

$$\overset{\text{by }(1)}{>}\frac{1}{4}\left(\sin x\sec^2\frac{x}{2}\tan\frac{x}{2}-2\sin^2\frac{x}{2}+2\sec^2\frac{x}{2}+2\cos^2\frac{x}{2}\right)$$

$$=\frac{1}{4}\left(\frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{\cos^2\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}-2\sin^2\frac{x}{2}+2\sec^2\frac{x}{2}+2\cos^2\frac{x}{2}\right)$$

$$=\frac{1}{4}\left(2\sin^2\frac{x}{2}\left(\sec^2\frac{x}{2}-1\right)+2\sec^2\frac{x}{2}+2\cos^2\frac{x}{2}\right)>$$

$$>\frac{1}{4}\left(2\sec^2\frac{x}{2}+2\cos^2\frac{x}{2}\right)\left(\because \sec^2\frac{x}{2}>1\right)>0\Rightarrow g(x)\ \textit{is convex}$$

$$\therefore\sum\left(\sin^2\frac{A}{2}+\frac{1}{2}\mu(A)\tan\frac{A}{2}\right)\overset{\text{Jensen}}{\geq}3\left(\sin^2\frac{\pi}{6}+\frac{\pi}{6}\tan\frac{\pi}{6}\right)=\frac{3}{4}+\frac{\pi}{2\sqrt{3}}\Rightarrow$$

$$\Rightarrow\sum\left(\sin^2\frac{A}{2}+\frac{1}{2}\mu(A)\tan\frac{A}{2}\right)\overset{(2)}{\geq}\frac{3}{4}+\frac{\pi}{2\sqrt{3}}$$

Now, WLOG, we may assume:

$$\max\{\mu^2(A),\mu^2(B),\mu^2(C)\}=\mu^2(A)\ \therefore\ 3\mu^2(A)\geq\sum\mu^2(A)\Rightarrow$$

$$\Rightarrow\max\{\mu^2(A),\mu^2(B),\mu^2(C)\}\geq\frac{1}{3}\sum\mu^2(A)\overset{\text{Jensen}}{\geq}\frac{3}{3}\left(\frac{\pi}{3}\right)^2$$

$(\because\ h(x)=x^2$ ***is convex for all real values of*** $x)$

$$\therefore\max\{\mu^2(A),\mu^2(B),\mu^2(C)\}\overset{(3)}{\geq}\frac{\pi^2}{9}$$

$\therefore$ ***(2)+(3)*** $\Rightarrow \text{LHS}\geq\frac{\pi^2}{9}+\frac{3}{4}+\frac{\pi}{2\sqrt{3}}\approx 2.7535>\frac{5\pi^2}{18}(\approx 2.74)$ ***(Proved)***

Solution 3 by Ravi Prakash-New Delhi-India

Let's assume that: $\mu(A)\geq\mu(B)\geq\mu(C)$. ***Then*** $\mu(A)\geq\frac{\pi}{3}$. ***Let, for*** $0\leq x<\frac{\pi}{2}$

$$f(x)=\sin^2 x+x\tan x-\frac{1}{2}x^2$$

$$f'(x)=\sin 2x+\tan x-x+x\sec^2 x>0\ \textit{for}\ 0<x<\frac{\pi}{2}$$

$$\Rightarrow f(x)\ \textit{is increasing on}\ \left[0,\frac{\pi}{2}\right)\Rightarrow f(x)>f(0)\ \textit{for}\ 0<x<\frac{\pi}{2}$$

$$\Rightarrow f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)>0\Rightarrow\sum\left[\sin^2\left(\frac{A}{2}\right)+\frac{A}{2}\tan\left(\frac{A}{2}\right)\right]\geq\frac{1}{2}\sum(A^2+B^2+C^2)$$

$$\Rightarrow \frac{1}{6}(A^2+B^2+C^2+2A^2+2B^2+2C^2) \geq \frac{1}{6}(A^2+B^2+C^2+2BC+2CA+2AB) =$$

$$= \frac{\pi^2}{6} \Rightarrow \max(\mu(A)^2, \mu(B)^2, \mu(C)^2) + \sum\left(\sin^2\left(\frac{A}{2}\right) + \frac{A}{2}\tan\left(\frac{A}{2}\right)\right) \geq \frac{\pi^2}{9} + \frac{\pi^2}{6} = \frac{5\pi^2}{18}$$

1350. In ΔABC the following relationship holds:

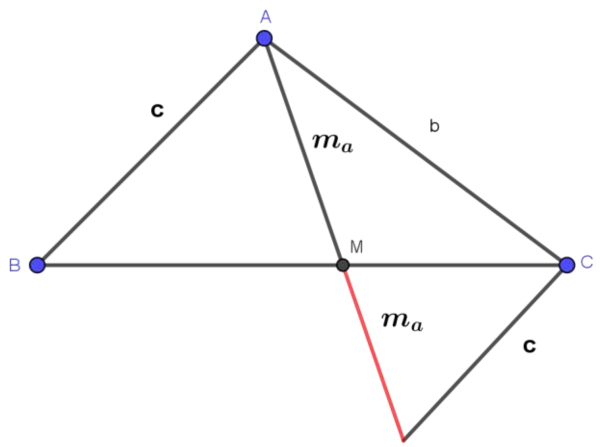
$$s_a + s_b + s_c < \frac{RS}{r^2}$$

Proposed by Ionuț Florin Voinea-Romania

Solution 1 by Adrian Popa-Romania

$$s_a + s_b + s_c < \frac{RS}{r^2}$$

$$\left.\begin{array}{l} s_a = \dfrac{2bc}{b^2+c^2} \cdot m_a \leq m_a \\ Similarly\ s_b \leq m_b; s_c \leq m_c \end{array}\right\} \Rightarrow s_a + s_b + s_c \leq m_a + m_b + m_c$$



$$\Rightarrow 2m_a < b + c \Rightarrow m_a < \frac{b+c}{2}$$

$$Similarly: m_b < \frac{a+c}{2}$$

$$m_c < \frac{a+c}{2}$$

$$-----------$$

$$m_a + m_b + m_c < a + b + c = 2s$$

$$We\ will\ have\ to\ prove\ that: 2s < \frac{Rs}{r^2} \Leftrightarrow 2s < \frac{R}{r} \cdot \frac{\overbrace{S}^{s}}{r} = \frac{Rs}{r}\Big| : s \Leftrightarrow$$

$$\Leftrightarrow 2 < \frac{R}{r} \text{ (true) (Euler's inequality)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum s_a \overset{CBS}{\leq} \sqrt{3}\sqrt{\sum s_a^2} = \sqrt{3}\sqrt{\sum\left(\frac{2bcm_a}{b^2+c^2}\right)^2} \overset{AM-GM}{\leq} \sqrt{3}\sqrt{\sum m_a^2} = \sqrt{3}\sqrt{\frac{3}{4}\sum a^2} =$$

$$= \frac{3}{2}\sqrt{\sum a^2} \overset{Leibnitz}{\leq} \frac{9R}{2} \overset{?}{<} \frac{Rrs}{r^2} \Leftrightarrow s \overset{?}{>} \frac{9r}{2} \to true \because s \overset{Mitrinovic}{\geq} 3\sqrt{3}r$$

1351. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\left(\sum_{cyc} m_a^2\right)\left(\sum_{cyc}\frac{AI}{\mu(A)}\right) \geq \frac{486r^3}{\pi}$$

Proposed by Radu Diaconu – Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \sum m_a^2 \overset{AM-GM}{\geq} 3\left(\prod m_a^2\right)^{\frac{1}{3}} \overset{m_a \geq \sqrt{s(s-a)}}{\geq} 3\left(s^3(s-a)(s-b)(s-c)\right)^{\frac{1}{3}}$$

$$= 3(s^2\Delta^2)^{\frac{1}{3}} = 3(s^4r^2)^{\frac{1}{3}} \overset{Mitrinovic}{\geq} 3(r^6 3^6)^{\frac{1}{3}} = 27r^2 \quad (i)$$

$$\Leftrightarrow \sum \mu(A) \overset{AM-GM}{\geq} 3\left(\prod \mu(A)\right)^{\frac{1}{3}} \Rightarrow 27\prod \mu(A) \leq \left(\sum \mu(A)\right)^3 = \pi^3 \Rightarrow \prod \mu(A) \leq \frac{\pi^3}{27} \quad (ii)$$

$$\Leftrightarrow \sum \frac{AI}{\mu(A)} \geq 3\left(\prod \frac{AI}{\mu(A)}\right)^{\frac{1}{3}} = 3\left(\frac{27}{\pi^3}\cdot\prod\frac{(s-a)}{\cos\frac{A}{2}}\right)^{\frac{1}{3}} \quad [From\ (ii)]$$

$$= \frac{9}{\pi}\left(\frac{(s-a)(s-b)(s-c)\cdot abc}{\sqrt{s^3(s-a)(s-b)(s-c)}}\right)^{\frac{1}{3}} = \frac{9}{\pi}\left(\frac{abc(s-a)(s-b)(s-c)}{s\Delta}\right)^{\frac{1}{3}} = \frac{9}{\pi}\left(\frac{abc\cdot\Delta^2}{s^2\Delta}\right)^{\frac{1}{3}}$$

$$= \frac{9}{\pi}\left(\frac{4Rrs\cdot r^2s^2}{s^2rs}\right)^{\frac{1}{3}} = \frac{9}{\pi}(4Rr^2)^{\frac{1}{3}} \overset{Euler}{\geq} \frac{9}{\pi}(4\cdot 2r\cdot r^2)^{\frac{1}{3}} = \frac{18r}{\pi} \quad (iii)$$

$$\Leftrightarrow \left(\sum m_a^2\right)\left(\sum\frac{AI}{\mu(A)}\right) \geq 27r^2\cdot\frac{18r}{\pi} \quad [From\ (i)\ and\ (iii)] \Leftrightarrow \Omega \geq \frac{486r^3}{\pi} \quad (proved)$$

Solution 2 by Șerban George Florin-Romania

$$\sum_{cyc}\frac{AI}{\mu(A)} = \sum\frac{r}{\mu(A)\sin\frac{A}{2}} = r\sum\frac{1}{\mu(A)\cdot\sin\frac{A}{2}}$$

$$\mu(A) \le \mu(B) \le \mu(C) \Rightarrow \frac{1}{\mu(A)} \ge \frac{1}{\mu(B)} \ge \frac{1}{\mu(C)}$$

$$\mu\left(\frac{A}{2}\right) \le \mu\left(\frac{B}{2}\right) \le \mu\left(\frac{C}{2}\right) \Rightarrow \sin\frac{A}{2} \le \sin\frac{B}{2} \le \sin\frac{C}{2} \Rightarrow \frac{1}{\sin\frac{A}{2}} \ge \frac{1}{\sin\frac{B}{2}} \ge \frac{1}{\sin\frac{C}{2}}$$

Applying Chebyshev's inequality

$$\sum \frac{AI}{\mu(A)} = r\sum \frac{1}{\mu(A)\sin\frac{A}{2}} \ge \frac{r}{3}\sum \frac{1}{\mu(A)} \cdot \sum \frac{1}{\sin\frac{A}{2}} \ge$$

$$\ge \frac{r}{3} \cdot \frac{9}{\sum \mu(A)} \cdot 6 = \frac{18r}{\pi} \quad \left(\sum \frac{1}{\sin\frac{A}{2}} \ge 6\right)$$

$$\sum_{cyc} m_a^2 = \frac{3}{4}(a^2+b^2+c^2)$$

$$\left(\sum m_a^2\right) \cdot \left(\sum \frac{AI}{\mu(A)}\right) \ge \frac{3}{4}\sum a^2 \cdot \frac{18r}{\pi} \ge \frac{486r^3}{\pi}$$

$$\frac{54r}{4\pi}\sum a^2 \ge \frac{486r^3}{\pi}, \sum a^2 \ge 36r^2$$

Applying Ionescu – Weitzenböck

$$a^2+b^2+c^2 \ge 4S\sqrt{3} \ge 36r^2 \Rightarrow rs\sqrt{3} \ge 9r^2$$

$$\Rightarrow s\sqrt{3} \ge 9r \Rightarrow s \ge \frac{9r}{\sqrt{3}} = \frac{9\sqrt{3}r}{3} = 3\sqrt{3}r$$

$$\Rightarrow s \ge 3\sqrt{3}r, \textit{ true (Mitrinovic's inequality)}$$

1352. In ΔABC the following relationship holds:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 6\sqrt{12Rr+3r^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 6\sqrt{12Rr+3r^2} \quad (1)$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \ge 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\right) \textit{ then:}$$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ac} \ge 2\sqrt{12Rr+3r^2}$$

after we make the substitution $a = x + y, b = y + z, c = x + z$

and $Rr = \frac{(x+y)(y+z)(x+z)}{4(x+y+z)}, r^2 = \frac{xyz}{x+y+z}$

the inequality becomes:

$$\sqrt{(x+y)(x+z)} + \sqrt{(y+x)(y+z)} + \sqrt{(z+x)(z+y)} \geq 2\sqrt{3(xy+yz+xy)}$$

or

$$\sum \sqrt{x^2 + xy + yz + xz} \geq 2\sqrt{3(xy+yz+xz)} \quad (2)$$

$$xy + yz + xz = k^2 \Rightarrow (x+y+z)^2 \geq 3k^2$$

we must show that

$$\sum \sqrt{x^2 + k^2} \geq 2k\sqrt{3} \quad (3)$$

$$LHS \overset{MITRINOVIC}{\geq} \sqrt{(x+y+z)^2 + 9k^2} \geq \sqrt{12k^2} = 2k\sqrt{3}$$

Solution 2 by Marian Ursărescu-Romania

Because $\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \geq 3\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ac}\right) \Rightarrow$ *we must show:*

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ac} \geq 2\sqrt{12Rr + 3r^2} \quad (1)$$

Because in any ΔABC *we have* $b + c - a + 2\sqrt{bc} > 0 \Rightarrow$

$\sqrt{b} + \sqrt{c} > \sqrt{a} \Rightarrow$ *exists a triangle* $A'B'C'$ *with lengths* $a' = \sqrt{a}, b' = \sqrt{b}, c' = \sqrt{c} \Rightarrow$

we prove inequality with the help of $\Delta A'B'C'$

We have Gordon's inequality: $ab + ac + bc \geq 4\sqrt{3}S \Rightarrow$ *for* $\Delta A'B'C'$ *we can write:*

$$a'b' + a'c' + b'c' \geq 4\sqrt{3}S' \quad (2)$$

In our case $a' = \sqrt{a}, b' = \sqrt{b}, c' = \sqrt{c}$ *and by calculation:*

$$S' = \frac{1}{2}\sqrt{4Rr + r^2} \quad (3)$$

From (2)+(3)⇒ $a\sqrt{ab} + \sqrt{ac} + \sqrt{bc} \geq 4\sqrt{3} \cdot \frac{1}{2}\sqrt{4Rr + r^2} \Leftrightarrow$

$\sqrt{ab} + \sqrt{ac} + \sqrt{bc} \geq 2\sqrt{12Rr + 3r^2} \Rightarrow$ *(1) it is true.*

Solution 3 by proposer

$$\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab} > a + b > c = \left(\sqrt{c}\right)^2 \Rightarrow$$

$$\sqrt{a} + \sqrt{b} > \sqrt{c} - \text{and analogs.}$$

By Mitrinovic's inequality in the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$s_1 \geq 3\sqrt{3}r_1 \Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq 3\sqrt{3}\cdot\frac{S_1}{s_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq 3\sqrt{3}\cdot\frac{\frac{1}{2}\sqrt{4Rr+r^2}}{\frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \geq 3\sqrt{12Rr+3r^2} \Leftrightarrow$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \geq 6\sqrt{12Rr+3r^2}$$

1353. In acute ΔABC the following relationship holds:

$$\mu(A)e^{\mu(A)+secA}+\mu(B)e^{\mu(B)+secB}+\mu(C)e^{\mu(C)+secC} > e\pi$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Daniel Sitaru-Romania

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=e^{x+\frac{1}{cosx}}, f'(x)=\left(1+\frac{sinx}{cos^2x}\right)e^{x+\frac{1}{cosx}}>0$$

$$f(x) > \lim_{\substack{x\to 0\\ x>0}} f(x)=e \to \sum_{cyc} f(A)=\sum_{cyc} e^{\mu(A)+secA} > 3e \quad (1)$$

$$\text{WLOG: } \mu(A)\leq\mu(B)\leq\mu(C) \to \begin{cases} e^{\mu(A)}\leq e^{\mu(B)}\leq e^{\mu(C)} \\ cosA\geq cosB\geq cosC \\ secA\leq secB\leq secC \\ e^{sec(A)}\leq e^{sec(B)}\leq e^{sec(C)} \\ e^{\mu(A)+secA}\leq e^{\mu(B)+secB}\leq e^{\mu(C)+secC} \end{cases}$$

$$\sum_{cyc}\mu(A)e^{\mu(A)+secA} \overset{CEBYSHEV}{\geq} \frac{1}{3}\cdot\sum_{cyc}\mu(A)\cdot\sum_{cyc}e^{\mu(A)+secA} \overset{(1)}{>} \frac{1}{3}\cdot\sum_{cyc}\mu(A)\cdot 3e =$$

$$=\frac{1}{3}\cdot\pi\cdot 3e=e\pi$$

Solution 2 by Florentin Vișescu-Romania

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=xe^{x+\frac{1}{cosx}}$$

$$f''(x)=e^{x+\frac{1}{cosx}}\left(2+\frac{2sinx}{cos^2x}+x\left(1+\frac{sinx}{cos^2x}\right)^2+x\cdot\frac{1+sin^2x}{cos^3x}\right)>0$$

f-convexe. By Jensen's inequality:

$$\frac{1}{3}(f(a)+f(B)+f(C))\geq f\left(\frac{A+B+C}{3}\right)$$

$$\sum_{cyc} Ae^{A+\frac{1}{\cos A}}\geq 3\cdot\frac{\pi}{3}e^{\frac{\pi}{3}+\frac{1}{\cos\frac{\pi}{3}}}=\pi e^{2+\frac{\pi}{3}}>\pi e$$

1354. **In ΔABC the following relationship holds:**

$$(\mu(B)+\mu(C))^2\csc A+2\csc\frac{B}{2}+2\csc\frac{C}{2}>\frac{2s}{r}$$

Proposed by Emil Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$For\ simplicity, let\ us\ denote\ \mu(B)\ by\ B, \mu(C)\ by\ C\ and\ \ \mu(A)\ by\ A$$

$$b+c-a=4R\cos\frac{A}{2}\cos\frac{B-C}{2}-4R\cos\frac{A}{2}\sin\frac{A}{2}=$$

$$=4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)=8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Rightarrow$$

$$\Rightarrow s-a\overset{(1)}{=}4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$Again, AI=\frac{r}{\sin\frac{A}{2}}=\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}=4R\sin\frac{B}{2}\sin\frac{C}{2}\overset{by\ (1)}{=}\frac{s-a}{\cos\frac{A}{2}}\Rightarrow\cos\frac{A}{2}\overset{(2)}{=}\frac{s-a}{AI}$$

$$Now, \tan\frac{A}{4}\overset{(i)}{=}\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}\overset{by\ (2)}{=}\frac{1-\frac{s-a}{AI}}{\frac{r}{AI}}=\frac{AI-(s-a)}{r}\Rightarrow AI\overset{(a)}{=}s-a+r\tan\frac{A}{4}$$

$$Similarly, BI\overset{(b)}{=}s-b+r\tan\frac{B}{4}\ and\ CI\overset{(c)}{=}s-c+r\tan\frac{C}{4}\therefore(a)+(b)+(c)\Rightarrow$$

$$\Rightarrow\sum AI\overset{(3)}{=}s+r\sum\tan\frac{A}{4}$$

$$Now, LHS=\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-2\mathrm{cosec}\frac{A}{2}+2\sum\mathrm{cosec}\frac{A}{2}=$$

$$=\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-2\mathrm{cosec}\frac{A}{2}+\frac{2}{r}\sum AI\overset{\text{by (3)}}{\cong}\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-2\mathrm{cosec}\frac{A}{2}+\frac{2s}{r}+2\sum\tan\frac{A}{4}$$

$$=\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-2\mathrm{cosec}\frac{A}{2}+\frac{2s}{r}+2\tan\frac{A}{4}+2\tan\frac{B}{4}+2\tan\frac{C}{4}\overset{\text{by (i)}}{\cong}$$

$$=\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-2\mathrm{cosec}\frac{A}{2}+\frac{2s}{r}+2\left(\mathrm{cosec}\frac{A}{2}-\frac{2\cos^2\frac{A}{2}}{2\cos\frac{A}{2}\sin\frac{A}{2}}\right)+2\tan\frac{B}{4}+2\tan\frac{C}{4}$$

$$=\frac{(\pi-A)^2}{2\cos\frac{A}{2}\sin\frac{A}{2}}-\frac{4\cos^2\frac{A}{2}}{2\cos\frac{A}{2}\sin\frac{A}{2}}+\frac{2s}{r}+2\tan\frac{B}{4}+2\tan\frac{C}{4}>\frac{2s}{r}+\frac{(\pi-A)^2-4\cos^2\frac{A}{2}}{2\cos\frac{A}{2}\sin\frac{A}{2}}$$

$$\therefore LHS\overset{(4)}{\gtrsim}\frac{2s}{r}+\frac{(\pi-A)^2-4\cos^2\frac{A}{2}}{2\cos\frac{A}{2}\sin\frac{A}{2}}$$

$$(4)\Rightarrow \textit{it suffices to prove}:\ (\pi-A)^2>4\cos^2\frac{A}{2}\Leftrightarrow\pi-A\overset{(5)}{\gtrsim}2\cos\frac{A}{2}$$

$$\textit{Let }f(x)=2\cos\frac{x}{2}+x-\pi\ \forall x\in(0,\pi]\ \textit{Then},f'(x)=1-\sin\frac{x}{2}\geq 0$$

$$\therefore f(x)\textit{ is increasing on }(0,\pi]\Rightarrow f(x)\leq f(\pi)=2\cos\frac{\pi}{2}+\pi-\pi=0$$

$$\Rightarrow\forall x\in(0,\pi],2\cos\frac{x}{2}+x\leq\pi\Rightarrow\forall x\in(0,\pi),\pi-x>2\cos\frac{x}{2}\Rightarrow\pi-A>2\cos\frac{A}{2}$$

$$\Rightarrow(5)\Rightarrow\textit{proposed inequality is true (Proved)}$$

1355. In ΔABC the following relationships holds:

$$R\geq\left(\sum\frac{1}{m_a+m_b}\right)^{-1}\geq 2r,\quad R\geq\left(\sum\frac{1}{w_a+w_b}\right)^{-1}\geq 2r$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{1}{m_a+m_b}\overset{\text{Bergstrom}}{\geq}\frac{9}{2\sum m_a}\geq\frac{9}{8R+2r}\overset{\text{Euler}}{\geq}\frac{9}{8R+R}=\frac{1}{R}\Leftrightarrow$$

$$R\geq\left(\sum\frac{1}{m_a+m_b}\right)^{-1}\overset{\because m_a\geq w_a\text{ and analogs}}{\geq}\left(\sum\frac{1}{w_a+w_b}\right)^{-1}\to(1)$$

$$\because\ w_a \geq h_a \text{ and analogs},$$

$$\therefore \sum \frac{1}{w_a + w_b} \leq \sum \frac{1}{h_a + h_b} = \sum \frac{2R}{c(a+b)} \overset{AM-GM}{\leq} \sum \frac{2R}{2c\sqrt{ab}} =$$

$$= R\sum \frac{1}{\sqrt{ca}\sqrt{bc}} \overset{CBS}{\leq} R\sqrt{\sum \frac{1}{ca}}\sqrt{\sum \frac{1}{bc}} = R\sum \frac{1}{ab} = \frac{2sR}{4Rrs} = \frac{1}{2r}$$

$$\left(\sum \frac{1}{w_a + w_b}\right)^{-1} \geq 2r \text{ and } \because\ m_a \geq w_a \text{ and analogs},$$

$$\therefore \left(\sum \frac{1}{m_a + m_b}\right)^{-1} \geq \left(\sum \frac{1}{w_a + w_b}\right)^{-1} \geq 2r \to (2)$$

$$(1) \text{ and } (2) \Rightarrow R \geq \left(\sum \frac{1}{m_a + m_b}\right)^{-1} \geq 2r \text{ and } R \geq \left(\sum \frac{1}{w_a + w_b}\right)^{-1} \geq 2r \ \ (\textit{Proved})$$

1356. In ΔABC the following relationships holds:

$$108\sum \sin^2 A\cot B\cot C \leq \left(2\left(\frac{R}{r}\right)^2 + 1\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$108\sum \sin^2 A\cot B\cot C = 108\left(\prod \cot A\right)\sum \tan A(1 - \cos^2 A) =$$

$$= 108\left(\prod \cot A\right)\sum \tan A - 108\left(\prod \cot A\right)\sum \tan A\cos^2 A$$

$$= 108\sum \cot A\cot B - 54\left(\prod \cot A\right)\sum (2\sin A\cos A) =$$

$$= 108 - 54\left(\frac{\prod \cos A}{\prod \sin A}\right)\sum \sin 2A = 108 - 54\left(\frac{s^2 - (2R+r)^2}{4R^2\left(\prod \sin A\right)}\right)\left(4\prod \sin A\right)$$

$$= 108 - 54\left(\frac{s^2 - (2R+r)^2}{R^2}\right) = \frac{108R^2 + 54(2R+r)^2 - 54s^2}{R^2} \leq \left(2\left(\frac{R}{r}\right)^2 + 1\right)^2 = \frac{(2R^2 + r^2)^2}{r^4}$$

$$\Leftrightarrow \frac{108R^2 + 54(2R+r)^2}{R^2} \leq \frac{(2R^2 + r^2)^2}{r^4} + \frac{54s^2}{R^2} = \frac{R^2(2R^2 + r^2)^2 + 54s^2r^4}{R^2r^4}$$

$$\Leftrightarrow R^2(2R^2 + r^2)^2 + 54s^2r^4 \overset{(1)}{\geq} r^4(108R^2 + 54(2R+r)^2)$$

$$\textit{Now, LHS of } (1) \overset{Gerretsen}{\geq} R^2(2R^2 + r^2)^2 + 54(16Rr - 5r^2)r^4 \overset{?}{\geq} r^4(108R^2 + 54(2R+r)^2)$$

$$\Leftrightarrow 4t^6 + 4t^4 - 323t^2 + 648t - 324 \overset{?}{\geq} 0 \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(4t^4 + 16t^3 + 52t^2 + 144t + 45) + 252\} \overset{?}{\geq} 0 \to \text{true}, \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore 108\sum \sin^2 A\cot B\cot C \leq \left(2\left(\frac{R}{r}\right)^2 + 1\right)^2 \text{ (Proved)}$$

1357. If in acute ΔABC, N –nine-point center then:

$$\sqrt{NA} + \sqrt{NB} + \sqrt{NC} \leq \sqrt{\frac{15R + 6r}{2}}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$O \text{ –circumcenter}, H \text{ –orthocenter}, NA \text{ –median in } \Delta AOH$$

$$NA \leq \frac{OA + HA}{2} \ (\text{equality for } N \equiv O \equiv H)$$

$$NA \leq \frac{R + 2R\cos A}{2} \Rightarrow 2NA \leq R + 2R\cos A$$

$$2\sum_{cyc} NA \leq 3R + 2R\sum_{cyc}\cos A = 3R + 2R\left(1 + \frac{r}{R}\right) = 5R + 2r$$

$$\sum_{cyc} NA \leq \frac{5R + 2r}{2}$$

$$\sum_{cyc}\sqrt{NA} \overset{CBS}{\leq} \sqrt{(1^2 + 1^2 + 1^2)(NA + NB + NC)} \leq \sqrt{3 \cdot \frac{5R + 2r}{2}} = \sqrt{\frac{15R + 6r}{2}}$$

Equality holds for $a = b = c$.

1358. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{m_a}{s_a}\right)^n + \prod_{cyc}\left(\frac{2a}{b + c}\right)^n \geq 4, \quad n \geq 0$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Proof}: LHS \overset{A-G}{\geq} 4\sqrt[4]{\left[\prod\left(\frac{m_a}{s_a}\right)^n\right]\left[\prod\left(\frac{2a}{b+c}\right)^n\right]} = 4\sqrt[4]{\left[\prod\left(\frac{b^2+c^2}{2bc}\right)\frac{8abc}{\prod(b+c)}\right]^n} =$$

$$= 4\sqrt[4]{\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^n}$$

$$\therefore LHS \overset{(1)}{\geq} 4\sqrt[4]{\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^n}$$

$$\textit{Now}, \frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right) \geq \frac{1}{abc}\left(\prod\left(\frac{(b+c)^2}{2(b+c)}\right)\right) = \frac{1}{abc}\prod\left(\frac{b+c}{2}\right)$$

$$= \frac{(a+b)(b+c)(c+a)}{8abc} \overset{Cesaro}{\geq} 1 \therefore \frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right) \geq 1$$

$$\Rightarrow \left(\frac{n}{4}\right)\ln\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right] \geq 0\ (\because n \geq 0) \Rightarrow \ln\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^{\frac{n}{4}} \geq 0$$

$$\Rightarrow \left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^{\frac{n}{4}} \geq 1 \Rightarrow \sqrt[4]{\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^n} \geq 1$$

$$\Rightarrow 4\sqrt[4]{\left[\frac{1}{abc}\left(\prod\frac{b^2+c^2}{b+c}\right)\right]^n} \overset{(2)}{\geq} 4$$

$$(1),(2) \Rightarrow LHS \geq 4\ (\text{Proved})$$

1359. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a-r}{s_a}\sqrt{\frac{h_a}{r_a}} \geq \sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_a-r}{s_a}\sqrt{\frac{h_a}{r_a}} == \sum\left[\frac{\frac{rs}{s-a}-\frac{rs}{s}}{\left(\frac{2bc}{b^2+c^2}\right)m_a}\sqrt{\frac{2rs(s-a)}{ars}}\right] \overset{Tsintsifas}{\geq}$$

$$\geq \sum\left[\frac{\frac{rs(s-(s-a))}{s(s-a)}}{\left(\frac{2bc}{b^2+c^2}\right)\left(\frac{b^2+c^2}{2bc}\right)w_a}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{2abc}{sa^2}}\right]$$

$$=\sum\left(\frac{2ra(b+c)}{2abc\cos\frac{A}{2}(s-a)}\cos\frac{A}{2}\sqrt{\frac{2Rrs}{s}}\right)=\sqrt{2Rr}\left(\sum\frac{ra(b+c)}{4Rrs(s-a)}\right)$$

$$=\left(\frac{\sqrt{2Rr}}{4Rs}\right)\sum\frac{a(s+s-a)}{s-a}=\left(\frac{\sqrt{2Rr}}{4Rs}\right)\left(\sum\frac{s(a-s+s)}{s-a}+\sum a\right)$$

$$=\left(\frac{\sqrt{2Rr}}{4Rs}\right)\left[\sum(-s)+\frac{s^2\sum(s-b)(s-c)}{\prod(s-a)}+2s\right]=\left(\frac{\sqrt{2Rr}}{4Rs}\right)\left[-s+\frac{(4Rr+r^2)s^2}{sr^2}\right]$$

$$=\left(\frac{\sqrt{2Rr}}{4Rs}\right)s\left(\frac{4R+r}{r}-1\right)=\left(\frac{\sqrt{2Rr}}{4Rs}\right)\left(\frac{4Rs}{r}\right)$$

$$=\sqrt{\frac{2R}{r}}\text{(Proved)}$$

1360. ADIL ABDULLAYEV'S REFINEMENT FOR IONESCU – WEITZENBOCK'S INEQUALITY

In ΔABC the following relationship holds:

$$\frac{a^2+b^2+c^2}{4\sqrt{3}S}\geq\sqrt[3]{\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}\geq 1$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Ravi Prakash-New Delhi-India

We have

$$a^2+b^2+c^2=\frac{1}{2}(a^2+b^2)+\frac{1}{2}(b^2+c^2)+\frac{1}{2}(c^2+a^2)\geq ab+bc+ca$$

$$=2S\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)$$

As $f(x)=\frac{1}{\sin x}, 0<x<\pi$, is convex, we get:

$$f(A)+f(B)+f(C)\geq 3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)=\frac{(3)(2)}{\sqrt{3}}=2\sqrt{3}$$

Thus, $a^2+b^2+c^2\geq ab+bc+ca\geq 4\sqrt{3}S$. *Now, we have:*

$$(a^2+b^2+c^2)^3(ab+bc+ca)^2\geq (a^2+b^2+c^2)(a^2+b^2+c^2)^2(ab+bc+ca)^2$$

$$\geq (4\sqrt{3}S)(4\sqrt{3}S)^2(a^2+b^2+c^2)^2 \Rightarrow \frac{(a^2+b^2+c^2)^3}{(4\sqrt{3})^3S^3}\geq\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2$$

$$\Rightarrow \frac{a^2+b^2+c^2}{4\sqrt{3}S}\geq\left[\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2\right]^{\frac{1}{3}}\quad (1)$$

Also, $\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1$

$$\Rightarrow \left[\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2\right]^{\frac{1}{3}}\geq 1\quad (2)$$

The inequality follows, from (1) and (2).

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

In a triangle ABC, we have: $\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)\geq 1$

$$\Rightarrow\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^3\geq\frac{a^2+b^2+c^2}{(ab+bc+ca)}\Rightarrow\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\sqrt[3]{\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}\geq 1$$

$$\Rightarrow\frac{a^2+b^2+c^2}{4\sqrt{3}S}\geq\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\sqrt[3]{\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2}\geq 1\ ok$$

Therefore it is true.

Let $x=a+b-c, y=b+c-a, z=c+a-b$

Hence $a=\frac{x+z}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}$

We have $(x+y)(y+z)(z+x)\geq 8xyz$

$$\Rightarrow 2(x+y)(y+z)(z+x)(x+y+z)\geq 16xyz(x+y+z)$$

$$\Rightarrow (x+y)(y+z)(z+x)\big((x+y)+(y+z)+(z+x)\big)\geq 16xyz(x+y+z)$$

$$\Rightarrow (x+y)(y+z)(z+x)(x+y)+(x+y)(y+z)(z+x)(y+z)+$$

$$+(x+y)(y+z)(z+x)(z+x)\geq 16xyz(x+y+z)$$

$$\Rightarrow 3\big((x+y)(y+z)(z+x)(x+y)+(x+y)(y+z)(z+x)(y+z)+(x+y)(y+z)(z+x)(z+x)\big)\geq$$

$$\geq 16\times 3xyz(x+y+z)$$

$$\Rightarrow \left(\frac{(x+y)}{2}\frac{(y+z)}{2}+\frac{(y+z)}{2}\frac{(z+x)}{2}+\frac{(z+x)}{2}\frac{(x+y)}{2}\right)^2\geq 3xyz(x+y+z)$$

$$\Rightarrow \left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)+\left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right)+\left(\frac{z+x}{2}\right)\left(\frac{x+y}{2}\right)$$

$$\geq 4\sqrt{3}\sqrt{\frac{(x+y+z)}{2}\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)\left(\frac{z}{2}\right)}$$

$$\Rightarrow bc+ca+ab\geq 4\sqrt{3}\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{c+a-b}{2}\right)}$$

$$=4\sqrt{3}\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-c\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)}=4\sqrt{3}S$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\text{Hadwiger – Finsler}\Rightarrow 2\sum ab-\sum a^2\geq 4\sqrt{3}S\Rightarrow \sum ab\geq \frac{\sum a^2+x}{2}$$

$$(\text{where } x=4\sqrt{3}S)$$

$$\Rightarrow \left(\sum ab\right)^2\overset{(i)}{\geq}\frac{x^2+2x\sum a^2+\left(\sum a^2\right)^2}{4}$$

$$\text{Now, (1)}\Leftrightarrow \left(\frac{\sum a^2}{x}\right)^3\geq\left(\frac{\sum a^2}{\sum ab}\right)^2\Leftrightarrow \left(\sum a^2\right)\left(\sum ab\right)^2\overset{(2)}{\geq}x^3$$

$$\text{Now, LHS of (2)}\overset{by\ (i)}{\geq}\left(\sum a^2\right)\left(\frac{x^2+2x\sum a^2+\left(\sum a^2\right)^2}{4}\right)\overset{?}{\geq}$$

$$\geq x^3\Leftrightarrow y^3+2xy^2+yx^2-4x^3\overset{?}{\geq}0$$

$$(\text{where } y=\sum a^2)\Leftrightarrow t^3+2t^2+t-4\overset{?}{\geq}0$$

$$(\text{where } t=\frac{y}{x})\Leftrightarrow (t-1)(t^2+3t+4)\overset{?}{\geq}0\rightarrow true$$

$$\because t=\frac{y}{x}=\frac{\sum a^2}{4\sqrt{3}S}\overset{Ionescu-Weitzenbock}{\geq}1\Rightarrow$$

$(2) \Rightarrow (1)$ *is true and* $\because \sum a^2 \geq \sum ab \therefore \left(\frac{\sum a^2}{\sum ab}\right)^2 \geq 1 \Rightarrow \sqrt[3]{\left(\frac{\sum a^2}{\sum ab}\right)^2} \geq 1$ (*Proved*)

1361. In ΔABC the following relationship holds:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 2\sqrt{3r(h_a+h_b+h_c)}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Șerban George Florin-Romania

$$\sum\sqrt{ab}=\sqrt{abc}\cdot\sum\frac{1}{\sqrt{a}}$$

$$f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}, f'(x)=-\frac{1}{2}x^{-\frac{3}{2}}, f''(x)=\frac{3}{4}x^{-\frac{5}{2}}>0$$

$$\Rightarrow f \text{ convexe} \Rightarrow f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a)+f(b)+f(c)}{3}$$

$$\frac{3}{\sqrt{\frac{2s}{3}}} \leq \sum\frac{1}{\sqrt{a}} \Rightarrow \sum\frac{1}{\sqrt{a}} \geq \frac{3\sqrt{3}}{\sqrt{2s}}$$

$$\sum\sqrt{ab}=\sqrt{abc}\cdot\sum\frac{1}{\sqrt{a}} \geq 2\sqrt{3r\cdot\sum h_a}\Bigg|^2 \Rightarrow (abc)\cdot\left(\sum\frac{1}{\sqrt{a}}\right)^2 \geq 4\cdot 3r\cdot 2S\sum\frac{1}{a}$$

$$(abc)\cdot\left(\sum\frac{1}{\sqrt{a}}\right)^2 \geq 4RS\cdot\frac{27}{s} \geq 24sr^2\cdot\sum\frac{1}{a}$$

$$54Rr \geq 24sr^2\cdot\sum\frac{1}{a} \Rightarrow \sum\frac{1}{a} \leq \frac{54R}{24sr}=\frac{9R}{4sr}$$

$\sum\frac{1}{a} \leq \frac{9R}{4sr}$. *Applying Petrovnic inequality*

$$\sum\frac{1}{a} \leq \frac{s}{3Rr} \leq \frac{9R}{4sr} \Rightarrow 4s^2 \leq 27R^2 \Rightarrow (2s)^2 \leq \left(3\sqrt{3}R\right)^2 \Rightarrow 2s \leq 3\sqrt{3}R$$

$$\Rightarrow s \leq \frac{3\sqrt{3}R}{2} \text{ true, Mitrinovic's inequality}$$

Solution 2 by Marian Ursărescu-Romania

$$\sqrt{ab}+\sqrt{ac}+\sqrt{bc} \geq 3\sqrt[3]{abc} \Rightarrow \text{we must show:}$$

$$3\sqrt[3]{abc} \geq 2\sqrt{3r(h_a+h_b+h_c)} \leq 3^6(abc)^2 \geq 2^6\cdot 3^3r^3(h_a+h_b+h_c)^3 \quad (1)$$

$$\text{But } abc=4sRr \text{ and } h_a+h_b+h_c=\frac{s^2+r^2+4Rr}{2R} \quad (2)$$

From (1)+(2) we must show:

$$3^6 \cdot 2^4 s^2 R^2 r^2 \ge 2^6 \cdot 3^3 \cdot r^3 \frac{(s^2+r^2+4Rr)^3}{8R^3} \Leftrightarrow$$

$$\Leftrightarrow 3^3 \cdot 2s^2R^5 \ge r(s^2+r^2+4Rr)^3 \quad (3)$$

But $2s^2 \ge 27Rr$ *(Cosnita and Turtoiu) (4)*

From (3)+(4) we must show:

$$9^3R^6 \ge (s^2+r^2+4Rr)^3 \Leftrightarrow 9R^2 \ge s^2+r^2+4Rr \quad (5)$$

From Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2 \Rightarrow$

$$s^2 \le 4R^2+8Rr+4r^2 \Rightarrow s^2 \le 4(R+r)^2 \quad (6)$$

From (5)+(6) we must show: $9R^2 \ge 4(R+r)^2 \Leftrightarrow$

$\Leftrightarrow 3R \ge 2(R+r) \Leftrightarrow R \ge 2r$*, true, Euler's inequality.*

Solution 3 by Boris Colakovic-Belgrade-Serbie

$$\sqrt{abc}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right) \overset{Jensen}{\ge} \sqrt{abc}\frac{3}{\sqrt{\frac{a+b+c}{3}}} = \frac{3\sqrt{3}\sqrt{4Rrs}}{\sqrt{2S}} = 3\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{R}\cdot\sqrt{r}$$

$$3\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{R}\cdot\sqrt{r} \ge 2\sqrt{3}\cdot\sqrt{r}\cdot\sqrt{h_a+h_b+h_c} \Leftrightarrow$$

$$\Leftrightarrow 3\sqrt{2}\cdot\sqrt{R} \ge 2\sqrt{h_a+h_b+h_c} \Leftrightarrow 18\cdot R \ge 4(h_a+h_b+h_c) \Rightarrow$$

$$\Rightarrow \frac{9}{2R} \ge h_a+h_b+h_c$$

From well-known inequality $h_a+h_b+h_c \le 2R+5r \Rightarrow$

$$\Rightarrow h_a+h_b+h_c \le 2R+5r \le \frac{9}{2}R \Rightarrow 2R+5r \le \frac{9}{2}R \Leftrightarrow$$

$$\Leftrightarrow 4R+10r \le 9R \Rightarrow R \ge 2r \;\; Euler$$

1362. In ΔABC the following relationship holds:

$$\sum_{cyc} m_a \cdot \sum_{cyc}\frac{1}{m_a} \le 5 + \frac{4}{9S^2}\cdot\prod_{cyc} m_a \cdot \sum_{cyc} m_a$$

Proposed by Adil Abdullyev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$(\sum a)\left(\sum\frac{1}{a}\right)\le 5+\frac{abc(\sum a)}{4S^2}\Leftrightarrow\frac{s(s^2+4Rr+r^2)}{2Rrs}\le 5+\frac{8Rrs^2}{4r^2s^2}\Leftrightarrow s^2+4Rr+r^2$$

$$\le 4R^2+10Rr$$

$$\text{Now, } s^2+4Rr+r^2\overset{\text{Gerretsen}}{\le}4R^2+8Rr+4r^2\overset{\text{Euler}}{\le}4R^2+8Rr+2Rr=4R^2+10Rr$$

$$\therefore (\sum a)\left(\sum\frac{1}{a}\right)\overset{(1)}{\le}5+\frac{abc(\sum a)}{4S^2}$$

Applying (1) *on a triangle with sides*

$$\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}\ \textit{whose area of course}=\frac{S}{3},\textit{we get}:$$

$$\frac{2}{3}\cdot\frac{3}{2}(\sum m_a)\left(\sum\frac{1}{m_a}\right)\le 5+\frac{\frac{8}{27}m_am_bm_c\left(\frac{2}{3}(\sum m_a)\right)}{4\left(\frac{S^2}{9}\right)}\Rightarrow$$

$$\Rightarrow(\sum m_a)\left(\sum\frac{1}{m_a}\right)\le 5+\frac{4m_am_bm_c(\sum m_a)}{9S^2}\ \text{(Proved)}$$

1363. In ΔABC the following relationship holds:

$$\frac{8(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(a+b)^2(b+c)^2(c+a)^2}\le\left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\prod(a+b)=2abc+\sum ab(2s-c)=2s(s^2+4Rr+r^2)-4Rrs=$$

$$=2s(s^2+2Rr+r^2)\Rightarrow\prod(a+b)\overset{(i)}{=}2s(s^2+2Rr+r^2)$$

$$\textit{Now},(a+b)^4=(a^2+b^2+2ab)^2\overset{A-G}{\ge}8ab(a^2+b^2)$$

$$\therefore\frac{a^2+b^2}{(a+b)^2}\overset{(a)}{\le}\frac{(a+b)^2}{8ab}\ \therefore\textit{similarly},\frac{b^2+c^2}{(b+c)^2}\overset{(b)}{\le}\frac{(b+c)^2}{8bc}\ \textit{and}\ \frac{c^2+a^2}{(c+a)^2}\overset{(c)}{\le}\frac{(c+a)^2}{8ca}$$

$$(a).(b).(c)\Rightarrow\frac{8(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(a+b)^2(b+c)^2(c+a)^2}\le\frac{\prod(a+b)^2}{64(abc)^2}\overset{?}{\le}\left(\frac{R}{2r}\right)^2\Leftrightarrow$$

$$\Leftrightarrow \frac{\prod(a+b)}{8abc} \overset{?}{\leq} \frac{R}{2r} \overset{\text{by (i)}}{\Leftrightarrow} \frac{2s(s^2+2Rr+r^2)}{32Rrs} \overset{?}{\leq} \frac{R}{2r}$$

$$\Leftrightarrow s^2+2Rr+r^2 \overset{?}{\leq} 8R^2 \Leftrightarrow s^2 \underset{(1)}{\overset{?}{\leq}} 8R^2-2Rr-r^2$$

$$\text{Now, } s^2 \overset{\text{Gerretsen}}{\leq} 4R^2+4Rr+3r^2 \overset{?}{\leq} 8R^2-2Rr-r^2 \Leftrightarrow 2R^2-3Rr-2r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2R+r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (1) \text{ is true} \therefore \frac{8(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(a+b)^2(b+c)^2(c+a)^2} \leq \left(\frac{R}{2r}\right)^2 \text{ (Proved)}$$

1364. In ΔABC the following relationship holds:

$$4R\sum_{cyc} m_a w_a^2 \geq \sum_{cyc}(b^2+c^2)h_a^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$4R\sum m_a w_a^2 \overset{\text{Tereshin}}{\geq} 4R\sum\left[\left(\frac{b^2+c^2}{4R}\right)w_a^2\right] \overset{w_a \geq h_a}{\geq} \sum(b^2+c^2)h_a^2 \text{ (Proved)}$$

1365. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\left((\mu(A)+2^n)\cdot\sin\frac{A}{2^n}\right)^m\right)\left(\sum_{cyc}\left(\tan\frac{A}{2}\right)^{2q}\right) > \frac{\pi^m}{3^{m+q-2}}, m,n,q \geq 2$$

Proposed by Radu Diaconu-Romania

Solution by Remus Florin Stanca-Romania

$$\sum_{cyc}\left((A+2^n)\sin\frac{A}{2^n}\right)^m \geq \frac{\left(\sum_{cyc}(A+2^n)\sin\frac{A}{2^n}\right)^m}{3^{m-1}} \quad (1)$$

Let $f:\left(0,\frac{\pi}{4}\right) \to \mathbb{R}$ be a function such that

$$f(x) = (x+1)\sin x \Rightarrow f'(x) = \sin x + (x+1)\cos x \Rightarrow$$

$$\Rightarrow f''(x) = \cos x + \cos x - (x+1)\sin x = 2\cos x - (x+1)\sin x \Rightarrow$$

$$\Rightarrow f'''(x) = -2\sin x - \sin x - (x+1)\sin x < 0 \Rightarrow$$

$$\Rightarrow f''(x) \text{ is decreasing, let } f''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\cdot\frac{4-\pi}{4} > 0 \Rightarrow f''(x) > 0 \Rightarrow f \text{ is convex} \overset{Jensen}{\Rightarrow}$$

$$\overset{Jensen}{\Rightarrow} \frac{1}{3}\sum_{cyc}\left(\frac{A}{2^n}+1\right)\sin\left(\frac{A}{2^n}\right) \ge \left(\frac{\pi}{3\cdot 2^n}+1\right)\sin\frac{\pi}{3\cdot 2^n} \quad (2)$$

$$\text{Let's prove that } (x+1)\sin x - x \ge 0, \text{ let } g(x) = (x+1)\sin x - x \Rightarrow$$

$$\Rightarrow g'(x) = \sin x + (x+1)\cos x - 1 \Rightarrow g''(x) = 2\cos x - (x+1)\sin x \Rightarrow$$

$$\Rightarrow g'''(x) = -2\sin x - \sin x - (x+1)\cos x < 0 \Rightarrow g''(x) \text{ is decreasing}$$

$$g''\left(\frac{\pi}{4}\right) > 0 \Rightarrow g''(x) > 0 \Rightarrow g'(x) \text{ is increasing, } g'(0) = 0 \Rightarrow g'(x) > 0 \Rightarrow$$

$$\Rightarrow g \text{ is increasing, } g(0) = 0 \Rightarrow g(x) > 0 \Rightarrow (x+1)\sin x \ge x \Rightarrow$$

$$\Rightarrow \left(\frac{\pi}{3\cdot 2^n}+1\right)\sin\frac{\pi}{3\cdot 2^n} \ge \frac{\pi}{3\cdot 2^n} \Rightarrow$$

$$\Rightarrow \frac{1}{3}\sum_{cyc}\left(\frac{A}{2^n}+1\right)\sin\left(\frac{A}{2^n}\right) \ge \frac{\pi}{3\cdot 2^n} \Rightarrow \sum_{cyc}(A+2^n)\sin\frac{A}{2^n} \ge \pi \overset{(1)}{\Rightarrow}$$

$$\overset{(1)}{\Rightarrow} \sum_{cyc}\left((A+2^n)\sin\frac{A}{2^n}\right)^m \ge \frac{\pi^m}{3^{m-1}} \quad (3)$$

$$\sum_{cyc}\left(\tan\frac{A}{2}\right)^{2q} \ge \frac{\left(\sum_{cyc}\tan\frac{A}{2}\right)^{2q}}{3^{2q-1}} \ge \frac{3^q}{3^{2q-1}} = \frac{1}{3^{q-1}} \quad (5)$$

$$\underset{(3);(5)}{\overset{"\cdot"}{\Rightarrow}} \left(\sum_{cyc}\left((A+2^n)\sin\frac{A}{2^n}\right)^m\right)\left(\sum_{cyc}\left(\tan\frac{A}{2}\right)^{2q}\right) \ge \frac{\pi^m}{3^{m+q-2}} \quad (Q.E.D.)$$

1366. In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{a^2+b^2}{2ab} \ge \max\left(\prod_{cyc}\frac{(a+b)m_a}{2ar_a}, \prod_{cyc}\frac{m_a}{w_a}\right)$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\prod\frac{(a+b)m_a}{2ar_a} = \left(\frac{\prod(b+c)}{8abc\prod r_a}\right)\prod m_a \overset{Tsintsifas}{\le}$$

$$\le \frac{\prod(b+c)}{8abcrs^2}\prod\left(\frac{b^2+c^2}{2bc}w_a\right) = \frac{\prod(b+c)}{8abcrs^2}\left[\frac{\prod(b^2+c^2)}{8a^2b^2c^2}\right]\left[\prod\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\right]$$

$$=\frac{\prod(b+c)}{8abcrs^2}\left[\frac{\prod(b^2+c^2)}{8a^2b^2c^2}\right]\left(\frac{8a^2b^2c^2}{\prod(b+c)}\right)\left(\frac{s}{4R}\right)=$$

$$=\frac{\prod(b^2+c^2)}{8abc(4Rrs)}=\frac{\prod(b^2+c^2)}{8abc(abc)}=\frac{\prod(b^2+c^2)}{\prod(2bc)}=\prod\frac{a^2+b^2}{2ab}\Rightarrow$$

$$\Rightarrow\prod\frac{(a+b)m_a}{2ar_a}\overset{(1)}{\leq}\prod\frac{a^2+b^2}{2ab}$$

Again, by Tsintsifas,

$$\prod\frac{m_a}{w_a}\leq\prod\frac{b^2+c^2}{2bc}\Rightarrow\prod\frac{m_a}{w_a}\overset{(1)}{\leq}\prod\frac{a^2+b^2}{2ab}$$

$$\therefore (1),(2)\Rightarrow\boxed{\prod\frac{a^2+b^2}{2ab}\geq\prod\frac{(a+b)m_a}{2ar_a},\prod\frac{m_a}{w_a}}$$

$$\Rightarrow\prod\frac{a^2+b^2}{2ab}\geq max\left(\prod\frac{(a+b)m_a}{2ar_a},\prod\frac{m_a}{w_a}\right)\ (Proved)$$

Solution 2 by Bogdan Fuștei-Romania

$$s_a=\frac{2bc}{b^2+c^2}m_a\ \text{(and the analogs)}$$

$$s_a\leq w_a\ \text{(and the analogs)}$$

$$w_a=\frac{2bc}{b+c}\cos\frac{A}{2}\ \text{(and the analogs)}$$

$$\cos\frac{A}{2}=\sqrt{\frac{r_br_c}{bc}}\ \text{(and the analogs)}$$

$$s_as_bs_c=\frac{8a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(a^2+c^2)}m_am_bm_c\Rightarrow$$

$$\Rightarrow\frac{m_am_bm_c}{s_as_bs_c}=\frac{(a^2+b^2)(b^2+c^2)(a^2+c^2)}{8a^2b^2c^2}\geq\frac{m_am_bm_c}{w_aw_bw_c}\Leftrightarrow\prod\frac{a^2+b^2}{2ab}\geq\prod\frac{m_a}{w_a}\quad (1)$$

$$\left.\begin{array}{l} w_aw_bw_c=\dfrac{8a^2b^2c^2}{(a+b)(b+c)(a+c)}\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} \\ \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}=\dfrac{r_ar_br_c}{abc}\end{array}\right\}\Rightarrow$$

$$\Rightarrow w_aw_bw_c=\frac{8a^2b^2c^2}{(a+b)(b+c)(a+c)}\cdot\frac{r_ar_br_c}{abc}$$

$$\frac{w_aw_bw_c}{r_ar_br_c}=\frac{8abc}{(a+b)(b+c)(a+c)}$$

$$\frac{m_am_bm_c}{w_aw_bw_c}=\frac{m_am_bm_c}{r_ar_br_c}\cdot\frac{r_ar_br_c}{w_aw_bw_c}=\frac{m_am_bm_c}{r_ar_br_c}\cdot\frac{(a+b)(b+c)(a+c)}{8abc}$$

$$\frac{m_am_bm_c}{w_aw_bw_c}=\prod\frac{(a+b)}{2ar_a}\le\prod\frac{a^2+b^2}{2ab}\quad (2)$$

From (1) and (2) the inequality from enunciation is proved.

1367. **If in $\Delta ABC, R<2(r+1)$ then:**

$$w_aw_bw_c<(2+h_a)(2+h_b)(2+h_c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$w_a=\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}\overset{AM-GM}{\le}1\cdot\sqrt{s(s-a)}=\sqrt{s(s-a)}$$

$$\textit{Similarly: } w_b\le\sqrt{s(s-b)};\, w_c\le\sqrt{s(s-c)}$$

$$\Rightarrow w_aw_bw_c\le s\sqrt{s(s-a)(s-b)(s-c)}=s\cdot S=s\cdot s\cdot r=s^2r$$

$$RHS=(2+h_a)(2+h_b)(2+h_c)\quad (*)$$

$$>(1+h_a)(1+h_b)(1+h_c)=(h_a+h_b+h_c)+(h_ah_b+h_bh_c+h_ch_a)+h_ah_bh_c$$

$$>h_ah_bh_c+h_ah_b+h_ch_a+h_bh_c$$

$$=\frac{2s^2r}{R}+\frac{2s^2r^2}{R}=\frac{2s^2r+2s^2r^2}{R}$$

$$\textit{We must show that: } s^2r<\frac{2s^2r+2s^2r}{R}\Leftrightarrow Rs^2r<2s^2r+2s^2r^2$$

$$\textit{Which is true because: }\because Rs^2r\overset{R<2(r+1)}{<}2s^2r(1+r)=2s^2r+2s^2r^2.\textit{ Proved.}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow w_aw_bw_c=\prod\frac{2}{(b+c)}\sqrt{bc\cdot s(s-a)}=$$

$$=\frac{8abc}{(a+b)(b+c)(c+a)}\sqrt{s^3(s-a)(s-b)(s-c)}\overset{AM-GM}{\le}\frac{8abc}{2\cdot2\cdot2\sqrt{ab\cdot bc\cdot ca}}s\cdot\Delta$$

$$=s\cdot rs=s^2r\Leftrightarrow(2+h_a)(2+h_b)(2+h_c)$$

$$=8+4\sum h_a+2\sum h_ah_b+\prod h_a$$

$$=8+4\cdot2\Delta\sum\frac{1}{a}+2\cdot4\Delta^2\sum\frac{1}{ab}+\frac{(2\Delta)^3}{\prod a}$$

$$= 8 + 8\Delta \cdot \frac{\sum ab}{4R\Delta} + 8\Delta^2 \cdot \frac{\sum a}{4R\Delta} + \frac{8\left(\frac{abc}{4R}\right)^3}{abc}$$

$$= 8 + \frac{2\sum ab}{R} + \frac{2rs \cdot 2s}{R} + \frac{a^2b^2c^2}{8R^3}$$

$$= 8 + \frac{2(s^2 + r^2 + 4Rr)}{R} + \frac{4s^2r}{R} + \frac{16R^2s^2r^2}{8R^3}$$

$$= 8 + \frac{2s^2+2r^2+8Rr}{R} + \frac{4s^2r}{R} + \frac{2s^2r^2}{R} \Leftrightarrow \textit{Need to show}$$

$$\Rightarrow \prod w_a < \prod (2 + h_a)$$

$$\Rightarrow \prod w_a \le s^2r < 8 + \frac{2s^2 + 2r^2 + 8Rr + 4s^2r + 2s^2r^2}{R}$$

$$\Rightarrow s^2rR < s^2r \cdot 2(r+1) = 2s^2r^2 + 2s^2r < 8R + 2s^2 + 2r^2 + 8Rr + 4s^2r + 2s^2r^2$$

$$\Leftrightarrow 8R + 2s^2 + 2r^2 + 8Rr + 2s^2r > 0 \quad \textit{(*true) (proved)}$$

1368. In ΔABC the following relationship holds:

$$2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \le 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Şerban George Florin-Romania

$$\left(\sum_{cyc} \sqrt{a}\right)^2 \overset{CBS}{\le} 3\sum_{cyc} \sqrt{a}^2 = 3\sum a = 3 \cdot 2s = 6s$$

$$2\sum \sqrt{a} \le 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

$$\left(2\sum \sqrt{a}\right)^2 = 4\left(\sum \sqrt{a}\right)^2 \le 4 \cdot 6s \le 9 \cdot \frac{3abc}{4Rr + r^2}$$

$$24s \le \frac{27 \cdot 4Rrs}{4Rr + r^2} = \frac{108Rrs}{4Rr + r^2}$$

$$24s(4Rr + r^2) \le 108Rrs |: 12s$$

$$2(4Rr + r^2) \le 9Rr, 8Rr + 2r^2 \le 9Rr$$

$$Rr \geq 2r^2 |: 2r \Rightarrow R \geq 2r \text{ (Euler), True}$$

Solution 2 by Adrian Popa-Romania

$$2(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq 3\sqrt{\frac{3abc}{4Rr+r^2}}$$

$$\underbrace{\frac{a}{1}+\frac{b}{1}+\frac{c}{1}}_{2s} \overset{Bergstrom}{\geq} \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{3} \Rightarrow (\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \leq 6s \Rightarrow$$

$$\Rightarrow \sqrt{a}+\sqrt{b}+\sqrt{c} \leq \sqrt{6s} | \cdot 2 \Rightarrow 2(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq 2\sqrt{6s}$$

$$\text{We must show that } 2\sqrt{6s} \leq 3\sqrt{\frac{3\cdot 4Rrs}{4Rr+r^2}}\Big|^2 \Leftrightarrow$$

$$\Rightarrow 24s \leq \frac{108Rrs}{4Rr+r^2} \Leftrightarrow 9sRr + 24sr^2 \leq 108Rrs \Leftrightarrow$$

$$\Leftrightarrow 24sr^2 \leq 12Rrs |: 12sr \Leftrightarrow 2r \leq R \text{ (True)} \Rightarrow \text{Euler}$$

Solution 3 by proposer

$$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab} > a+b > c = (\sqrt{c})^2 \Rightarrow$$

$$\sqrt{a}+\sqrt{b} > \sqrt{c} \text{ - and analogs.}$$

By Mitrinovic's inequality in the triangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$s_1 \leq \frac{3\sqrt{3}}{2}R_1 \Leftrightarrow \frac{1}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq \frac{3\sqrt{3}}{2}\cdot\frac{\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}}{4S_1} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq \frac{3\sqrt{3abc}}{8\cdot\frac{1}{2}\sqrt{4Rr+r^2}} \Leftrightarrow$$

$$\Leftrightarrow 2(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq 3\sqrt{\frac{3abc}{4Rr+r^2}}$$

1369. In $\Delta ABC, g_a$ –Gergonne's cevian the following relationship holds:

$$\sqrt{2}m_a \geq g_a + \frac{|b-c|}{2}\cdot\sqrt{\frac{2h_a-3r}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^{\,2}+a(s-b)(s-c)$$
$$\textit{and } b^2(s-b)+c^2(s-c)=ag_a^{\,2}+a(s-b)(s-c)$$
$$\textit{Adding the above two, we get}:$$
$$(b^2+c^2)(2s-b-c)=an_a^{\,2}+ag_a^{\,2}+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^{\,2}+g_a^{\,2})+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)=$$
$$=2(n_a^{\,2}+g_a^{\,2})+a^2-(b-c)^2$$
$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^{\,2}+g_a^{\,2})$$
$$\Rightarrow 4m_a^{\,2}+(b-c)^2=2(n_a^{\,2}+g_a^{\,2})\Rightarrow 4m_a^{\,2}+(b-c)^2+4r_br_c=$$
$$=2(n_a^{\,2}+g_a^{\,2})+4r_br_c$$
$$\Rightarrow 4m_a^{\,2}+(b-c)^2+4s(s-a)=2(n_a^{\,2}+g_a^{\,2})+4s(s-a)$$
$$\Rightarrow 4m_a^{\,2}+4m_a^{\,2}=2(n_a^{\,2}+g_a^{\,2})+4s(s-a)\Rightarrow \boxed{n_a^{\,2}+g_a^{\,2}\overset{(1)}{=}4m_a^{\,2}-2s(s-a)}$$
$$\text{Now, } b^2(s-c)+c^2(s-b)=an_a^{\,2}+a(s-b)(s-c)\Rightarrow s(b^2+c^2)-bc(2s-a)=$$
$$=an_a^{\,2}+a(s^2-s(2s-a)+bc)$$
$$\Rightarrow s(b^2+c^2)-2sbc=an_a^{\,2}+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^{\,2}-as^2$$
$$\Rightarrow an_a^{\,2}=as^2+s(2bc\cos A-2bc)$$
$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}$$
$$=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\ \therefore\ \boxed{n_a^{\,2}\overset{(2)}{=}s^2-2h_ar_a}$$
$$\text{Now, } g_a+\frac{|b-c|}{2}\sqrt{\frac{2h_a-3r}{r}}\overset{CBS}{\leq}\sqrt{2}\sqrt{g_a^{\,2}+\frac{(b-c)^2}{4}\left(\frac{2h_a-3r}{r}\right)}$$
$$\overset{\text{by (1) and (2)}}{=}\sqrt{2}\sqrt{4m_a^{\,2}-2s(s-a)-s^2+2h_ar_a+\frac{(b-c)^2}{4}\left(\frac{4s}{a}-3\right)}$$
$$=\sqrt{2}\sqrt{4m_a^{\,2}-2s(s-a)-s^2+\frac{4s(s-a)(s-b)(s-c)}{a(s-a)}+\frac{(b-c)^2}{4}\left(\frac{4s}{a}-3\right)}$$

$$= \sqrt{2}\sqrt{4m_a{}^2 - 2s(s-a) - s^2 + \frac{s(c+a-b)(a+b-c)}{a} + \frac{(b-c)^2}{4}\left(\frac{4s}{a} - 3\right)}$$

$$= \sqrt{2}\sqrt{4m_a{}^2 - 2s(s-a) - s^2 + \frac{s(a^2-(b-c)^2)}{a} + \frac{s(b-c)^2}{a} - \frac{3(b-c)^2}{4}}$$

$$= \sqrt{2}\sqrt{4m_a{}^2 - 2s(s-a) - s^2 + sa - \frac{3(b-c)^2}{4}}$$

$$= \sqrt{2}\sqrt{4s(s-a) + (b-c)^2 - 2s(s-a) - s(s-a) - \frac{3(b-c)^2}{4}}$$

$$= \sqrt{2}\sqrt{s(s-a) + \frac{(b-c)^2}{4}} = \sqrt{2}\sqrt{\frac{4s(s-a)+(b-c)^2}{4}}$$

$$= \sqrt{2}\sqrt{\frac{4m_a{}^2}{4}} = m_a\sqrt{2} \Rightarrow m_a\sqrt{2} \ge g_a + \frac{|b-c|}{2}\sqrt{\frac{2h_a - 3r}{r}} \text{ (Proved)}$$

1370. In ΔABC the following relationship holds:

$$\frac{1}{\sqrt{2}}\sum_{cyc}\frac{n_a}{r_a} + \sum_{cyc}\sqrt{\frac{h_a}{r_a}} \le \frac{s}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Firstly, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) =$$

$$= an_a{}^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a{}^2 + a(s^2 - s(2s-a) + bc)$$

$$\Rightarrow s(b^2+c^2) - 2sbc = an_a{}^2 + a(as - s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc) = an_a{}^2 - as^2 \Rightarrow an_a{}^2 = as^2 + s(2bc\cos A - 2bc)$$

$$= as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore \boxed{n_a{}^2 \overset{(1)}{=} s^2 - 2h_ar_a}$$

$$Now, \frac{n_a}{\sqrt{2}r_a} + \sqrt{\frac{h_a}{r_a}} \overset{CBS}{\leq} \sqrt{2}\sqrt{\frac{n_a{}^2}{2r_a{}^2} + \frac{h_a}{r_a}} =$$

$$= \sqrt{2}\sqrt{\frac{n_a{}^2 + 2h_ar_a}{2r_a{}^2}} \overset{by\,(1)}{=} \sqrt{2}\sqrt{\frac{s^2 - 2h_ar_a + 2h_ar_a}{2r_a{}^2}} = \frac{s}{r_a} \Rightarrow \frac{n_a}{\sqrt{2}r_a} + \sqrt{\frac{h_a}{r_a}} \overset{(a)}{\leq} \frac{s}{r_a}$$

$$Similarly, \frac{n_b}{\sqrt{2}r_b} + \sqrt{\frac{h_b}{r_b}} \overset{(b)}{\leq} \frac{s}{r_b} \text{ and } \frac{n_c}{\sqrt{2}r_c} + \sqrt{\frac{h_c}{r_c}} \overset{(c)}{\leq} \frac{s}{r_c}$$

$$(a) + (b) + (c) \Rightarrow \sum\left(\frac{n_a}{\sqrt{2}r_a} + \sqrt{\frac{h_a}{r_a}}\right) \leq s\sum\frac{1}{r_a} = \frac{s}{rs}\sum(s-a) = \frac{3s-2s}{r} = \frac{s}{r}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sum\frac{n_a}{r_a} + \sum\sqrt{\frac{h_a}{r_a}} \leq \frac{s}{r} \text{ (Proved)}$$

1371. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian the following relationship holds:

$$\frac{2m_a + n_a + g_a}{h_a} + \sqrt{\frac{r_b + r_c}{h_a}} \leq \frac{(1+\sqrt{3})R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s.\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\prod\cos\frac{A}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\Rightarrow r_b + r_c \overset{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$Now, Stewart's\ theorem \Rightarrow b^2(s-c) + c^2(s-b)$$

$$= an_a{}^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) = ag_a{}^2 + a(s-b)(s-c)$$

$$\textit{Adding the above two, we get}: \ (b^2+c^2)(2s-b-c)=$$

$$=a{n_a}^2+a{g_a}^2+2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2)=2a({n_a}^2+{g_a}^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$

$$=2({n_a}^2+{g_a}^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2({n_a}^2+{g_a}^2)$$

$$\Rightarrow 4{m_a}^2+(b-c)^2=2({n_a}^2+{g_a}^2)\Rightarrow 4{m_a}^2+(b-c)^2+4r_br_c=$$

$$2({n_a}^2+{g_a}^2)+4r_br_c\Rightarrow 4{m_a}^2+(b-c)^2+4s(s-a)=2({n_a}^2+{g_a}^2)+4s(s-a)$$

$$\Rightarrow 4{m_a}^2+4{m_a}^2=2({n_a}^2+{g_a}^2)+4s(s-a)\Rightarrow \boxed{{n_a}^2+{g_a}^2\overset{(2)}{=}4{m_a}^2-2s(s-a)}$$

$$Now,\ \frac{2m_a+n_a+g_a}{h_a}+\sqrt{\frac{r_b+r_c}{h_a}}$$

$$=\frac{2m_a}{h_a}+\frac{n_a+g_a}{h_a}+\sqrt{\frac{r_b+r_c}{h_a}}\overset{m_a\le\frac{Rh_a}{2r}}{\le}\frac{R}{r}+\left(\frac{n_a}{h_a}+\frac{g_a}{h_a}+\sqrt{\frac{r_b+r_c}{h_a}}\right)\overset{CBS}{\le}$$

$$\le\frac{R}{r}+\sqrt{3}\sqrt{\frac{{n_a}^2}{{h_a}^2}+\frac{{g_a}^2}{{h_a}^2}+\frac{r_b+r_c}{h_a}}$$

$$\overset{\text{using (1)and (2)}}{=}\frac{R}{r}+\sqrt{3}\sqrt{\frac{4{m_a}^2-2s(s-a)+\left(\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\right)4R\cos^2\frac{A}{2}}{{h_a}^2}}=$$

$$=\frac{R}{r}+\sqrt{3}\sqrt{\frac{4{m_a}^2-2s(s-a)+\frac{2rs^2}{s\tan\frac{A}{2}}}{{h_a}^2}}=\frac{R}{r}+\sqrt{3}\sqrt{\frac{4{m_a}^2-2s(s-a)+\frac{2r_ar_br_c}{r_a}}{{h_a}^2}}$$

$$=\frac{R}{r}+\sqrt{3}\sqrt{\frac{4{m_a}^2-2s(s-a)+2s(s-a)}{{h_a}^2}}=\frac{R}{r}+\sqrt{3}\left(\frac{2m_a}{h_a}\right)\overset{m_a\le\frac{Rh_a}{2r}}{\le}$$

$$\le\frac{R}{r}+\sqrt{3}\left(\frac{R}{r}\right)=\frac{(1+\sqrt{3})R}{r}\ \text{(Proved)}$$

1372. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a(cosB+cosC)}{b+c}\ge\frac{3}{2}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution 1 by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a(cosB+cosC)}{b+c}=\sum_{cyc}\frac{2RsinA(cosB+cosC)}{2R(sinB+sinC)}=$$

$$=\sum_{cyc}\frac{sinA\cdot 2cos\frac{B+C}{2}cos\frac{B-C}{2}}{2sin\frac{B+C}{2}cos\frac{B-C}{2}}=\sum_{cyc}sin\left(2\cdot\frac{A}{2}\right)cot\frac{B+C}{2}=$$

$$=\sum_{cyc}2sin\frac{A}{2}cos\frac{A}{2}cot\frac{\pi-A}{2}=\sum_{cyc}2sin\frac{A}{2}cos\frac{A}{2}tan\frac{A}{2}=$$

$$=\sum_{cyc}2sin\frac{A}{2}cos\frac{A}{2}\cdot\frac{sin\frac{A}{2}}{cos\frac{A}{2}}=2\sum_{cyc}sin^2\frac{A}{2}=$$

$$=2\left(1-\frac{r}{2R}\right)=\frac{2R-r}{R}\overset{EULER}{\ge}\frac{2R-\frac{R}{2}}{R}=\frac{3R}{2R}=\frac{3}{2}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum_{cyc}\frac{a(cosB+cosC)}{b+c}=\sum_{cyc}\frac{(acosB+bcosA)+(acosC+ccosA)-(b+c)cosA}{b+c}=$$

$$=\sum_{cyc}\frac{c+b-(b+c)cosA}{b+c}=\sum_{cyc}(1-cosA)=3-\sum_{cyc}cosA=3-1-\frac{r}{R}\overset{EULER}{\ge}\frac{3}{2}$$

1373. If in $\Delta ABC, abc=1$ then:

$$\sum_{cyc}\left(2\sqrt{a}+\frac{1}{a}\right)+\sqrt{\sum_{cyc}\frac{\cos A}{a^3}}\ge 9+\frac{\sqrt{6}}{2}$$

Proposed by Radu Diaconu-Romania

Solution by Marian Ursărescu-Romania

$$2\sqrt{a}+\tfrac{1}{a}=\sqrt{a}+\sqrt{a}+\tfrac{1}{a}\ge 3\sqrt[3]{a\cdot\tfrac{1}{a}}=3\Rightarrow\sum\left(2\sqrt{a}+\tfrac{1}{a}\right)\ge 9\Rightarrow \textit{we must show:}$$

$$\sqrt{\sum\frac{\cos A}{a^3}}\ge\frac{\sqrt{6}}{2}\Leftrightarrow\sum\frac{\cos A}{a^3}\ge\frac{3}{2}\ \ (1)$$

$$\text{But } \cos A=\frac{b^2+c^2-a^2}{2bc}\ \ (2)$$

From (1)+(2) we must show:

$$\frac{1}{2}\sum\frac{b^2+c^2-a^2}{a^3bc}\ge\frac{3}{2}\Leftrightarrow\sum\frac{b^2+c^2-a^2}{a^2}\ge 3\Leftrightarrow$$

$$\Leftrightarrow\frac{b^2}{a^2}+\frac{a^2}{b^2}+\frac{a^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-3\ge 3\Leftrightarrow$$

$$\Leftrightarrow\frac{b^2}{a^2}+\frac{a^2}{b^2}+\frac{a^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}\ge 6,\ \text{true because } x+\frac{1}{x}\ge 2,\forall x>0$$

1374. In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b}\right)^4+\left(\frac{r_b}{r_c}\right)^4+\left(\frac{r_c}{r_a}\right)^4+\frac{2nr}{R}\ge n+3,n\le 8$$

Proposed by Marin Chirciu-Romania

Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$n=8$$

$$r_a=x,r_b=y,r_c=z\ \text{inequality becomes:}$$

$$\left(\frac{x}{y}\right)^4+\left(\frac{y}{z}\right)^4+\left(\frac{z}{x}\right)^4+\frac{64xyz}{(x+y)(y+z)(x+z)}\ge 11\ \ (1)$$

$$(1)\Rightarrow\frac{x^4+y^4}{y^4}+\frac{y^4+z^4}{z^4}+\frac{z^4+x^4}{x^4}+\frac{64xyz}{(x+y)(y+z)(x+z)}\ge 14\ \ (2)$$

Lemma.

$$x^4+y^4\ge\frac{1}{8}(x+y)^4\ \ (3)$$

$$(2)\Rightarrow WLOG\ M=\frac{16xyz}{(x+y)(y+z)(x+z)}$$

$$LHS=\sum\frac{x^4+y^4}{y^4}+M+M+M+M\ge$$

$$\ge 7\cdot\sqrt[7]{\frac{x^4+y^4}{y^4}\cdot\frac{y^4+z^4}{z^4}\cdot\frac{z^4+x^4}{x^4}\cdot\frac{16^4x^4y^4z^4}{(x+y)^4(y+z)^4(z+x)^4}}\ge$$

$$\ge 7\sqrt[7]{\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot 16^4}=72=14$$

We propose the general form:

$$\left(\frac{x}{y}\right)^n+\left(\frac{y}{z}\right)^n+\left(\frac{z}{x}\right)^n+\frac{16nxyz}{(x+y)(y+z)(x+z)}\ge 2n+3$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$Let\ s-a=x, s-b=y, s-c=z$$

$$\therefore 3s-2s=s=\sum x\Rightarrow a=y+z, b=z+x, c=x+y$$

$$Now, \sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}\ge 7\Leftrightarrow\sum\left(\frac{s-b}{s-a}\right)^2+8\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right)$$

$$\overset{via\ above\ transformation}{\Leftrightarrow}\sum\frac{y^2}{x^2}+\frac{32s(s-a)(s-b)(s-c)}{s\prod(x+y)}\ge 7$$

$$\overset{via\ above\ transformation}{\Leftrightarrow}\sum\frac{y^2}{x^2}+\frac{32xyz}{\prod(x+y)}\ge 7\Leftrightarrow$$

$$\sum\frac{y^2}{x^2}+3+\frac{32xyz}{\prod(x+y)}\ge 10\Leftrightarrow$$

$$\sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}\overset{(i)}{\ge}10$$

$$Now, \sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}=$$

$$\sum\frac{y^2+x^2}{x^2}+\frac{16xyz}{\prod(x+y)}+\frac{16xyz}{\prod(x+y)}\overset{A-G}{\ge}$$

$$5\sqrt[5]{\left(\frac{2^8(xyz)^2}{\prod x^2}\right)\left(\frac{\prod(x^2+y^2)}{\prod(x+y)^2}\right)}\ge 5\sqrt[5]{2^8\frac{\prod\left(\frac{1}{2}(x+y)^2\right)}{\prod(x+y)^2}}=5\sqrt[5]{2^5}$$

$$=10\Rightarrow(i)\ is\ true\ \therefore\sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}\ge 7\Rightarrow\boxed{\sum\frac{r_a^2}{r_b^2}\overset{(1)}{\ge}7-\frac{8r}{R}}$$

$$Now, \sum\frac{r_a^4}{r_b^4}+\frac{16r}{R}-11\ge\frac{1}{3}\left(\sum\frac{r_a^2}{r_b^2}\right)^2+\frac{16r}{R}-11\overset{by\ (1)}{\ge}$$

$$\frac{1}{3}\left(\frac{7R-8r}{R}\right)^2+\frac{16r-11R}{R}=\frac{(7R-8r)^2+48Rr-33R^2}{3R^2}=\frac{16(R-2r)^2}{3R^2}\ge 0$$

$$\Rightarrow \sum \frac{r_a^4}{r_b^4} + \frac{16r}{R} - 11 + \frac{2nr}{R} - \frac{2nr}{R} - (n+3) + (n+3) \geq 0 \Rightarrow$$

$$\sum \frac{r_a^4}{r_b^4} + \frac{2nr}{R} - (n+3) \geq \frac{2nr}{R} - \frac{16r}{R} + 11 - (n+3)$$

$$= \frac{2r}{R}(n-8) - (n-8) = (n-8)\left(\frac{2r}{R} - 1\right) \geq 0$$

$$\left(\because n - 8 \leq 0 \text{ and } \frac{2r}{R} - 1 \overset{Euler}{\leq} 0\right) \Rightarrow$$

$$\sum \frac{r_a^4}{r_b^4} + \frac{2nr}{R} \geq n + 3 \ \forall\, n \leq 8 \ (Proved)$$

1375. In ΔABC the following relationship holds:

$$\sum_{cyc} \left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)^3 \geq \frac{3s}{4R}$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$We\ have: \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$$

$$Let\ x = \cos\frac{A}{2};\ y = \cos\frac{B}{2};\ z = \cos\frac{C}{2}$$

$$(x, y, z > 0)$$

We just check:

$$\sum_{cyc} (x + y - z)^3 \geq 3xyz$$

$$\sum_{cyc} (x + y - z)^3 = (x + y - z)^3 + (x + z - y)^3 + (y + z - x)^3$$

$$= x^3 + y^3 + z^3 + 3xy^2 + 3yx^2 + 3xz^2 + 3zx^2 + 3yz^2 + 3zy^2 - 18xyz$$

So, we prove:

$$x^3 + y^3 + z^3 + 3xy^2 + 3yx^2 + 3xz^2 + 3zx^2 + 3yz^2 + 3zy^2 - 18xyz \geq 3xyz$$

$$\Leftrightarrow \sum x^3 + 3\left(\sum xy^2 + \sum yx^2\right) \geq 21xyz$$

$$It\ is\ true\ because: \sum x^3 \overset{AM-GM}{\geq} 3xyz$$

$$3(xy^2+yx^2+xz^2+zx^2+yz^2+zy^2)\ge 3\cdot 6\sqrt[6]{x^6y^6z^6}=18xyz.\ \textit{Proved.}$$

1376. In $\Delta ABC, n_a -$ Nagel's cevian, the following relationship holds:

$$\frac{n_a}{a}+\frac{n_b}{b}+\frac{n_c}{c}\le\frac{s}{2r}\left(\frac{R}{r}-1\right)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Marian Ursărescu-Romania

From Cauchy's inequality we have:

$$\left(\frac{n_a}{a}+\frac{n_b}{b}+\frac{n_c}{c}\right)^2\le\left(n_a^2+n_b^2+n_c^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\Rightarrow \textit{we must show:}$$

$$\left(n_a^2+n_b^2+n_c^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\le\frac{s^2}{4r^2}\left(\frac{R}{r}-1\right)^2\quad (1)$$

But from Steining inequality: $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$ *(2)*

From (1)+(2) we must show: $n_a^2+n_b^2+n_c^2\le\frac{s^2(R-r)^2}{r^2}$ *(3)*

From Stewart relation we have:

$$n_a^2=s^2-\frac{4s(s-b)(s-c)}{a}\Rightarrow\sum n_a^2=3s^2-4s\sum\frac{(s-b)(s-c)}{a}=$$

$$=3s^2-4s\cdot\frac{r[s^2+(4R+r)^2]}{4sR}=3s^2-\frac{r[s^2+(4Rr+r)^2]}{R}\quad (4)$$

From (3)+(4) we must show: $3s^2-\frac{r[s^2+(4R+r)^2]}{R}\le\frac{s^2(R-r)^2}{r^2}$ *(5)*

From Doucet inequality we have $(4R+r)^2\ge 3s^2$ *(6)*

From (5)+(6) we must show: $s^2\left(3-\frac{4r}{R}\right)\le\frac{s^2(R-r)^2}{r^2}\Leftrightarrow$

$r^2(3R-4r)\le R(R-r)^2$*. But* $r\le\frac{R}{2}\Rightarrow$ *we must show:* $r(3R-4r)\le 2(R-r)^2$

$$\Leftrightarrow 3Rr-4r^2\le 2R^2-4Rr+2r^2\Leftrightarrow 2R^2-7Rr+6r^2\Leftrightarrow$$

$(2R-3r)(R-2r)\ge 0$*, true because* $R\ge 2r$*.*

Solution 2 by Soumava Chakraborty-Kolkata-India

Stewart's theorem

$$\Rightarrow b^2(s-c)+c^2(s-b)=a{n_a}^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=$$

$$= a{n_a}^2 + a(s^2 - s(2s - a) + bc) \Rightarrow$$

$$\Rightarrow s(b^2 + c^2) - 2sbc = a{n_a}^2 + a(as - s^2)$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a{n_a}^2 - as^2 \Rightarrow$$

$$\Rightarrow a{n_a}^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} =$$

$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) =$$

$$= as^2 - 2ah_ar_a \therefore {n_a}^2 \overset{(1)}{=} s^2 - 2h_ar_a$$

$$\text{Now, } \frac{n_a}{h_a} \le \frac{R}{r} - 1 \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} + 1 \ge \frac{n_a^2}{h_a^2} \overset{\text{by }(1)}{\Leftrightarrow}$$

$$\frac{R^2}{r^2} - \frac{2R}{r} + 1 \ge \frac{s^2 - 2h_ar_a}{h_a^2} = \frac{s^2a^2}{4r^2s^2} - \frac{2r_a}{h_a} =$$

$$= \frac{a^2}{4r^2} - \left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right) = \frac{a^2}{4r^2} - \frac{(a-s)+s}{s-a}$$

$$= \frac{a^2}{4r^2} + 1 - \frac{s}{s-a} = 1 + \frac{a^2(s-a) - 4(sr^2)}{4(s-a)r^2} =$$

$$= 1 + \frac{a^2(s-a) - 4(s-a)(s-b)(s-c)}{4(s-a)r^2} =$$

$$= 1 + \frac{a^2 - (a^2 - (b-c)^2)}{4r^2} = 1 + \frac{(b-c)^2}{4r^2}$$

$$\Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} \ge \frac{(b-c)^2}{4r^2} \Leftrightarrow$$

$$\frac{R(R-2r)}{r^2} \ge \frac{b^2 + c^2 - 2bc}{4r^2} \Leftrightarrow$$

$$R - 2r \ge \frac{b^2 + c^2}{4R} - \frac{bc}{2R}$$

$$\Leftrightarrow R\left(1 - \frac{2r}{R}\right) \ge \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2\sin B\sin C}{2R} \Leftrightarrow$$

$$\Leftrightarrow 1 - \frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} \ge \sin^2 B + \sin^2 C - 2\sin B\sin C = (\sin B - \sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\geq\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2\Leftrightarrow$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\geq 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\geq 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}\Leftrightarrow$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\geq 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\geq 0\rightarrow \text{true}\Rightarrow$$

$$\frac{n_a}{h_a}\leq\frac{R}{r}-1=\frac{R-r}{r}\Rightarrow\boxed{\frac{n_a}{a}\leq\left(\frac{R-r}{r}\right)\left(\frac{2rs}{a^2}\right)\text{ and analogs}}$$

$$\therefore\sum\frac{n_a}{a}\leq\left(\frac{R-r}{r}\right)\left(\frac{2rs}{16R^2r^2s^2}\right)\left(\sum a^2b^2\right)\overset{\text{Goldstone}}{\leq}$$

$$\left(\frac{R-r}{r}\right)\left(\frac{2rs}{16R^2r^2s^2}\right)(4R^2s^2)=\frac{s}{2r}\left(\frac{R}{r}-1\right)\ \text{(Proved)}$$

1377. In ΔABC the following relationship holds:

$$\frac{1}{sinA}\sqrt{\frac{2\cdot\sum_{cyc}sinA\cdot\prod_{cyc}(sinA+sinB-sinC)}{3+cos2A-2cos2B-2cos2C}}\leq 1$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Daniel Sitaru-Romania

$$2\cdot\sum_{cyc}sinA\cdot\prod_{cyc}(sinA+sinB-sinC)=\frac{2}{16R^4}\sum_{cyc}2RsinA\cdot\prod_{cyc}(2RsinA+2RsinB-2RsinC)=$$

$$=\frac{2\cdot 2s\cdot 2(s-a)\cdot 2(s-b)\cdot 2(s-c)}{16R^4}=\frac{2S^2}{R^4}$$

$$3+cos2A-2cos2B-2cos2C=3+1-2sin^2A-2+4sin^2B-$$

$$-2+4sin^2C=2\left(\frac{2b^2+2c^2-a^2}{4R^2}\right)=\frac{2m_a^2}{R^2}$$

$$LHS=\frac{1}{sinA}\sqrt{\frac{2S^2}{R^4}\cdot\frac{R^2}{2m_a^2}}=\frac{S}{sinA\cdot Rm_a}\leq\frac{S}{sinA\cdot Rh_a}=\frac{ah_a}{\frac{a}{2R}\cdot Rh_a}=1$$

1378. If in $\Delta ABC, AA_1, BB_1, CC_1$ –medians, AA_2, BB_2, CC_2 –circumcevians of centroid, $F = [ABC]$ then the following relationship holds:

$$\sum_{cyc} A_1A_2 \cdot B_1B_2 \cdot \sin^2 C \le \frac{\sqrt{3}}{4} \cdot F$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Daniel Sitaru-Romania

$$\rho(A_1) = AA_1 \cdot A_1A_2 = A_1B \cdot A_1C \to m_a \cdot A_1A_2 = \frac{a^2}{4}$$

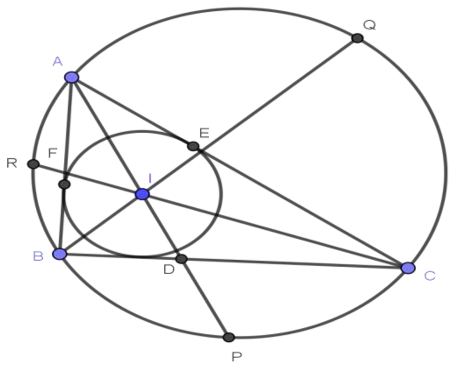
$$A_1A_2 = \frac{a^2}{4m_a}, B_1B_2 = \frac{b^2}{4m_b}, C_1C_2 = \frac{c^2}{4m_c}$$

$$\sum_{cyc} A_1A_2 \cdot B_1B_2 \cdot \sin^2 C = \sum_{cyc} \frac{a^2b^2\sin^2 C}{16m_am_b} = \sum_{cyc} \frac{(2F)^2}{16m_am_b} =$$

$$= \frac{F^2}{4}\sum_{cyc}\frac{1}{m_am_b} \le \frac{F^2}{4}\sum_{cyc}\frac{1}{\sqrt{s(s-a)\cdot s(s-b)}} = \frac{F^2}{4\sqrt{s}}\sum_{cyc}\frac{\sqrt{s-c}}{F}$$

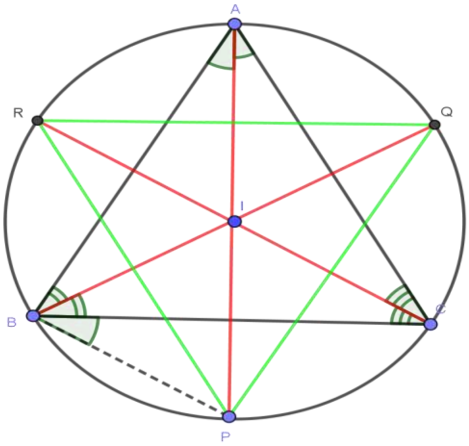
$$= \frac{F}{4\sqrt{s}}\sum_{cyc}\left(1\cdot\sqrt{s-c}\right) \overset{CBS}{\le} \le \frac{F}{4\sqrt{s}}\sqrt{3(s-a+s-b+s-c)} = \frac{\sqrt{3}}{4}F$$

1379. Prove that: $\left(\frac{AP}{DP}\right)\left(\frac{BQ}{EQ}\right)\left(\frac{CR}{FR}\right) \le 2^2\left(\frac{2}{3}\right)^2\left(\frac{2s}{3}\right)^2\left(\frac{2s}{3Rr}\right)^2$



Proposed by Thanasis Gakopoulos-Larisa-Greece

Solution by Soumava Chakraborty-Kolkata-India



$\because$ *∠PBC and ∠PAC are ∠s on same arc and on the same side of it,*

$$\angle PBC = \angle PAC = \frac{A}{2}$$

$$\therefore \angle ABP = B + \frac{A}{2}$$

$$= \frac{2B + A}{2} = \frac{B + (180^\circ - C)}{2} \overset{(1)}{=} 90^\circ + \frac{B - C}{2}$$

Using sine rule on ΔABP

$$AP = 2R\sin(\angle ABP) \overset{by\ (1)}{=} 2R\sin\left(90^\circ + \frac{B - C}{2}\right) = 2R\cos\frac{B - C}{2}$$

$$= \frac{R\left(2\sin\frac{B + C}{2}\cos\frac{B - C}{2}\right)}{\cos\frac{A}{2}} = \frac{R(\sin B + \sin C)}{\cos\frac{A}{2}}$$

$$= \frac{R(b + c)}{2R\cos\frac{A}{2}} = \frac{b + c}{2\cos\frac{A}{2}} \Rightarrow AP \overset{(a)}{=} \frac{b + c}{2\cos\frac{A}{2}}$$

$$\textit{Similarly, } BQ \overset{(a)}{=} \frac{c+a}{2\cos\frac{B}{2}} \textit{ and } CR \overset{(c)}{=} \frac{a+b}{2\cos\frac{C}{2}}$$

$$(a) \Rightarrow AP = bc\left(\frac{b+c}{2bc\cos\frac{A}{2}}\right) \overset{(i)}{=} \frac{bc}{w_a} \Rightarrow DP = AP - AD = \frac{bc}{w_a} - w_a \overset{(ii)}{=} \frac{bc - w_a^2}{w_a}$$

$$(i), (ii) \Rightarrow \frac{AP}{DP} = \frac{\left(\frac{bc}{w_a}\right)}{\left(\frac{bc - w_a^2}{w_a}\right)} = \frac{bc}{bc - \frac{4bcs(s-a)}{(b+c)^2}}$$

$$= \frac{(b + c)^2}{(b + c)^2 - (b + c + a)(b + c - a)} = \frac{(b + c)^2}{(b + c)^2 - (b + c)^2 + a^2} = \left(\frac{b + c}{a}\right)^2$$

$$\therefore \frac{AP}{DP} \overset{(d)}{=} \left(\frac{b+c}{a}\right)^2$$

$$\text{Similarly, (b)} \Rightarrow \frac{BQ}{EQ} \overset{(e)}{=} \left(\frac{c+a}{b}\right)^2 \text{ and (c)} \Rightarrow \frac{CR}{FR} \overset{(f)}{=} \left(\frac{a+b}{c}\right)^2$$

$$\text{(d).(e).(f)} \Rightarrow \frac{AP}{DP}\cdot\frac{BQ}{EQ}\cdot\frac{CR}{FR} = \left(\prod\left(\frac{b+c}{a}\right)\right)^2 \le 2^2\left(\frac{2}{3}\right)^2\left(\frac{2s}{3}\right)^2\left(\frac{2s}{3Rr}\right)^2$$

$$\Leftrightarrow \prod\left(\frac{b+c}{a}\right) \le 2\left(\frac{2}{3}\right)\left(\frac{2s}{3}\right)\left(\frac{2s}{3Rr}\right)$$

$$\Leftrightarrow \frac{2s(s^2+2Rr+r^2)}{4Rrs} \le 2\left(\frac{2}{3}\right)\left(\frac{2s}{3}\right)\left(\frac{2s}{3Rr}\right) \Leftrightarrow 32s^2 \ge 27(s^2+2Rr+r^2)$$

$$\Leftrightarrow 5s^2 \overset{(2)}{\ge} 54Rr + 27r^2$$

$$\text{Now, } 5s^2 \overset{Gerretsen}{\ge} 80Rr - 25r^2 \overset{?}{\ge} 54Rr + 27r^2 \Leftrightarrow 26Rr \overset{?}{\ge} 52r^2 \Leftrightarrow R \overset{?}{\ge} 2r$$

$\rightarrow$ ***true (Euler)*** $\Rightarrow$ ***(2) is true (Proved)***

1380. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})}{\sqrt[4]{bc}} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})}{\sqrt[4]{bc}} \overset{(1)}{\le} \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab} > c+2\sqrt{ab} > c$$

$$\Rightarrow \sqrt{a}+\sqrt{b} > \sqrt{c} \textit{ and similarly, } \sqrt{b}+\sqrt{c} > \sqrt{a} \textit{ and } \sqrt{c}+\sqrt{a} > \sqrt{b}$$

$$\Rightarrow \sqrt{a}, \sqrt{b}, \sqrt{c} \textit{ are sides of a triangle with semiperimeter, circumradius and inradius}$$

$$= p, x, y \textit{ respectively (say)}$$

$$\textit{and let } \alpha = \sqrt{a}, \beta = \sqrt{b} \textit{ and } \gamma = \sqrt{c} \textit{ and } \therefore (1) \Leftrightarrow \sum \frac{4(p-\beta)(p-\gamma)}{\sqrt{\beta\gamma}} \le 2p \Leftrightarrow$$

$$\Leftrightarrow (p-\alpha)(p-\beta)(p-\gamma)\sum \frac{1}{\sqrt{\beta\gamma}(p-\alpha)} \le \frac{p}{2}$$

$$\Leftrightarrow y^2p\sum\frac{1}{\sqrt{\beta\gamma}(p-\alpha)}\le\frac{p}{2}\Leftrightarrow\sum\frac{1}{\sqrt{\beta\gamma}(p-\alpha)}\overset{(2)}{\le}\frac{1}{2y^2}$$

$$\text{Now,}\sum\frac{1}{\sqrt{\beta\gamma}(p-\alpha)}=\sum\frac{1}{\sqrt{p-\alpha}\sqrt{\beta\gamma(p-\alpha)}}\overset{CBS}{\le}\sqrt{\sum\frac{1}{p-\alpha}}\sqrt{\sum\frac{1}{\beta\gamma(p-\alpha)}}=$$

$$=\sqrt{\frac{\sum(p-\beta)(p-\gamma)}{y^2p}}\sqrt{\frac{\sum\alpha(p-\beta)(p-\gamma)}{\alpha\beta\gamma(p-\alpha)(p-\beta)(p-\gamma)}}$$

$$=\sqrt{\frac{4xy+y^2}{y^2p}}\sqrt{\frac{\sum\alpha(p^2-p(\beta+\gamma)+\beta\gamma)}{4xyp.y^2p}}=$$

$$=\sqrt{\frac{4x+y}{yp}}\sqrt{\frac{2p.p^2-2p(p^2+4xy+y^2)+12xyp}{4xyp.y^2p}}=\sqrt{\frac{4x+y}{yp}}\sqrt{\frac{2py(2x-y)}{4xyp.y^2p}}$$

$$=\frac{1}{y}\sqrt{\frac{4x+y}{yp}}\sqrt{\frac{2x-y}{2xp}}=\frac{1}{py}\sqrt{\frac{4x+y}{y}}\sqrt{\frac{2x-y}{2x}}\therefore\sum\frac{1}{\sqrt{\beta\gamma}(p-\alpha)}\overset{(i)}{\le}\frac{1}{py}\sqrt{\frac{4x+y}{y}}\sqrt{\frac{2x-y}{2x}}$$

$$(i)\Rightarrow \textit{in order to prove } (2), \textit{it suffices to prove}: \frac{1}{py}\sqrt{\frac{4x+y}{y}}\sqrt{\frac{2x-y}{2x}}\le\frac{1}{2y^2}$$

$$\Leftrightarrow\frac{p^2}{4y^2}\ge\left(\frac{4x+y}{y}\right)\left(\frac{2x-y}{2x}\right)\Leftrightarrow xp^2\overset{(3)}{\ge}2y(4x+y)(2x-y)$$

$$\textit{Now, by Rouche}, xp^2\ge$$

$$\ge x\left(2x^2+10xy-y^2-2(x-2y)\sqrt{x^2-2xy}\right)\overset{?}{\ge}2y(4x+y)(2x-y)$$

$$\Leftrightarrow 2x^3-6x^2y+3xy^2+2y^3\overset{?}{\ge}2x(x-2y)\sqrt{x^2-2xy}$$

$$\Leftrightarrow (x-2y)(2x^2-2xy-y^2)\underset{(4)}{\overset{?}{\ge}}2x(x-2y)\sqrt{x^2-2xy}$$

$$\because x-2y\overset{Euler}{\ge}0,\therefore \textit{in order to prove } (4), \textit{it suffices to prove}:$$

$$2x^2-2xy-y^2>2x\sqrt{x^2-2xy}$$

$$\Leftrightarrow(2x^2-2xy-y^2)^2>4x^2(x^2-2xy)\Leftrightarrow 4xy^3+y^4>0\to$$

$$\to \textit{true}\Rightarrow(4)\Rightarrow(3)\Rightarrow(2)\Rightarrow(1)\ \textit{is true (Proved)}$$

1381. In any scalene ΔABC, n_a –Nagel's cevian the following relationship holds:

$$\left(\sum_{cyc}\frac{1}{n_a-h_a}\right)\left(\sum_{cyc}\frac{1}{m_a-n_a}\right)>\frac{2}{(R-2r)^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Stewart's\ theorem \Rightarrow b^2(s-c)+c^2(s-b)=a{n_a}^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=a{n_a}^2+a(s^2-s(2s-a)+bc)$$

$$\Rightarrow s(b^2+c^2)-2sbc=a{n_a}^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=a{n_a}^2-as^2\Rightarrow a{n_a}^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\ \therefore\ {n_a}^2\overset{(1)}{=}s^2-2h_ar_a$$

$$\text{Now, }\frac{n_a}{h_a}\le\frac{R}{r}-1\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{n_a^2}{h_a^2}\overset{\text{by (1)}}{\Leftrightarrow}\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{s^2-2h_ar_a}{h_a^2}=$$

$$=\frac{s^2a^2}{4r^2s^2}-\frac{2r_a}{h_a}=\frac{a^2}{4r^2}-\left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right)=\frac{a^2}{4r^2}-\frac{(a-s)+s}{s-a}$$

$$=\frac{a^2}{4r^2}+1-\frac{s}{s-a}=1+\frac{a^2(s-a)-4(sr^2)}{4(s-a)r^2}$$

$$=1+\frac{a^2(s-a)-4(s-a)(s-b)(s-c)}{4(s-a)r^2}=1+\frac{a^2-(a^2-(b-c)^2)}{4r^2}=$$

$$=1+\frac{(b-c)^2}{4r^2}$$

$$\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}\ge\frac{(b-c)^2}{4r^2}\Leftrightarrow\frac{R(R-2r)}{r^2}\ge\frac{b^2+c^2-2bc}{4r^2}\Leftrightarrow R-2r\ge\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\ge\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\ge$$

$$\sin^2 B+\sin^2 C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\ge\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\ge 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\ge 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\ge 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\ge 0\to\text{true}\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1\Rightarrow\frac{n_a-h_a}{h_a}\le\frac{R-2r}{r}$$

$$\Rightarrow n_a-h_a\le\left(\frac{R-2r}{r}\right)\left(\frac{2rs}{a}\right)$$

$$\Rightarrow\frac{1}{n_a-h_a}\ge\frac{a}{2s(R-2r)}\text{ and analogs}\Rightarrow$$

$$\Rightarrow\sum\frac{1}{n_a-h_a}\ge\frac{2s}{2s(R-2r)}\Rightarrow\sum\frac{1}{n_a-h_a}\overset{(i)}{\ge}\frac{1}{R-2r}$$

$$\text{Again, }\frac{m_a}{h_a}\overset{\text{Panaitopol}}{\le}\frac{R}{2r}\Rightarrow\frac{m_a-h_a}{h_a}\le\frac{R-2r}{2r}\Rightarrow m_a-h_a\le\left(\frac{R-2r}{2r}\right)\left(\frac{2rs}{a}\right)$$

$$\Rightarrow\frac{1}{m_a-h_a}\ge\frac{a}{s(R-2r)}\text{ and analogs}$$

$$\Rightarrow\sum\frac{1}{m_a-h_a}\ge\frac{2s}{s(R-2r)}\Rightarrow\sum\frac{1}{m_a-h_a}\overset{(ii)}{\ge}\frac{2}{R-2r}$$

$$(i).(ii)\Rightarrow\left(\sum\frac{1}{n_a-h_a}\right)\left(\sum\frac{1}{m_a-h_a}\right)\ge\frac{2}{(R-2r)^2}\ (\text{Proved})$$

1382. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$2r\le\left(\sum_{cyc}\frac{1}{n_a+h_a}\right)^{-1}\le R$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Perpendicular distance is least among all line segments from A to BC,

$$\therefore h_a \le n_a \Rightarrow n_a + h_a \ge 2h_a \Rightarrow \frac{1}{n_a + h_a} \overset{(i)}{\le} \frac{1}{2h_a}$$

$$\text{Similarly}, \frac{1}{n_b + h_b} \overset{(ii)}{\le} \frac{1}{2h_b} \text{ and } \frac{1}{n_c + h_c} \overset{(iii)}{\le} \frac{1}{2h_c} \therefore (i) + (ii) + (iii) \Rightarrow$$

$$\sum \frac{1}{n_a + h_a} \le \frac{1}{2}\sum \frac{1}{h_a} = \frac{1}{2}\left(\sum \frac{a}{2rs}\right) = \frac{2s}{4rs} = \frac{1}{2r} \Rightarrow$$

$$\Rightarrow \left(\sum \frac{1}{n_a + h_a}\right)^{-1} \overset{(m)}{\ge} 2r$$

Now, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b) = a{n_a}^2 + a(s - b)(s - c)$

$$\Rightarrow s(b^2 + c^2) - bc(2s - a) = a{n_a}^2 + a(s^2 - s(2s - a) + bc) \Rightarrow$$

$$\Rightarrow s(b^2 + c^2) - 2sbc = a{n_a}^2 + a(as - s^2)$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a{n_a}^2 - as^2 \Rightarrow a{n_a}^2 = as^2 + s(2bc\cos A - 2bc)$$

$$= as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}$$

$$= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_ar_a \therefore {n_a}^2 \overset{(1)}{=} s^2 - 2h_ar_a$$

$$\text{Now}, \frac{n_a}{h_a} \le \frac{R}{r} - 1 \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} + 1 \ge \frac{n_a^2}{h_a^2} \overset{\text{by }(1)}{\Leftrightarrow} \frac{R^2}{r^2} - \frac{2R}{r} + 1 \ge \frac{s^2 - 2h_ar_a}{h_a^2} =$$

$$= \frac{s^2a^2}{4r^2s^2} - \frac{2r_a}{h_a} = \frac{a^2}{4r^2} - \left(\frac{2rs}{s - a}\right)\left(\frac{a}{2rs}\right) = \frac{a^2}{4r^2} - \frac{(a - s) + s}{s - a}$$

$$= \frac{a^2}{4r^2} + 1 - \frac{s}{s - a} = 1 + \frac{a^2(s - a) - 4(sr^2)}{4(s - a)r^2} =$$

$$= 1 + \frac{a^2(s - a) - 4(s - a)(s - b)(s - c)}{4(s - a)r^2} =$$

$$= 1 + \frac{a^2 - (a^2 - (b - c)^2)}{4r^2} = 1 + \frac{(b - c)^2}{4r^2}$$

$$\Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} \ge \frac{(b - c)^2}{4r^2} \Leftrightarrow \frac{R(R - 2r)}{r^2} \ge \frac{b^2 + c^2 - 2bc}{4r^2} \Leftrightarrow R - 2r \ge \frac{b^2 + c^2}{4R} - \frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\geq\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}\Leftrightarrow$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\geq\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\geq\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\geq 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\geq 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\geq 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\geq 0\rightarrow\text{true}\Rightarrow\frac{n_a}{h_a}\leq\frac{R}{r}-1\Rightarrow\frac{n_a+h_a}{h_a}\leq\frac{R}{r}$$

$$\Rightarrow n_a+h_a\leq\frac{R}{r}\left(\frac{2rs}{a}\right)=\frac{2Rs}{a}\Rightarrow\frac{1}{n_a+h_a}\overset{(a)}{\geq}\frac{a}{2Rs}$$

$$\textit{Similarly},\frac{1}{n_b+h_b}\overset{(b)}{\geq}\frac{b}{2Rs}\text{ and }\frac{1}{n_c+h_c}\overset{(c)}{\geq}\frac{c}{2Rs}$$

$$\therefore (a)+(b)+(c)\Rightarrow\sum\frac{1}{n_a+h_a}\geq\frac{2s}{2Rs}=\frac{1}{R}\Rightarrow\left(\sum\frac{1}{n_a+h_a}\right)^{-1}\overset{(n)}{\leq}R$$

$$(m),(n)\Rightarrow 2r\leq\left(\sum\frac{1}{n_a+h_a}\right)^{-1}\leq R\ (\text{Proved})$$

1383. **In $\Delta ABC, n_a$ –Nagel's cevian the following relationship holds:**

$$\frac{1}{n_a}+\frac{1}{n_b}+\frac{1}{n_c}\geq\frac{1}{R-r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=a{n_a}^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=a{n_a}^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=a{n_a}^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=a{n_a}^2-as^2\Rightarrow a{n_a}^2=as^2+s(2bc\cos A-2bc)=$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore \boxed{{n_a}^2\overset{(1)}{=}s^2-2h_ar_a}$$

Now, $\frac{n_a}{h_a}\le\frac{R}{r}-1\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{n_a^2}{h_a^2}\overset{\text{by }(1)}{\Leftrightarrow}\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{s^2-2h_ar_a}{h_a^2}=\frac{s^2a^2}{4r^2s^2}-\frac{2r_a}{h_a}$

$=$

$$\frac{a^2}{4r^2}-\left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right)=\frac{a^2}{4r^2}-\frac{(a-s)+s}{s-a}$$

$$=\frac{a^2}{4r^2}+1-\frac{s}{s-a}=1+\frac{a^2(s-a)-4(sr^2)}{4(s-a)r^2}=$$

$$1+\frac{a^2(s-a)-4(s-a)(s-b)(s-c)}{4(s-a)r^2}=1+\frac{a^2-(a^2-(b-c)^2)}{4r^2}=1+\frac{(b-c)^2}{4r^2}$$

$$\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}\ge\frac{(b-c)^2}{4r^2}\Leftrightarrow\frac{R(R-2r)}{r^2}\ge\frac{b^2+c^2-2bc}{4r^2}\Leftrightarrow R-2r\ge\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\ge\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\ge$$

$$\ge\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\ge\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\ge 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\ge 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\ge 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\ge 0\to\text{true}\Rightarrow$$

$$\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1=\frac{R-r}{r}\Rightarrow\frac{h_a}{n_a}\ge\frac{r}{R-r}\Rightarrow\frac{1}{n_a}\ge\left(\frac{r}{R-r}\right)\left(\frac{a}{2rs}\right)=\left(\frac{1}{R-r}\right)\left(\frac{a}{2s}\right)$$

$$\Rightarrow \frac{1}{n_a} \overset{(a)}{\geq} \left(\frac{1}{R-r}\right)\left(\frac{a}{2s}\right) \text{ and } \therefore \frac{1}{n_b} \overset{(b)}{\geq} \left(\frac{1}{R-r}\right)\left(\frac{b}{2s}\right) \text{ and } \frac{1}{n_c} \overset{(c)}{\geq} \left(\frac{1}{R-r}\right)\left(\frac{c}{2s}\right)$$

$$\therefore (a)+(b)+(c) \Rightarrow \frac{1}{n_a}+\frac{1}{n_b}+\frac{1}{n_c} \geq \left(\frac{1}{R-r}\right)\left(\frac{a+b+c}{2s}\right)$$

$$\Rightarrow \frac{1}{n_a}+\frac{1}{n_b}+\frac{1}{n_c} \geq \frac{1}{R-r} \text{ (Proved)}$$

1384. In ΔABC the following relationship holds:

$$\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}+\frac{54r}{4R+r} \geq 9$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } s-a=x, s-b=y, s-c=z$$

$$\therefore 3s-2s=s=\sum x \Rightarrow a=y+z, b=z+x, c=x+y$$

$$\textit{Now}, \sum \frac{r_a^2}{r_b^2}+\frac{8r}{R} \geq 7 \Leftrightarrow$$

$$\Leftrightarrow \sum\left(\frac{s-b}{s-a}\right)^2+8\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right) \overset{\textit{via above transformation}}{\Leftrightarrow}$$

$$\Leftrightarrow \sum \frac{y^2}{x^2}+\frac{32s(s-a)(s-b)(s-c)}{s\prod(x+y)} \geq 7$$

$$\overset{\textit{via above transformation}}{\Leftrightarrow} \sum \frac{y^2}{x^2}+\frac{32xyz}{\prod(x+y)} \geq 7 \Leftrightarrow \sum \frac{y^2}{x^2}+3+\frac{32xyz}{\prod(x+y)} \geq 10 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)} \overset{(i)}{\geq} 10$$

$$\textit{Now}, \sum \frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)} = \sum \frac{y^2+x^2}{x^2}+\frac{16xyz}{\prod(x+y)}+\frac{16xyz}{\prod(x+y)} \overset{A-G}{\geq}$$

$$\geq 5\sqrt[5]{\left(\frac{2^8(xyz)^2}{\prod x^2}\right)\left(\frac{\prod(x^2+y^2)}{\prod(x+y)^2}\right)} \geq 5\sqrt[5]{2^8\frac{\prod\left(\frac{1}{2}(x+y)^2\right)}{\prod(x+y)^2}} = 5\sqrt[5]{2^5}$$

$$= 10 \Rightarrow (i)\ is\ true \therefore \sum \frac{r_a^2}{r_b^2} + \frac{8r}{R} \geq 7 \Rightarrow \boxed{\sum \frac{r_a^2}{r_b^2} \overset{(1)}{\geq} 7 - \frac{8r}{R}}$$

$$Again, \sum \frac{r_a}{r_b} + \frac{4r}{R} \geq 5 \Leftrightarrow \sum \left(\frac{s-b}{s-a}\right) + 4\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right) \geq 5$$

$$\overset{\text{via above transformation}}{\Leftrightarrow} \sum \frac{y}{x} + \frac{16s(s-a)(s-b)(s-c)}{s\prod(x+y)} \geq 5$$

$$\overset{\text{via above transformation}}{\Leftrightarrow} \sum \frac{y}{x} + \frac{16xyz}{\prod(x+y)} \geq 5 \Leftrightarrow \sum \frac{y}{x} + 3 + \frac{16xyz}{\prod(x+y)} \geq 8 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{y+x}{x} + \frac{16xyz}{\prod(x+y)} \overset{(ii)}{\geq} 8$$

$$Now, \sum \frac{y+x}{x} + \frac{16xyz}{\prod(x+y)} \overset{A-G}{\geq} 4\sqrt[4]{\left(\frac{\prod(x+y)}{xyz}\right)\left(\frac{16xyz}{\prod(x+y)}\right)} = 8 \Rightarrow (ii)\ is\ true$$

$$\therefore \sum \frac{r_a}{r_b} + \frac{4r}{R} \geq 5 \Rightarrow \boxed{\sum \frac{r_a}{r_b} \overset{(2)}{\geq} 5 - \frac{4r}{R}}$$

$$Now, \sum \frac{r_a^3}{r_b^3} + \frac{54r}{4R+r} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum \frac{r_a}{r_b}\right)\left(\sum \frac{r_a^2}{r_b^2}\right) + \frac{54r}{4R+r} \overset{by\ (1) and\ (2)}{\geq}$$

$$\geq \frac{1}{3}\left(7 - \frac{8r}{R}\right)\left(5 - \frac{4r}{R}\right) + \frac{54r}{4R+r} \overset{?}{\geq} 9$$

$$\Leftrightarrow (7R-8r)(5R-4r)(4R+r) + 162R^2r \overset{?}{\geq} 27(4R+r)R^2$$

$$\Leftrightarrow 16t^3 - 51t^2 + 30t + 16 \overset{?}{\geq} 0 \left(where\ t = \frac{R}{r}\right) \Leftrightarrow (t-2)[(t-2)(16t+13)+18] \overset{?}{\geq} 0$$

$$\rightarrow true \because t \overset{Euler}{\geq} 2 \therefore \sum \frac{r_a^3}{r_b^3} + \frac{54r}{4R+r} \geq 9\ (Proved)$$

1385. In ΔABC the following relationship holds:

$$m_a\sqrt{\frac{2}{3 + \cos 2A - 2\cos 2B - 2\cos 2C}} = R$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbajan

Solution by Daniel Sitaru-Romania

$$m_a\sqrt{\frac{2}{3+\cos 2A-2\cos 2B-2\cos 2C}}=$$

$$=m_a\sqrt{\frac{2}{3+1-2\sin^2 A-2+4\sin^2 B-2+4\sin^2 C}}=$$

$$=m_a\sqrt{\frac{2}{-2\sin^2 A+4\sin^2 B+4\sin^2 C}}=$$

$$=m_a\sqrt{\frac{4R^2}{-4R^2\sin^2 A+8R^2\sin^2 B+8R^2\sin^2 C}}=$$

$$=m_a\sqrt{\frac{4R^2}{-a^2+2b^2+2c^2}}=m_a\sqrt{\frac{R^2}{\frac{-a^2+2b^2+2c^2}{4}}}==m_a\sqrt{\frac{R^2}{m_a^2}}=R$$

1386. **In ΔABC the following relationship holds:**

$$m_a\geq\frac{1}{2}\left(\frac{h_b+h_c}{2}+|b-c|\sin^2\frac{A}{2}\right)\sqrt{\frac{n_a+h_a}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a\overset{(i)}{\gtrsim}\frac{1}{2}\left(\frac{h_b+h_c}{2}+|b-c|\sin^2\frac{A}{2}\right)\sqrt{\frac{n_a+h_a}{r_a}}$$

$$\textit{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=a{n_a}^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=a{n_a}^2+a(s^2-s(2s-a)+bc)\Rightarrow$$

$$\Rightarrow s(b^2+c^2)-2sbc=a{n_a}^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=a{n_a}^2-as^2\Rightarrow a{n_a}^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore \boxed{{n_a}^2 \overset{(1)}{\cong} s^2 - 2h_ar_a}$$

$$\textit{Now we prove}: \ m_a \overset{(2)}{\geq} \frac{1}{2\sqrt{2}}\left((b+c)\cos\frac{A}{2} + |b-c|\sin\frac{A}{2}\right)$$

Upon squaring both sides, (1)

$$\Leftrightarrow 8m_a^2 \geq (b+c)^2\left(\cos\frac{A}{2}\right)^2 + (b-c)^2\left(\sin\frac{A}{2}\right)^2 + 2(b+c)|b-c|\cos\frac{A}{2}\sin\frac{A}{2}$$

$$\Leftrightarrow 8m_a^2 \geq (b-c)^2\left(\left(\cos\frac{A}{2}\right)^2 + \left(\sin\frac{A}{2}\right)^2\right) + 4bc\left(\cos\frac{A}{2}\right)^2 + \left(\frac{a}{2R}\right)(b+c)|b-c|$$

$$\Leftrightarrow 8m_a^2 \geq (b-c)^2 + \frac{4bcs(s-a)}{bc} + \left(\frac{a}{2R}\right)(b+c)|b-c|$$

$$\Leftrightarrow 8m_a^2 \geq (b-c)^2 + (b+c+a)(b+c-a) + \left(\frac{a}{2R}\right)(b+c)|b-c| \Leftrightarrow$$

$$\Leftrightarrow 8m_a^2 \geq (b-c)^2 + (b+c)^2 - a^2 + \left(\frac{a}{2R}\right)(b+c)|b-c|$$

$$\Leftrightarrow 8m_a^2 \geq 4m_a^2 + \left(\frac{a}{2R}\right)(b+c)|b-c| \Leftrightarrow 8Rm_a^2 \geq a(b+c)|b-c| \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{2abc}{4\Delta}\right)4m_a^2 \geq a(b+c)|b-c|$$

$$\Leftrightarrow 4a^2b^2c^2(2b^2+2c^2-a^2)^2 \geq$$

$$\geq (a+b+c)(b+c-a)(c+a-b)(a+b-c)a^2(b^2-c^2)^2$$

$$\Leftrightarrow 4b^2c^2(2b^2+2c^2-a^2)^2 \geq \left(2\sum a^2b^2 - \sum a^4\right)(b^2-c^2)^2$$

$$\overset{\text{(expanding and re-arranging)}}{\Leftrightarrow} a^4(b^2+c^2)^2 - 2a^2(b^6+c^6) - 14a^2b^2c^2(b^2+c^2) +$$

$$+(b^2+c^2)^4 + 8b^2c^2(b^2+c^2)^2 + 16b^4c^4 \geq 0$$

$$\Leftrightarrow \{a^4(b^2+c^2)^2 + 16b^4c^4 - 8a^2b^2c^2(b^2+c^2)\} -$$

$$-6a^2b^2c^2(b^2+c^2) + (b^2+c^2)^4 + 8b^2c^2(b^2+c^2)^2$$

$$-2a^2(b^2+c^2)(b^4+c^4-b^2c^2) \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2) - 4b^2c^2\}^2 - 6a^2b^2c^2(b^2+c^2) + (b^2+c^2)^4 +$$

$$+8b^2c^2(b^2+c^2)^2 - 2a^2(b^2+c^2)\{(b^2+c^2)^2 - 3b^2c^2\} \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2) - 4b^2c^2\}^2 + (b^2+c^2)^4 + 8b^2c^2(b^2+c^2)^2 - 2a^2(b^2+c^2)^3 \geq 0$$

$$\Leftrightarrow \{a^2(b^2+c^2) - 4b^2c^2\}^2 + (b^2+c^2)^4 - 2(b^2+c^2)^2\{a^2(b^2+c^2) - 4b^2c^2\} \geq 0$$

$$\Leftrightarrow [\{a^2(b^2+c^2)-4b^2c^2\}-(b^2+c^2)^2]^2 \geq 0 \to \text{ true} \Rightarrow (2)\text{is true}$$

$$Now, (i) \Leftrightarrow 2m_a \geq \left(\frac{a(b+c)}{4R}+|b-c|\sin^2\frac{A}{2}\right)\sqrt{\frac{n_a+h_a}{r_a}} =$$

$$\left(\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}(b+c)}{4R}+|b-c|\sin^2\frac{A}{2}\right)\sqrt{\frac{n_a+h_a}{r_a}}$$

$$\Leftrightarrow 2m_a \overset{(ii)}{\geq} \sin\frac{A}{2}\left((b+c)\cos\frac{A}{2}+|b-c|\sin\frac{A}{2}\right)\sqrt{\frac{n_a+h_a}{r_a}}$$

$$Now, RHS\ of\ (ii) \overset{by\ (2)}{\leq} 2\sqrt{2}m_a\sin\frac{A}{2}\sqrt{\frac{n_a+h_a}{r_a}}$$

$$\therefore in\ order\ to\ prove\ (ii), it\ suffices\ to\ prove: 2\sqrt{2}m_a\sin\frac{A}{2}\sqrt{\frac{n_a+h_a}{r_a}} \leq 2m_a$$

$$\Leftrightarrow \frac{1}{2\sin^2\frac{A}{2}} \geq \frac{n_a+h_a}{r_a} \Leftrightarrow \frac{abc(s-a)}{2a(s-a)(s-b)(s-c)} \geq \frac{n_a}{r_a}+\left(\frac{2rs}{a}\right)\left(\frac{s-a}{rs}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{4Rrs(s-a)}{2asr^2} \geq \frac{n_a}{r_a}+\frac{2(s-a)}{a} \Leftrightarrow \frac{2(s-a)}{a}\left(\frac{R}{r}-1\right) \geq \frac{n_a}{r_a}$$

$$\Leftrightarrow \frac{2(s-a)}{a}\left(\frac{R}{r}-1\right)\left(\frac{rs}{s-a}\right) \geq n_a \Leftrightarrow \frac{2rs}{a}\left(\frac{R}{r}-1\right) \geq n_a \Leftrightarrow h_a\left(\frac{R}{r}-1\right) \geq n_a \Leftrightarrow$$

$$\Leftrightarrow \frac{R}{r}-1 \geq \frac{n_a}{h_a} \Leftrightarrow \frac{R^2}{r^2}-\frac{2R}{r}+1 \geq \frac{n_a^2}{h_a^2}$$

$$\overset{by\ (1)}{\Leftrightarrow} \frac{R^2}{r^2}-\frac{2R}{r}+1 \geq \frac{s^2-2h_ar_a}{h_a^2} = \frac{s^2a^2}{4r^2s^2}-\frac{2r_a}{h_a} = \frac{a^2}{4r^2}-\left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right) =$$

$$= \frac{a^2}{4r^2}-\frac{(a-s)+s}{s-a} = \frac{a^2}{4r^2}+1-\frac{s}{s-a}$$

$$= 1+\frac{a^2(s-a)-4(sr^2)}{4(s-a)r^2} = 1+\frac{a^2(s-a)-4(s-a)(s-b)(s-c)}{4(s-a)r^2} =$$

$$= 1+\frac{a^2-(a^2-(b-c)^2)}{4r^2} = 1+\frac{(b-c)^2}{4r^2}$$

$$\Leftrightarrow \frac{R^2}{r^2}-\frac{2R}{r} \geq \frac{(b-c)^2}{4r^2} \Leftrightarrow \frac{R(R-2r)}{r^2} \geq \frac{b^2+c^2-2bc}{4r^2} \Leftrightarrow R-2r \geq \frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\ge\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\ge$$

$$\ge\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\ge\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\ge 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\ge 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\ge 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\ge 0\to \textit{true}\Rightarrow(ii)\Rightarrow(i)\textit{ is true }(\textit{Proved})$$

1387. In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b}\right)^3+\left(\frac{r_b}{r_c}\right)^3+\left(\frac{r_c}{r_a}\right)^3+\frac{2nr}{R}\ge n+3, n\le 6$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

First, we will prove: $\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}+\frac{12r}{R}\ge 9$ *(1)*

$$\Leftrightarrow\sum_{cyc}\left(\frac{s-b}{s-a}\right)^3+12\left(\frac{S}{s}\right)\cdot\left(\frac{4S}{abc}\right)\ge 9$$

(Let $x=s-a; y=s-b; z=s-c\Rightarrow x+y+z=s; a=y+z; b=x+z; c=x+y$)

$$\Leftrightarrow\sum_{cyc}\frac{y^3}{x^3}+48\cdot\frac{xyz}{\prod_{cyc}(x+y)}\ge 9$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{y^3+x^3}{x^3}\right)+48\cdot\frac{xyz}{\prod_{cyc}(x+y)}\ge 12\quad (2)$$

$$\sum\frac{y^3+x^3}{x^3}+16\cdot\frac{xyz}{\prod_{cyc}(x+y)}+16\cdot\frac{xyz}{\prod_{cyc}(x+y)}+16\cdot\frac{xyz}{\prod_{cyc}(x+y)}\ge$$

$$\geq 6\sqrt[6]{16^3\cdot\frac{\prod_{cyc}(x^3+y^3)\cdot(xyz)^3}{\prod x^3\prod_{cyc}(x+y)^3}}=6\sqrt[6]{16^3\cdot\frac{\prod_{cyc}(x^3+y^3)}{\prod_{cyc}(x+y)^3}}$$

$$\overset{x^3+y^3\geq\frac{1}{4}(x+y)^3\ (etc)}{\geq} 6\sqrt[6]{16^3\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}}=6\sqrt[6]{2^6}=12\Rightarrow (2)\ true.$$

$$Now,\ \sum\frac{r_a^3}{r_b^3}+\frac{2nr}{R}=\sum\frac{r_a^3}{r_b^3}+\frac{(2n+12-12)r}{R}=$$

$$\sum\frac{r_a^3}{r_b^3}+\frac{12r}{R}+\frac{2(n-6)r}{R}\overset{(1)}{\geq}9+\frac{2(n-6)r}{R}$$

We must show that: $9+\frac{2(n-6)r}{R}\geq n+3\Leftrightarrow\frac{2(n-6)r}{R}\geq n-6$

It is true because: $\begin{cases}n\leq 6\Rightarrow n-6\leq 0\\ 2r\leq R\Rightarrow\frac{2r}{R}\leq 1\end{cases}\Rightarrow\frac{2r(n-6)}{R}\geq n-6$

Solution 2 by Marian Ursărescu-Romania

First we prove this: $\left(\frac{r_a}{r_b}\right)^3+\left(\frac{r_b}{r_c}\right)^3+\left(\frac{r_c}{r_a}\right)^3+\frac{12r}{R}\geq 9$ *(1)*

Because $\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_ar_br_c}=\frac{4R}{r}\Rightarrow$ *we must show:*

$$\left(\frac{r_a}{r_b}\right)^3+\left(\frac{r_b}{r_c}\right)^3+\left(\frac{r_c}{r_a}\right)^3+\frac{48r_ar_br_c}{(r_a+r_b)(r_b+r_c)(r_c+r_a)}\geq 9\Leftrightarrow$$

$$\left(\frac{r_a}{r_b}\right)^3+\left(\frac{r_b}{r_c}\right)^3+\left(\frac{r_c}{r_a}\right)^3+\frac{48}{\left(\frac{r_a}{r_b}+1\right)\left(\frac{r_b}{r}+1\right)\left(\frac{r_c}{r_a}+1\right)}\geq 9\quad (2)$$

$$Let\ \frac{r_a}{r_b}=x,\frac{r_b}{r_c}=y,\frac{r_c}{r_a}=z\ ;xyz=1\quad (3)$$

From (2)+(3) we must show: $x^3+y^3+z^3+\frac{48}{(x+1)(y+1)(z+1)}\geq 9\Leftrightarrow$

$$x^3+1+y^3+1+z^3+1+\frac{48}{(x+1)(y+1)(z+1)}\geq 12\quad (4)$$

But $x^3+1\geq\frac{(x+1)^3}{4}$ *and similarly (5)*

From (4)+(5) we must show:

$$\frac{(x+1)^3}{4}+\frac{(y+1)^3}{4}+\frac{(z+1)^3}{4}+\frac{48}{(x+1)(y+1)(z+1)}\geq 12\quad (6)$$

$$\frac{(x+1)^3}{4}+\frac{(y+1)^3}{4}+\frac{(z+1)^3}{4}+\frac{16}{(x+1)(y+1)(z+1)}+$$

$$+\frac{16}{(x+1)(y+1)(z+1)}+\frac{16}{(x+1)(y+1)(z+1)}\geq$$

$$\geq 6\sqrt[6]{\frac{16\cdot16\cdot16}{4\cdot4\cdot4}}=6\cdot 2=12\Rightarrow 6\ \textit{it is true}\Rightarrow (1)\ \textit{it is true.}$$

From (1) $\Rightarrow$ *we must show:*

$$9-\frac{12}{R}\geq n+3-\frac{2nr}{R}\Leftrightarrow 6-n+\frac{2nr}{R}-\frac{12r}{R}\geq 0$$

$$\Leftrightarrow 6-n-\frac{2r}{R}(6-n)\geq 0\Leftrightarrow (6-n)\left(1-\frac{2r}{R}\right)\geq 0$$

$$\Leftrightarrow (6-n)\frac{(R-2r)}{R}\geq 0,\ \textit{true because } n\leq 6 \textit{ and } R\geq 2r\ (\textit{Euler})$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\textit{Let } s-a=x, s-b=y, s-c=z$$

$$\therefore 3s-2s=s=\sum x\Rightarrow a=y+z, b=z+x, c=x+y$$

$$\textit{Now}, \sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}\geq 7\Leftrightarrow \sum\left(\frac{s-b}{s-a}\right)^2+8\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right)\Leftrightarrow$$

$$\overset{\textit{via above transformation}}{\Leftrightarrow}\sum\frac{y^2}{x^2}+\frac{32s(s-a)(s-b)(s-c)}{s\prod(x+y)}\geq 7\Leftrightarrow$$

$$\overset{\textit{via above transformation}}{\Leftrightarrow}\sum\frac{y^2}{x^2}+\frac{32xyz}{\prod(x+y)}\geq 7\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{y^2}{x^2}+3+\frac{32xyz}{\prod(x+y)}\geq 10\Leftrightarrow\sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}\overset{(i)}{\geq}10$$

$$\textit{Now}, \sum\frac{y^2+x^2}{x^2}+\frac{32xyz}{\prod(x+y)}=$$

$$=\sum\frac{y^2+x^2}{x^2}+\frac{16xyz}{\prod(x+y)}+\frac{16xyz}{\prod(x+y)}\overset{A-G}{\geq}5\sqrt[5]{\left(\frac{2^8(xyz)^2}{\prod x^2}\right)\left(\frac{\prod(x^2+y^2)}{\prod(x+y)^2}\right)}\geq$$

$$\geq 5\sqrt[5]{2^8\frac{\prod\left(\frac{1}{2}(x+y)^2\right)}{\prod(x+y)^2}}=5\sqrt[5]{2^5}=10\Rightarrow (i)\ \textit{is true}$$

$$\therefore\sum\frac{r_a^2}{r_b^2}+\frac{8r}{R}\geq 7\Rightarrow \boxed{\sum\frac{r_a^2}{r_b^2}\overset{(1)}{\geq}7-\frac{8r}{R}}$$

$$Again, \sum\frac{r_a}{r_b}+\frac{4r}{R}\ge 5 \Leftrightarrow \sum\left(\frac{s-b}{s-a}\right)+4\left(\frac{\Delta}{s}\right)\left(\frac{4\Delta}{abc}\right)\ge 5$$

$$\overset{\text{via above transformation}}{\Leftrightarrow} \sum\frac{y}{x}+\frac{16s(s-a)(s-b)(s-c)}{s\prod(x+y)}\ge 5 \Leftrightarrow$$

$$\overset{\text{via above transformation}}{\Leftrightarrow} \sum\frac{y}{x}+\frac{16xyz}{\prod(x+y)}\ge 5 \Leftrightarrow$$

$$\sum\frac{y}{x}+3+\frac{16xyz}{\prod(x+y)}\ge 8 \Leftrightarrow \sum\frac{y+x}{x}+\frac{16xyz}{\prod(x+y)}\overset{(ii)}{\ge} 8$$

$$Now, \sum\frac{y+x}{x}+\frac{16xyz}{\prod(x+y)}\overset{A-G}{\ge} 4\sqrt[4]{\left(\frac{\prod(x+y)}{xyz}\right)\left(\frac{16xyz}{\prod(x+y)}\right)}=8 \Rightarrow$$

$$(ii)\text{ is true} \therefore \sum\frac{r_a}{r_b}+\frac{4r}{R}\ge 5 \Rightarrow \boxed{\sum\frac{r_a}{r_b}\overset{(2)}{\ge} 5-\frac{4r}{R}}$$

$$Now, \sum\frac{r_a^3}{r_b^3}+\frac{12r}{R}-9 \overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum\frac{r_a^2}{r_b^2}\right)\left(\sum\frac{r_a}{r_b}\right)+\frac{12r}{R}-9$$

$$\overset{\text{by (1) and (2)}}{\ge} \frac{1}{3}\left(\frac{7R-8r}{R}\right)\left(\frac{5R-4r}{R}\right)+\frac{12r-9R}{R}$$

$$=\frac{(7R-8r)(5R-4r)+36Rr-27R^2}{3R^2}=\frac{8(R-2r)^2}{3R^2}\ge 0$$

$$\Rightarrow \sum\frac{r_a^3}{r_b^3}+\frac{12r}{R}-9+\frac{2nr}{R}-\frac{2nr}{R}-(n+3)+(n+3)\ge 0 \Rightarrow$$

$$\Rightarrow \sum\frac{r_a^3}{r_b^3}+\frac{2nr}{R}-(n+3)\ge\frac{2nr}{R}-\frac{12r}{R}+9-(n+3)=$$

$$=\frac{2r}{R}(n-6)-(n-6)=(n-6)\left(\frac{2r}{R}-1\right)\ge 0$$

$$\left(\because n-6\le 0 \text{ and } \frac{2r}{R}-1\overset{Euler}{\le} 0\right)\Rightarrow \sum\frac{r_a^3}{r_b^3}+\frac{2nr}{R}\ge n+3\ \forall\, n\le 6\ (Proved)$$

1388. In ΔABC the following relationship holds:

$$\frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}}+\frac{b^2}{\frac{1}{c}+\frac{1}{a}}+\frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a+m_b+m_c}\geq\frac{3R}{2s-6\sqrt{3}r+9r}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{\frac{1}{b}+\frac{1}{c}}+\frac{b^2}{\frac{1}{c}+\frac{1}{a}}+\frac{c^2}{\frac{1}{a}+\frac{1}{b}}=\sum\frac{a^2bc}{b+c}=4Rrs\sum\frac{a}{b+c}\overset{\text{Nesbitt}}{\geq}6Rrs\Rightarrow$$

$$\Rightarrow\frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}}+\frac{b^2}{\frac{1}{c}+\frac{1}{a}}+\frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a+m_b+m_c}\geq\frac{3R}{\sum m_a}\overset{?}{\geq}\frac{3R}{2s-6\sqrt{3}r+9r}$$

$$\Leftrightarrow\sum m_a\overset{?}{\leq}2s-6\sqrt{3}r+9r\Leftrightarrow\left(\sum m_a\right)^2\overset{?}{\leq}4s^2+\left(6\sqrt{3}-9\right)^2r^2-4sr\left(6\sqrt{3}-9\right)$$

$$\Leftrightarrow\left(\sum m_a\right)^2+4sr\left(6\sqrt{3}-9\right)\underset{(1)}{\overset{?}{\leq}}4s^2+\left(6\sqrt{3}-9\right)^2r^2$$

$$\text{Now, }\left(\sum m_a\right)^2+4sr\left(6\sqrt{3}-9\right)\overset{\text{Chu and Yang, Blundon}}{\leq}$$

$$\leq 4s^2-16Rr+5r^2+8Rr\left(6\sqrt{3}-9\right)+4\left(3\sqrt{3}-4\right)\left(6\sqrt{3}-9\right)r^2\overset{?}{\leq}4s^2+\left(6\sqrt{3}-9\right)^2r^2$$

$$\Leftrightarrow 16R-8R\left(6\sqrt{3}-9\right)\overset{?}{\geq}\left(6\sqrt{3}-9\right)\left\{4\left(3\sqrt{3}-4\right)-\left(6\sqrt{3}-9\right)\right\}r+5r$$

$$\Leftrightarrow 8R\left(11-6\sqrt{3}\right)\overset{?}{\geq}\left\{\left(6\sqrt{3}-9\right)\left(6\sqrt{3}-7\right)+5\right\}r$$

$$\Leftrightarrow 8R\left(11-6\sqrt{3}\right)\overset{?}{\geq}\left(176-96\sqrt{3}\right)r\Leftrightarrow R\left(11-6\sqrt{3}\right)\overset{?}{\geq}2\left(11-6\sqrt{3}\right)r$$

$$\Leftrightarrow\left(11-6\sqrt{3}\right)(R-2r)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because R\overset{\text{Euler}}{\geq}2r\text{ and }\left(11-6\sqrt{3}\right)>0\Rightarrow$$

$$\Rightarrow (1)\ is\ true \therefore \frac{\frac{1}{2S}\left(\frac{a^2}{\frac{1}{b}+\frac{1}{c}}+\frac{b^2}{\frac{1}{c}+\frac{1}{a}}+\frac{c^2}{\frac{1}{a}+\frac{1}{b}}\right)}{m_a+m_b+m_c} \geq \frac{3R}{2s-6\sqrt{3}r+9r}\ (\text{Proved})$$

1389. In acute $\Delta ABC, H$ –orthocenter, I –incenter, G –centroid, the following relationship holds:

$$am_a cosA + bm_b cosB + cm_c cosC \leq \frac{3s}{2R}(HI^2 + GI^2 + 4Rr)$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\bullet\ \Omega = acos^2A + bcos^2B + ccos^2C$$

$$= a(1-sin^2A) + b(1-sin^2B) + c(1-sin^2C)$$

$$= (a+b+c) - 2R(sin^3A + sin^3B + sin^3C)$$

$$= 2s - 2R\left[\frac{s(s^2-6Rr+3r^2-s^2)}{2R^2}\right] = \frac{s(4R^2+6Rr+3r^2-s^2)}{2R^2}$$

$$= 2s - \frac{s(s^2-6Rr-3r^2)}{2R^2} = \frac{s(4R^2+6Rr+3r^2-s^2)}{2R^2}$$

$$\bullet\ \Psi = am_a^2 + bm_b^2 + cm_c^2 = \frac{s}{2}(s^2+2Rr+5r^2)\ (*)$$

Now,

$$LHS = \sum_{cyc}(am_a cosA) = \frac{3}{1}\cdot\frac{1}{2R}\sum_{cyc}\left(a\cdot 2RcosA\cdot\frac{2}{3}m_a\right)$$

$$\underset{\Delta ABC - acute}{\overset{AM-GM}{\leq}} \frac{3}{4R}\left(a\cdot\frac{[2RcosA]^2+\frac{4}{9}m_a^2}{2} + b\cdot\frac{[2RcosB]^2+\frac{4}{9}m_b^2}{2} + c\cdot\frac{[2RcosC]^2+\frac{4}{9}m_c^2}{2}\right]$$

$$= \frac{3}{8R}\left(4R^2\Omega + \frac{4}{9}\Psi\right) = \frac{3}{8R}\left[2s(4R^2+6Rr+3r^2-s^2) + \frac{4}{9}\cdot\frac{s}{2}(s^2+2Rr+5r^2)\right]$$

$$= \frac{3s}{4R}\left(4R^2+6Rr+3r^2-s^2+\frac{s^2}{9}+\frac{2R}{9}+\frac{5}{9}r^2\right)$$

$$= \frac{3s}{4R}\left(4R^2+\frac{56}{9}Rr-\frac{8s^2}{9}+\frac{32r^2}{9}\right) = \frac{3s}{4R}\cdot\frac{36R^2+56Rr+32r^2-8s^2}{9}$$

$$Now,$$

$$HI^2 = 2r^2 + 4R^2 + 4Rr - p^2;$$

$$GI^2 = \frac{p^2 - 16Rr + 5r^2}{9}$$

$$\Rightarrow HI^2 + GI^2 + 4Rr = \frac{36R^2 + 32R^2 + 56Rr - 8s^2}{9}$$

$$\Rightarrow RHS = \frac{3s}{2R} \cdot \frac{36R^2 + 32R^2 + 56Rr - 8s^2}{9}$$

$$\Rightarrow 2LHS \le RHS \ (Proved)$$

1390. In $\Delta ABC, n_a$ –Nagel's cevian the following relationship holds:

$$\frac{n_a + m_a + w_b + w_c + \sqrt{2r_a h_a}}{h_a + h_b + h_c} \le \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Firstly, Stewart's\ theorem \Rightarrow b^2(s-c) + c^2(s-b) = a{n_a}^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = a{n_a}^2 + a(s^2 - s(2s-a) + bc) \Rightarrow$$

$$\Rightarrow s(b^2+c^2) - 2sbc = a{n_a}^2 + a(as - s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc) = a{n_a}^2 - as^2 \Rightarrow a{n_a}^2 = as^2 + s(2bc\cos A - 2bc) =$$

$$= as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore {n_a}^2 \overset{(1)}{=} s^2 - 2h_a r_a$$

$$\text{Now}, n_a + \sqrt{2r_a h_a} \overset{\text{CBS}}{\le} \sqrt{2}\sqrt{{n_a}^2 + 2h_a r_a} \overset{\text{by (1)}}{=}$$

$$= \sqrt{2}\sqrt{s^2 - 2h_a r_a + 2h_a r_a} \Rightarrow n_a + \sqrt{2r_a h_a} \overset{(i)}{\le} \sqrt{2}s$$

$$\text{Also}, m_a + w_b + w_c \underbrace{\overset{\text{Lessel-Pelling}}{\le}}_{(ii)} \sqrt{3}s \therefore (i) + (ii) \Rightarrow$$

$$\Rightarrow \frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{h_a+h_b+h_c} \le \frac{(\sqrt{2}+\sqrt{3})2Rs}{\sum ab} \overset{?}{\lesseqgtr} \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}$$

$$= \frac{(\sqrt{2}+\sqrt{3})}{\sqrt{6}}\sqrt{\frac{R}{r}} \Leftrightarrow 24R^2s^2 \overset{?}{\lesseqgtr} \left(\frac{R}{r}\right)\left(\sum ab\right)^2 \Leftrightarrow (s^2+4Rr+r^2)^2 \overset{?}{\gtrsim} 24Rrs^2$$

$$\Leftrightarrow s^4+r^2(4R+r)^2+2s^2(4Rr+r^2) \overset{?}{\gtrsim} 24Rrs^2$$

$$\Leftrightarrow s^4+r^2(4R+r)^2 \underset{(a)}{\overset{?}{\gtrsim}} s^2(16Rr-2r^2)$$

$$\text{Now, LHS of (a)} \overset{\text{Gerretsen}}{\gtrsim} s^2(16Rr-5r^2)+r^2(4R+r)^2 \overset{?}{\gtrsim} s^2(16Rr-2r^2)$$

$$\Leftrightarrow r^2(4R+r)^2 \overset{?}{\gtrsim} 3r^2s^2 \Leftrightarrow 4R+r \overset{?}{\gtrsim} \sqrt{3}s \to \text{true (Trucht)}$$

$$\Rightarrow \text{(a) is true} \therefore \frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{h_a+h_b+h_c} \le \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}} \text{ (Proved)}$$

1391. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian the following relationship holds:

$$2s^2+\sum a^2 \ge 4S\sqrt{4-\frac{2r}{R}}+\sum(n_a{}^2+g_a{}^2)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a{}^2+a(s-b)(s-c)$$

$$\text{and } b^2(s-b)+c^2(s-c) = ag_a{}^2+a(s-b)(s-c)$$

$$\text{Adding the above two, we get:}$$

$$(b^2+c^2)(2s-b-c) = an_a{}^2+ag_a{}^2+2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2) = 2a(n_a{}^2+g_a{}^2)+a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2) =$$

$$= 2(n_a{}^2+g_a{}^2)+a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2 = 2(n_a{}^2+g_a{}^2)$$

$$\Rightarrow 4m_a{}^2+(b-c)^2 = 2(n_a{}^2+g_a{}^2) \Rightarrow 4m_a{}^2+(b-c)^2+4r_br_c =$$

$$2(n_a^2+g_a^2)+4r_br_c$$

$$\Rightarrow 4m_a^2+(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)+4s(s-a)$$

$$\Rightarrow 4m_a^2+4m_a^2=2(n_a^2+g_a^2)+4s(s-a)$$

$$\Rightarrow \boxed{n_a^2+g_a^2=4m_a^2-2s(s-a)\text{ and analogs}}$$

$$\therefore \sum(n_a^2+g_a^2)=4\sum m_a^2-2s^2=3\sum a^2-2s^2\Rightarrow 2s^2+\sum a^2-\sum(n_a^2+g_a^2)$$

$$=2s^2+\sum a^2-3\sum a^2+2s^2=\left(\sum a\right)^2-2\sum a^2$$

$$=2\sum ab-\sum a^2=2[(s^2+4Rr+r^2)-(s^2-4Rr-r^2)]=4r(4R+r)$$

$$\therefore 2s^2+\sum a^2-\sum(n_a^2+g_a^2)\overset{(1)}{=}4r(4R+r)$$

$$\therefore (1)\Rightarrow \text{it suffices to prove}: 4r(4R+r)\ge 4rs\sqrt{\frac{4R-2r}{R}}$$

$$\Leftrightarrow R(4R+r)^2\overset{(i)}{\ge}(4R-2r)s^2$$

$$\text{Now, RHS of (i)}\overset{\text{Rouche}}{\le}(4R-2r)\left(2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right)\overset{?}{\le}R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2-(2R^2+10Rr-r^2)(4R-2r)\overset{?}{\ge}2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2)\underset{(ii)}{\overset{?}{\ge}}2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\because R-2r\overset{\text{Euler}}{\ge}\therefore \text{in order to prove (ii), it suffices to prove}: 8R^2-12Rr+r^2$$

$$>2(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (8R^2-12Rr+r^2)^2-4(R^2-2Rr)(4R-2r)^2>0\Leftrightarrow r^2(4R+r)^2>0$$

$$\rightarrow \textit{true}\Rightarrow (ii)\Rightarrow (i)\textit{ is true}$$

$$\therefore 2s^2+\sum a^2\ge 4S\sqrt{4-\frac{2r}{R}}+\sum(n_a^2+g_a^2)\ \text{(Proved)}$$

1392. In ΔABC the following relationship holds:

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$$\sum_{cyc}\frac{h_a}{\sqrt{h_a-2r}}\geq\frac{4R+s+(10-3\sqrt{3})r}{\sqrt{2R}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$s\sec\frac{A}{2}\geq w_a+r_a\Leftrightarrow s\geq\frac{2bc\cos^2\frac{A}{2}}{b+c}+s\tan\frac{A}{2}\cos\frac{A}{2}\Leftrightarrow$$

$$\Leftrightarrow s\geq\frac{2bcs(s-a)}{bc(b+c)}+s\sin\frac{A}{2}=\frac{(b+c-a)(s-a)}{b+c}+s\sin\frac{A}{2}$$

$$=s-a+\frac{a(s-a)}{b+c}+s\sin\frac{A}{2}\Leftrightarrow a\left(1-\frac{s-a}{b+c}\right)\geq s\sin\frac{A}{2}\Leftrightarrow$$

$$\Leftrightarrow a\left(\frac{s+s-a-s+a}{b+c}\right)\geq s\sin\frac{A}{2}\Leftrightarrow\frac{a}{b+c}\geq\sin\frac{A}{2}$$

$$\Leftrightarrow 4R\sin\frac{A}{2}\cos\frac{A}{2}\geq 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\Leftrightarrow\cos\frac{B-C}{2}\leq 1\rightarrow\text{true}$$

$$\therefore \boxed{s\sec\frac{A}{2}\overset{(1)}{\geq} w_a+r_a}\ \textit{and analogs}$$

$$\text{Now, } b+c-a=4R\cos\frac{A}{2}\cos\frac{B-C}{2}-4R\cos\frac{A}{2}\sin\frac{A}{2}$$

$$=4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)=8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow s-a\overset{(2)}{=}4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\textit{Again, } AI=\frac{r}{\sin\frac{A}{2}}=\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}=4R\sin\frac{B}{2}\sin\frac{C}{2}\overset{\text{by }(2)}{=}\frac{s-a}{\cos\frac{A}{2}}\Rightarrow\cos\frac{A}{2}\overset{(3)}{=}\frac{s-a}{AI}$$

$$\textit{We have, } \tan\frac{A}{4}=\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}\overset{\text{by }(3)}{=}\frac{1-\frac{s-a}{AI}}{\frac{r}{AI}}=\frac{AI-(s-a)}{r}$$

$$\Rightarrow AI\overset{(a)}{=}s-a+r\tan\frac{A}{4}$$

$$\textit{Similarly, } BI\overset{(b)}{=}s-b+r\tan\frac{B}{4}\ \textit{and } CI\overset{(c)}{=}s-c+r\tan\frac{C}{4}\therefore (a)+(b)+(c)$$

$$\Rightarrow \sum AI \overset{(4)}{=} s + r\sum \tan\frac{A}{4}$$

Let $f(x) = \tan\left(\frac{x}{4}\right)\ \forall x \in (0,\pi) \therefore f''(x) = \frac{\tan\left(\frac{x}{4}\right)\sec^2\left(\frac{x}{4}\right)}{8} > 0 \Rightarrow f(x)$ *is convex*

$$\text{Let } \tan\frac{\pi}{12} = m \therefore \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{2m}{1-m^2} \Rightarrow$$

$$\Rightarrow m^2 + 2\sqrt{3}m - 1 = 0 \Rightarrow m = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = 2-\sqrt{3} \Rightarrow \tan\frac{\pi}{12} \overset{(4)}{=} 2-\sqrt{3}$$

$$\text{Now, by } (4), \sum AI = s + r\sum \tan\frac{A}{4} \overset{\text{Jensen}}{\geq}$$

$$s + 3r\tan\frac{\pi}{12}\left(\text{as } f(x) = \tan\left(\frac{x}{4}\right) \text{ is convex which has been proved earlier}\right)$$

$$= s + 3r\left(2-\sqrt{3}\right) = s - 3\sqrt{3}r + 6r \Rightarrow \boxed{\sum AI \overset{(5)}{\geq} s - 3\sqrt{3}r + 6r}$$

$$\text{Now, } \frac{\sqrt{2Rh_a}}{\sqrt{h_a - 2r}} = \sqrt{2R}\left(\frac{2rs}{a}\right)\frac{1}{\sqrt{\frac{2rs}{a} - 2r}} = \sqrt{2R}\left(\frac{2rs}{a}\right)\sqrt{\frac{a}{2r(s-a)}} =$$

$$= \sqrt{\frac{R}{r}}\left(\frac{2rs}{a}\right)\sqrt{\left(\frac{sa}{bc}\right)\frac{bc}{s(s-a)}} = \sqrt{\frac{Rsa^2}{abcr}}\left(\frac{2rs}{a}\right)\sec\frac{A}{2}$$

$$= \left(2rs\sec\frac{A}{2}\right)\sqrt{\frac{Rs}{4Rr^2s}} = s\sec\frac{A}{2} = \frac{s-a+a}{\cos\frac{A}{2}}$$

$$= \frac{s-a}{\cos\frac{A}{2}} + \left(\frac{a}{s}\right)s\sec\frac{A}{2} \overset{\text{by }(2)}{=} \frac{4R\cos\frac{A}{2}\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}{\sin\frac{A}{2}\cos\frac{A}{2}} + \left(\frac{a}{s}\right)s\sec\frac{A}{2}$$

$$= \frac{4R\left(\frac{r}{4R}\right)}{\sin\frac{A}{2}} + \left(\frac{a}{s}\right)s\sec\frac{A}{2} = AI + \left(\frac{a}{s}\right)s\sec\frac{A}{2} \therefore \boxed{\frac{\sqrt{2Rh_a}}{\sqrt{h_a - 2r}} \overset{(d)}{=} AI + \left(\frac{a}{s}\right)s\sec\frac{A}{2}}$$

$$\text{Similarly, } \boxed{\frac{\sqrt{2Rh_b}}{\sqrt{h_b - 2r}} \overset{(e)}{=} BI + \left(\frac{b}{s}\right)s\sec\frac{B}{2}} \text{ and } \boxed{\frac{\sqrt{2Rh_c}}{\sqrt{h_c - 2r}} \overset{(f)}{=} CI + \left(\frac{c}{s}\right)s\sec\frac{C}{2}}$$

$$(\mathbf{d})+(\mathbf{e})+(\mathbf{f})\Rightarrow\sum\frac{\sqrt{2R}h_a}{\sqrt{h_a-2r}}\overset{\text{using (1)and its analogs}}{\geq}$$

$$\sum AI+\sum\left(\frac{a}{s}\right)(w_a+r_a)\overset{\text{using } w_a\geq h_a \text{ and its analogs}}{\geq}\sum AI+\sum\left(\frac{a}{s}\right)\left(\frac{2rs}{a}+\frac{rs}{s-a}\right)$$

$$=\sum AI+r\sum\left(a\left(\frac{2}{a}+\frac{1}{s-a}\right)\right)=\sum AI+r\sum\left(\frac{b+c-a+a}{s-a}\right)=\sum AI+r\sum\left(\frac{s+s-a}{s-a}\right)$$

$$=\sum AI+\frac{rs}{r^2s}\sum(s-b)(s-c)+3r$$

$$\overset{\text{by (5)}}{\geq} s-3\sqrt{3}r+9r+\frac{rs(4Rr+r^2)}{r^2s}=s-3\sqrt{3}r+9r+4R+r=$$

$$=4R+s+(10-3\sqrt{3})r$$

$$\Rightarrow\sum\frac{h_a}{\sqrt{h_a-2r}}\geq\frac{4R+s+(10-3\sqrt{3})r}{\sqrt{2R}}\ (\text{Proved})$$

1393. In ΔABC the following relationship holds:

$$\frac{m_am_bm_c}{r_ar_br_c}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-2a^2)(2c^2+2a^2-2b^2)(2a^2+2b^2-2c^2)\overset{(1)}{\triangleq}$$

$$\frac{1}{64}\{-4\sum a^6+6(\sum a^4b^2+\sum a^2b^4)+3a^2b^2c^2\}$$

$$Now,\sum a^6=(\sum a^2)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)=$$

$$=(\sum a^2)^3-3\left(2a^2b^2c^2+\sum a^2b^2(\sum a^2-c^2)\right)$$

$$=(\sum a^2)^3+3a^2b^2c^2-3(\sum a^2b^2)\sum a^2\ \therefore\sum a^6\overset{(2)}{\triangleq}(\sum a^2)^3+3a^2b^2c^2-3(\sum a^2b^2)\sum a^2$$

$$Again,\sum a^4b^2+\sum a^2b^4=\sum a^2b^2(\sum a^2-c^2)\overset{(3)}{\triangleq}(\sum a^2b^2)\sum a^2-3a^2b^2c^2$$

$$\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2=$$

$$\frac{1}{64}\{-4(\sum a^2)^3-12a^2b^2c^2+12(\sum a^2b^2)\sum a^2+6(\sum a^2b^2)\sum a^2-18a^2b^2c^2+3a^2b^2c^2\}$$

$$=\frac{1}{64}\{-4(\sum a^2)^3+18(\sum a^2b^2)\sum a^2-27a^2b^2c^2\}$$

$$=\frac{1}{64}\{-4(\sum a^2)^3+18((\sum ab)^2-2abc(2s))(\sum a^2)-27a^2b^2c^2\}$$

$$=\frac{1}{64}\left\{-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}\leq\frac{R^2s^4}{4}$$

$$\Leftrightarrow s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(i)}{\overbrace{\leq}}0$$

$$Now, LHS\ of\ (i)\overset{Gerretsen}{\overbrace{\leq}}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{?}{\overbrace{\leq}}0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underset{(ii)}{\underbrace{\overset{?}{\geq}}}20rs^4$$

$$Now, LHS\ of\ (ii)\underset{(a)}{\underbrace{\overset{Gerretsen}{\geq}}}s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$

$$and\ RHS\ of\ (ii)\underset{(b)}{\underbrace{\overset{Geretsen}{\leq}}}20rs^2(4R^2+4Rr+3r^2)$$

$$(a),(b)\Rightarrow in\ order\ to\ prove\ (ii), it\ suffices\ to\ prove:$$

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$

$$\geq 20rs^2(4R^2+4Rr+3r^2)\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\geq 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(iii)}{\overbrace{\geq}}27r^2s^2$$

$$Now, LHS\ of\ (iii)\underset{(c)}{\underbrace{\overset{Gerretsen}{\geq}}}(108R^2-256Rr+80r^2)(16Rr-5r^2)$$

$$+r(4R+r)^3\ and\ RHS\ of\ (iii)\underset{(d)}{\underbrace{\overset{Geretsen}{\leq}}}27r^2(4R^2+4Rr+3r^2)$$

$$(c),(d)\Rightarrow in\ order\ to\ prove\ (iii), it\ suffices\ to\ prove:$$

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\geq 27r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(where\ t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{(t-2)(224t+309)+648\}\geq 0 \to true \because t \overset{Euler}{\geq} 2 \Rightarrow (iii) \Rightarrow (ii)$$

$$\Rightarrow (i)\ is\ true \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow \frac{m_a m_b m_c}{r_a r_b r_c} \leq \frac{Rs^2}{2rs^2}$$

$$\Rightarrow \frac{m_a m_b m_c}{r_a r_b r_c} \leq \frac{R}{2r}\ (Proved)$$

1394. **In $\Delta ABC, O$ –circumcenter, I –incenter the following relationship holds:**

$$(h_a - h_b)^2 + (h_b - h_c)^2 + (h_c - h_a)^2 \leq n \cdot OI^2, \qquad \forall\, n \geq \frac{33}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = 2\sum h_a^2 - 2\sum h_a h_b = 2\sum \frac{b^2c^2}{4R^2} - 2\sum \frac{bc.ca}{4R^2} = \frac{2\{\sum a^2b^2 + 2abc(2s)\} - 6abc(2s)}{4R^2} =$$

$$= \frac{2(\sum ab)^2 - 48Rrs^2}{4R^2} \leq \frac{33}{4} OI^2$$

$$\Leftrightarrow 33R(R-2r) \geq 2(s^2+4Rr+r^2)^2 - 48Rrs^2 \Leftrightarrow$$

$$\Leftrightarrow 2s^4 - s^2(32Rr - 4r^2) + 2r^2(4R+r)^2 - 33R(R-2r) \overset{(1)}{\leq} 0$$

$$Now, Rouche \Rightarrow s^2 - (m-n) \geq 0 \text{ and } s^2 - (m+n) \leq 0,$$

$$where\ m = 2R^2 + 10Rr - r^2 \text{ and } n = 2(R-2r)\sqrt{R^2 - 2Rr}$$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow 2s^4 - s^2(8R^2 + 40Rr - 4r^2) + 2r(4R+r)^3 \overset{(i)}{\leq} 0$$

$$(i) \Rightarrow in\ order\ to\ prove\ (1), it\ suffices\ to\ prove:$$

$$2s^4 - s^2(32Rr - 4r^2) + 2r^2(4R+r)^2 - 33R(R-2r)$$

$$\leq 2s^4 - s^2(8R^2 + 40Rr - 4r^2) + 2r(4R+r)^3$$

$$\Leftrightarrow s^2(8R^2 + 8Rr) + 2r^2(4R+r)^2 - 2r(4R+r)^3 \overset{(2)}{\leq} 33R(R-2r)$$

$$Now, LHS\ of\ (2) \overset{Gerretsen}{\leq} (4R^2 + 4Rr + 3r^2)(8R^2 + 8Rr) +$$

$$+2r^2(4R+r)^2-2r(4R+r)^3\overset{?}{\lesseqgtr}33R(R-2r)$$

$$\Leftrightarrow R^4-2R^3r+8R^2r^2-16Rr^3\overset{?}{\geq}0\Leftrightarrow$$

$$\Leftrightarrow R(R-2r)(R^2+8r^2)\overset{?}{\geq}0\to \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow(2)\Rightarrow(1)\ is\ true$$

$$\therefore (h_a-h_b)^2+(h_b-h_c)^2+(h_c-h_a)^2\leq\frac{33}{4}OI^2\overset{\frac{33}{4}\leq n}{\leq}n.OI^2\ (Proved)$$

1395. **In ΔABC the following relationship holds:**

$$\frac{27R^2}{4s^2}+\frac{3ns^2}{(4R+r)^2}\geq n+1,\qquad \forall\, n\leq\frac{11}{16}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{27R^2}{4s^2}+\frac{3ns^2}{(4R+r)^2}\overset{(1)}{\geq}n+1,\qquad \forall\, n\leq\frac{11}{16}$$

$$(1)\Leftrightarrow\frac{27R^2}{4s^2}-1\geq n-\frac{3ns^2}{(4R+r)^2}\Leftrightarrow\frac{27R^2-4s^2}{4s^2}\overset{(2)}{\geq}n\left[\frac{(4R+r)^2-3s^2}{(4R+r)^2}\right]$$

$$\frac{27R^2-4s^2}{4s^2}\geq\frac{11}{16}\left[\frac{(4R+r)^2-3s^2}{(4R+r)^2}\right]$$

$$\Leftrightarrow 4(27R^2-4s^2)(4R+r)^2\geq$$

$$\geq 11s^2\{(4R+r)^2-3s^2\}\left(\because 27R^2-4s^2\overset{\text{Mitrinovic}}{\geq}0\text{ and }(4R+r)^2-3s^2\overset{\text{Trucht}}{\geq}0\right)$$

$$\Leftrightarrow 11s^4+36R^2(4R+r)^2\overset{(i)}{\geq}9s^2(4R+r)^2$$

$$\text{Now, LHS of (i)}\overset{\text{Gerretsen}}{\geq}11(16Rr-5r^2)s^2+36R^2(4R+r)^2\overset{?}{\geq}9s^2(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow 36R^2(4R+r)^2\overset{?}{\geq}s^2\{9(4R+r)^2-11(16Rr-5r^2)\}$$

$$\Leftrightarrow 9R^2(4R+r)^2\underset{(ii)}{\overset{?}{\geq}}s^2(36R^2-26Rr+16r^2)$$

$$\text{Now, RHS of (ii)}\overset{\text{Gerretsen}}{\leq}(36R^2-26Rr+16r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}9R^2(4R+r)^2$$

$$\Leftrightarrow 32t^3-59t^2+14t-48\overset{?}{\gtrless}0$$

$$\Leftrightarrow (t-2)(32t^2+5t+24)\overset{?}{\gtrless}0\to \text{true} \because t\overset{\text{Euler}}{\gtrless}2\Rightarrow(\text{ii})\Rightarrow(\text{i})\text{is true} \therefore \frac{27R^2-4s^2}{4s^2}$$

$$\geq\frac{11}{16}\left[\frac{(4R+r)^2-3s^2}{(4R+r)^2}\right]\overset{\frac{11}{16}\geq n}{\gtrless} n\left[\frac{(4R+r)^2-3s^2}{(4R+r)^2}\right]$$

$$\Rightarrow(2)\Rightarrow(1)\text{ is true (Proved)}$$

1396. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^2 A+\cos^2 B}{\cos A+\cos B}\geq\frac{3}{2}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\sum a^3=3abc+2s\left(\sum a^2-\sum ab\right)=$$

$$12Rrs+2s\left(2(s^2-4Rr-r^2)-(s^2+4Rr+r^2)\right)\overset{(\text{i})}{=}2s(s^2-6Rr-3r^2)$$

$$\text{Also, }\frac{b+c}{a}\sin\frac{A}{2}=\frac{4R\sin\frac{A}{2}\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}=\cos\frac{B-C}{2}$$

$$\Rightarrow\cos\frac{B-C}{2}\overset{(\text{ii})}{=}\frac{b+c}{a}\sin\frac{A}{2}$$

$$\sum\frac{\cos^2A+\cos^2B}{\cos A+\cos B}=\sum\frac{(\cos A+\cos B)^2-2\cos A\cos B}{\cos A+\cos B}=$$

$$=\sum(\cos A+\cos B)-\sum\frac{2\cos B\cos C}{\cos B+\cos C}\Rightarrow \text{LHS}\overset{(1)}{=}2\sum\cos A-\sum\frac{2\cos B\cos C}{\cos B+\cos C}$$

$$Now,\ \frac{2\cos B\cos C}{\cos B+\cos C}=\frac{\cos(B+C)+\cos(B-C)}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}=\frac{-\cos A+2\cos^2\left(\frac{B-C}{2}\right)-1}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}=$$

$$=\frac{2\sin^2\frac{A}{2}-1}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}+\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}-\frac{1}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}$$

$$=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}+\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}-\frac{1}{\sin\frac{A}{2}\cos\frac{B-C}{2}}\overset{\text{by (ii)}}{\cong}\frac{a}{b+c}+\frac{b+c}{a}-\frac{a}{(b+c)\sin^2\frac{A}{2}}=$$

$$=\frac{a}{b+c}+\frac{b+c}{a}-\frac{abc(s-a)}{(b+c)(s-b)(s-c)(s-a)}$$

$$=\frac{a}{b+c}+\frac{b+c}{a}-\left(\frac{4Rrs}{sr^2}\right)\left(\frac{s-a}{b+c}\right)=\frac{a}{b+c}+\frac{b+c}{a}-\frac{2R}{r}\left(\frac{b+c-a}{b+c}\right)=$$

$$=\frac{a}{b+c}\left(\frac{2R+r}{r}\right)+\frac{b+c}{a}-\frac{2R}{r}$$

$$\therefore \boxed{\frac{2\cos B\cos C}{\cos B+\cos C}=\frac{a}{b+c}\left(\frac{2R+r}{r}\right)+\frac{b+c}{a}-\frac{2R}{r}}\ \textit{and analogs}\Rightarrow$$

$$\sum\frac{2\cos B\cos C}{\cos B+\cos C}\boxed{\overset{(2)}{\cong}}\left(\frac{2R+r}{r}\right)\sum\frac{a}{b+c}+\sum\frac{b+c}{a}-\frac{6R}{r}$$

$$\textit{Now,}\sum\frac{a}{b+c}=\frac{\sum a(c+a)(a+b)}{2abc+\sum ab(2s-c)}$$

$$=\frac{\sum a\left(\sum ab+a^2\right)}{2s(s^2+4Rr+r^2)-4Rrs}\overset{\text{by (i)}}{\cong}\frac{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)}$$

$$\Rightarrow\sum\frac{a}{b+c}\boxed{\overset{(iii)}{\cong}}\frac{2(s^2-2Rr-r^2)}{s^2+2Rr+r^2}$$

$$\text{Also,}\sum\frac{b+c}{a}=\frac{\sum bc(2s-a)}{4Rrs}=\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}\Rightarrow$$

$$\Rightarrow\sum\frac{b+c}{a}\overset{(iv)}{\cong}\frac{s^2-2Rr+r^2}{2Rr}$$

$$(1),(2),(iii),(iv)\Rightarrow\boxed{LHS-\frac{3}{2}}=$$

$$=2\left(1+\frac{r}{R}\right)-\frac{3}{2}-\left(\frac{2R+r}{r}\right)\left[\frac{2(s^2-2Rr-r^2)}{s^2+2Rr+r^2}\right]-\frac{s^2-2Rr+r^2}{2Rr}+\frac{6R}{r}$$

$$=\frac{1}{2}+\frac{2r}{R}+\frac{6R}{r}-\frac{s^2-2Rr+r^2}{2Rr}-\left(\frac{2R+r}{r}\right)\left[\frac{2(s^2-2Rr-r^2)}{s^2+2Rr+r^2}\right]=$$

$$=\frac{12R^2+3Rr+3r^2-s^2}{2Rr}-\left(\frac{2R+r}{r}\right)\left[\frac{2(s^2-2Rr-r^2)}{s^2+2Rr+r^2}\right]$$

$$=\frac{(12R^2+3Rr+3r^2-s^2)(s^2+2Rr+r^2)-4R(2R+r)(s^2-2Rr-r^2)}{2Rr(s^2+2Rr+r^2)}$$

$$\boxed{=\frac{s^2(4R^2-3Rr+2r^2)+r(32R^3+30R^2r+13Rr^2+3r^3)-s^4}{2Rr(s^2+2Rr+r^2)}}\geq 0$$

$$\Leftrightarrow s^4 \overset{(a)}{\overbrace{\leq}} s^2(4R^2-3Rr+2r^2)+r(32R^3+30R^2r+13Rr^2+3r^3)$$

$$\text{Now, } s^4 \overset{\text{Gerretsen}}{\overbrace{\leq}} s^2(4R^2+4Rr+3r^2) \overset{?}{\overbrace{\leq}}$$

$$\leq s^2(4R^2-3Rr+2r^2)+r(32R^3+30R^2r+13Rr^2+3r^3)$$

$$\Leftrightarrow s^2(7R+r) \underset{(b)}{\underbrace{\overset{?}{\overbrace{\leq}}}} 32R^3+30R^2r+13Rr^2+3r^3$$

$$\textit{Again, LHS of } (b) \overset{\text{Gerretsen}}{\overbrace{\leq}} (4R^2+4Rr+3r^2)(7R+r) \overset{?}{\overbrace{\leq}}$$

$$\leq 32R^3+30R^2r+13Rr^2+3r^3 \Leftrightarrow 2R^2-Rr-6r^2 \overset{?}{\overbrace{\geq}} 0$$

$$\Leftrightarrow (2R+3r)(R-2r) \overset{?}{\overbrace{\geq}} 0 \to \text{true} \Rightarrow (b) \Rightarrow (a) \textit{ is true}$$

$$\Rightarrow \sum \frac{\cos^2A+\cos^2B}{\cos A+\cos B} \geq \frac{3}{2} \ (\textit{Proved})$$

1397. If in $\Delta ABC, m(A) < 152^\circ$ then the following relationship holds:

$$h_a < \frac{7}{50}(b+c)$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$h_a < \frac{7}{50}(b+c) \Leftrightarrow \frac{bc}{2R} < \frac{7}{50}(b+c) \Leftrightarrow \frac{4R^2\sin B\sin C}{2R} < \left(\frac{7}{50}\right)2R(\sin B+\sin C) \Leftrightarrow$$

$$\Leftrightarrow \frac{2\sin B\sin C}{\sin B+\sin C} \overset{(1)}{\overbrace{<}} \frac{7}{25}$$

$$\because HM \leq GM \therefore \frac{2\sin B\sin C}{\sin B+\sin C} \leq \sqrt{\sin B\sin C} \overset{?}{\overbrace{<}} \frac{7}{25} \Leftrightarrow 2\sin B\sin C \overset{?}{\overbrace{<}} 2\left(\frac{7}{25}\right)^2$$

$$\Leftrightarrow \cos(B-C)-\cos(B+C) \underset{(2)}{\underbrace{\overset{?}{\overbrace{<}}}} 2\left(\frac{7}{25}\right)^2$$

$$\because \cos(B-C)\le 1 \therefore \cos(B-C)-\cos(B+C)\le 1+\cos A = 2\cos^2\frac{A}{2}$$

$$\le 2\cos^2 76^\circ \left(\because 76^\circ \le \frac{A}{2} < 90^\circ\right) < 2\cos^2 75^\circ$$

$$\therefore \text{LHS of }(2) \overset{(3)}{<} 2\cos^2 75^\circ$$

$$\text{Now}, \because \tan 30^\circ = \frac{1}{\sqrt{3}} \therefore \frac{2x}{1-x^2} = \frac{1}{\sqrt{3}}\ (\text{where } x=\tan 15^\circ) \Rightarrow 1-x^2 = 2\sqrt{3}x$$

$$\Rightarrow x^2 + 2\sqrt{3}x - 1 = 0$$

$$\Rightarrow x = \frac{-2\sqrt{3}\pm\sqrt{12+4}}{2} = 2-\sqrt{3} \Rightarrow \cot^2 15^\circ = \left(\frac{1}{2-\sqrt{3}}\right)^2 = \left(2+\sqrt{3}\right)^2 = 7+4\sqrt{3}$$

$$\Rightarrow \mathrm{cosec}^2 15^\circ = 8+4\sqrt{3}$$

$$\Rightarrow \sin^2 15^\circ = \frac{(2+\sqrt{3})(2-\sqrt{3})}{4(2+\sqrt{3})} \overset{(4)}{=} \frac{2-\sqrt{3}}{4}$$

$$(3) \Rightarrow \text{LHS of }(2) < 2\sin^2 15^\circ \overset{\text{by }(4)}{=} \frac{2-\sqrt{3}}{2} < 2\left(\frac{7}{25}\right)^2 \Rightarrow (2) \Rightarrow (1)$$

$$\Rightarrow \textit{proposed inequality is true (Proved)}$$

1398. In ΔABC the following relationship holds:

$$\frac{w_a^2}{bc}+\frac{w_b^2}{ca}+\frac{w_c^2}{ab}+\frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)} \le 3$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{In any } \Delta\, ABC, \sum\frac{w_a^2}{bc}+\frac{3\sum a^2}{4\sum ab} \overset{(a)}{\le} 3$$

$$: w_a^2 = \frac{4b^2c^2}{(b+c)^2}\left\{\frac{s(s-a)}{bc}\right\} = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{bc\{(b+c)^2-a^2\}}{(b+c)^2} =$$

$$= bc - \frac{a^2bc}{(b+c)^2} \Rightarrow \frac{w_a^2}{bc} = 1-\frac{a^2}{(b+c)^2}\ \&\ \textit{analogs}$$

$$\Rightarrow \text{LHS} = 3-\sum\frac{a^2}{(b+c)^2}+\frac{3\sum a^2}{4\sum ab} \le 3 \Leftrightarrow \frac{3\sum a^2}{4\sum ab} \overset{(1)}{\le} \sum\frac{a^2}{(b+c)^2}$$

$$\text{Now}, \sum\frac{a^2}{(b+c)^2}=\sum\frac{a^4}{(ab+ac)^2}\overset{\text{Bergstrom}}{\geq}\frac{(\sum a^2)^2}{\sum(a^2b^2+a^2c^2+2a^2bc)}$$

$$=\frac{(\sum a^2)^2}{2\sum a^2b^2+2abc(\sum a)}\overset{?}{\geq}\frac{3\sum a^2}{4\sum ab}$$

$$\Leftrightarrow 4(\sum ab)(\sum a^2)\overset{?}{\geq}6\sum a^2b^2+6abc(\sum a)\Leftrightarrow 2\sum a^3b+2\sum ab^3\underset{(2)}{\overset{?}{\geq}}3\sum a^2b^2+abc(\sum a)$$

$$\text{Now}, \frac{1}{2}(\sum a^3b+\sum ab^3)\overset{A-G}{\geq}\sum a^2b^2\overset{(i)}{\geq}abc(\sum a)\text{and}, \frac{3}{2}(\sum a^3b+\sum ab^3)\underset{(ii)}{\overset{A-G}{\geq}}3\sum a^2b^2$$

$$\therefore (i)+(ii)\Rightarrow(2)\Rightarrow(1)\Rightarrow(a)\text{ is true (Proved)}$$

1399. In ΔABC the following relationship holds:

$$\left(\sqrt{\frac{r_a}{w_a}}+\sqrt{\frac{r_b}{w_b}}+\sqrt{\frac{r_c}{w_c}}\right)^2\geq 4+5\sqrt[5]{\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\right)^6}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$w_a\leq\sqrt{s(s-a)}=\sqrt{r_br_c}\ \textit{and analogs}$$

$$\Rightarrow w_aw_bw_c\leq\sqrt{r_br_c}\sqrt{r_cr_a}\sqrt{r_ar_b}=r_ar_br_c\Rightarrow\prod w_a\overset{(1)}{\leq}\prod r_a$$

$$\textit{Now}, \sum r_aw_a=\sum\left[\left(s\tan\frac{A}{2}\right)\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\right]$$

$$=\sum\left[\left(s\sin\frac{A}{2}\right)\left(\frac{2bc}{b+c}\right)\right]\overset{HM\leq GM}{\leq}\sum\left[\left(s\sin\frac{A}{2}\right)\sqrt{bc}\right]=\sum s\sqrt{(s-b)(s-c)}$$

$$=\sum\left(\sqrt{s(s-b)}\sqrt{s(s-c)}\right)$$

$$\leq\sum m_bm_c\leq\frac{(\sum m_a)^2}{3}\therefore\frac{1}{\sum r_aw_a}\overset{(2)}{\geq}\frac{3}{(\sum m_a)^2}$$

$$\left(\sum\sqrt{\frac{r_a}{w_a}}\right)^2=\sum\frac{r_a}{w_a}+2\sum\sqrt{\frac{r_br_c}{w_bw_c}}\overset{AM\geq GM}{\geq}\sum\frac{r_a}{w_a}+6\sqrt[3]{\prod\left(\sqrt{\frac{r_br_c}{w_bw_c}}\right)}=$$

$$= \sum \frac{r_a}{w_a} + 6\sqrt[3]{\frac{\prod r_a}{\prod w_a}} \overset{\text{by (1)}}{\geq} \sum \frac{r_a}{w_a} + 6 = \sum \frac{r_a^{\,2}}{r_a w_a} + 6$$

$$\overset{\text{Bergstrom}}{\geq} \frac{(\sum r_a)^2}{\sum r_a w_a} + 6 \overset{\text{by (2)}}{\geq} \frac{3(\sum r_a)^2}{(\sum m_a)^2} + 6 \Rightarrow \left(\sum \sqrt{\frac{r_a}{w_a}}\right)^2 \overset{(3)}{\geq} \frac{3(\sum r_a)^2}{(\sum m_a)^2} + 6$$

$$(3) \Rightarrow \textit{it suffices to prove}: 3\left(\frac{\sum r_a}{\sum m_a}\right)^2 + 6 \overset{(i)}{\geq} 4 + 5\left(\frac{\sum r_a}{\sum m_a}\right)^{\frac{6}{5}}$$

$$\textit{Let } \left(\frac{\sum r_a}{\sum m_a}\right)^{\frac{1}{5}} = t \ \therefore (i) \Leftrightarrow 3t^{10} - 5t^6 + 2 \geq 0 \Leftrightarrow$$

$$(t-1)^2(t+1)^2(3t^6 + 6t^4 + 4t^2 + 2) \geq 0 \rightarrow \text{true} \because t \geq 1 \text{ as } \sum r_a \geq \sum m_a$$

$$\Rightarrow (i) \textit{ is true}$$

$$\therefore \left(\sum \sqrt{\frac{r_a}{w_a}}\right)^2 \geq 4 + 5\sqrt[5]{\left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^6} \ (\textit{Proved})$$

1400. In ΔABC the following relationship holds:

$$\frac{h_a}{h_b} + \frac{h_b}{h_c} + \frac{h_c}{h_a} \leq \frac{1}{27}\left(1 + \frac{4R}{r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$h_a = \frac{2S}{a} \Rightarrow \text{we must show}:$$

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq \frac{1}{27}\left(1 + \frac{4R}{r}\right)^2 \Leftrightarrow \frac{a^2b + b^2c + c^2a}{abc} \leq \frac{1}{27}\left(1 + \frac{4R}{r}\right)^2 \ldots (1)$$

From rearanjament inequality we have:

$$a^2b + b^2c + c^2a \leq a^3 + b^3 + c^3 \ldots (2)$$

From (1)+(2) we must show:

$$\frac{a^3 + b^3 + c^3}{abc} \leq \frac{1}{27}\left(1 + \frac{4R}{r}\right)^2 \ldots (3)$$

$$\text{But } \frac{a^3 + b^3 + c^3}{abc} + 1 \leq \frac{2R}{r} \ldots (4) \text{because} \sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$$

$$\text{and } abc = 4Rrs \text{ then } (4) \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \ (\text{true from Gerretsen})$$

From (3)+(4) we must show:

$$\frac{2R}{r}-1\le\frac{1}{27}\left(1+\frac{4R}{r}\right)^2\ \dots(5)$$

$$\text{Let: } \frac{2R}{r}=x, x\ge 4$$

$$(5)\Leftrightarrow x-1\le\frac{1}{27}(1+2x)^2\Leftrightarrow 27x-27\le 1+4x+4x^2\Leftrightarrow$$

$$4x^2-23x+28\ge 0\Leftrightarrow(4x-7)(x-4)\ge 0,\text{ true because } x\ge 4$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru