

RMM - Calculus Marathon 801 - 900

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801. Prove that:

$${}_4F_3\left(1, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{5}{2}, \frac{11}{2}, \frac{7}{2}; -1\right) = \frac{3(1260G - 105\pi - 754)}{224}$$

G: Catalan's constant

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Abdul Hafeez Ayinde-Nigeria

$$\text{Let } \Omega = {}_4F_3\left(1, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{5}{2}, \frac{11}{2}, \frac{7}{2}; -1\right)$$

$$\Omega = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{9}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{9}{2}\right)}{\Gamma\left(k+\frac{5}{2}\right)\Gamma\left(k+\frac{11}{2}\right)\Gamma\left(k+\frac{7}{2}\right)} \cdot \frac{(-1)^k}{k!}$$

$$\Omega = \frac{\frac{3}{2} \cdot \frac{9}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}}{2} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{9}{2}\right)}{\Gamma\left(k+\frac{5}{2}\right)\Gamma\left(k+\frac{11}{2}\right)\Gamma\left(k+\frac{7}{2}\right)} \cdot \frac{(-1)^k}{k!}$$

$$\Omega = \frac{405}{32} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{9}{2}\right)}{\Gamma\left(k+\frac{5}{2}\right)\Gamma\left(k+\frac{11}{2}\right)\Gamma\left(k+\frac{7}{2}\right)} \cdot \frac{(-1)^k}{k!}$$

$$\Omega = \frac{405}{32} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{9}{2}\right)}{\Gamma\left(k+\frac{5}{2}\right)\Gamma\left(k+\frac{11}{2}\right)\Gamma\left(k+\frac{7}{2}\right)} \cdot (-1)^k$$

$$\Omega = \frac{405}{32} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)\left(k+\frac{9}{2}\right)\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)\left(k+\frac{1}{2}\right)} \cdot (-1)^k$$

$$\Omega = 405 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(2k+9)(2k+5)(2k+3)(2k+1)}$$

$$\Omega = 405 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)^2(2k+9)(2k+5)(2k+1)}$$



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$$\Omega = 405 \left(\frac{1}{128} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} - \frac{1}{24} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)^2} - \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+5} + \frac{1}{1152} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+9} + \frac{1}{144} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+3} \right)$$

$$\Omega = 405 \left\{ \frac{\pi}{512} + \frac{1}{24} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 1 \right) + \frac{1}{144} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+3} - \right. \\ \left. - \frac{1}{64} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} - 1 + \frac{1}{3} \right) + \frac{1}{1152} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} \right) \right\}$$

$$\Omega = 405 \left\{ \frac{\pi}{512} + \frac{C}{24} - \frac{\pi}{256} + \frac{\pi}{4608} + \frac{1}{144} - \frac{\pi}{576} - \frac{1}{24} + \frac{1}{64} - \frac{1}{192} + \frac{1}{1152} \left(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} \right) \right\}$$

$$\Omega = 405 \left\{ \frac{\pi}{512} + \frac{C}{24} - \frac{\pi}{256} + \frac{\pi}{4608} - \frac{\pi}{576} - \frac{377}{15120} \right\}$$

$$\Omega = 405 \left(\frac{C}{24} - \frac{\pi}{288} - \frac{377}{15120} \right)$$

$$\Omega = 3 \left(\frac{1260C - 105\pi - 754}{224} \right)$$

C → Catalan's constant.

802. Let's define the function

$$g(x) = \int_{\log(x)}^{\infty} \frac{x}{e^x - 1} dx$$

Prove that:

$$\int_0^1 \left(g\left(\frac{1}{x}\right) + \log(x) \right) g(x) dx = -\frac{\pi^4}{30}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$g(x) = \int_{\log x}^{\infty} \frac{y}{e^y - 1} dy = \int_0^{\infty} \frac{y}{e^y - 1} dy - \int_0^{\log x} \frac{y}{e^y - 1} dy = \xi(2) - \int_0^{\log x} ye^{-y} \sum_{n=0}^{\infty} e^{-ny} dy =$$



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$$\begin{aligned}
 &= \xi(2) - \sum_{n=0}^{\infty} \int_0^{\log x} ye^{-(n+1)y} dy = \xi(2) + \sum_{n=0}^{\infty} \left\{ \frac{e^{-(n+1)y}}{(n+1)^2} + \frac{ye^{-(n+1)y}}{n+1} \right\} \Big|_0^{\log x} = \\
 &= \xi(2) + \sum_{n=0}^{\infty} \left(\frac{1}{(n+1)^2 x^{n+1}} + \frac{\log x}{(n+1)x^{n+1}} - \frac{1}{(n+1)^2} \right) \\
 &= Li_2\left(\frac{1}{x}\right) + \log\left(\frac{1}{x}\right) \log\left(1 - \frac{1}{x}\right) = \xi(2) - Li_2\left(1 - \frac{1}{x}\right) \\
 &\quad g\left(\frac{1}{x}\right) = \xi(2) - Li_2(1-x)
 \end{aligned}$$

$$g(x) == \xi(2) - Li_2\left(1 - \frac{1}{x}\right) = \xi(2) + Li_2(1-x) + \frac{1}{2} \log^2 x$$

Now:

$$\begin{aligned}
 M &= \int_0^1 \left(g\left(\frac{1}{x}\right) + \log x \right) g(x) dx = \int_0^1 \left(g\left(\frac{1}{x}\right) + \log x \right) g(x) dx \\
 &= \int_0^1 \left(\xi^2(x) + \xi(2) \cos x + \frac{\xi(2)}{2} \log^2 x + \log(x) Li_2(1-x) + \frac{1}{2} \log^3 x - Li_2^2(1-x) - \frac{1}{2} \log^2(x) Li_2(1-x) \right) dx \\
 &= \xi^2(2) - 3 + N - P - \frac{1}{2} \Omega
 \end{aligned}$$

$$\begin{aligned}
 N &= \int_0^1 \log(x) Li_2(1-x) dx = \int_0^1 \log(x) \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2} dx =
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 (1-x)^n \log x dx = - \sum_{n=1}^{\infty} \frac{H_{n+1}}{n^2(n+1)}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{n} - \frac{H_{n+1}}{n} - \frac{H_{n+1}}{n+1} \right) = \sum_{n=1}^{\infty} \left(\frac{H_n}{n} + \frac{1}{n(n+1)} - \frac{H_n}{n^2} - \frac{1}{n^2(n+1)} - \frac{H_{n+1}}{n+1} \right)
 \end{aligned}$$



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$$= 1 + \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+1} - \frac{1}{n^2} - \frac{H_n}{n^2} \right) = 3 - \xi(2) - 2\xi(3)$$

$$\begin{aligned} P &= \int_0^1 Li_2^2(1-x) dx = \overbrace{\int_0^1 Li_2^2(x) dx}^{IBP} = \{xLi_2^2(x)\}_0^1 + 2 \int_0^1 \log(1-x) Li_2(x) dx = \xi^2(2) + 2N \\ &\quad = \xi^2(2) + 6 - 2\xi(2) - 4\xi(3) \end{aligned}$$

$$\begin{aligned} \Omega &= \overbrace{\int_0^1 \log^2(x) L_{i_2}(1-x) dx}^{IBP} = \overbrace{\{xL_{i_2}(1-x)(\log^2 x - 2\log x + 2)\}_0^1}^{=0} + \\ &\quad + \int_0^1 \left(\frac{\log x}{1-x} x(2\log x - \log^2 x - 2) \right) dx \\ &= \sum_{n=0}^{\infty} \int_0^1 x^{n+1} (2\log^2 x - \log^3 x - 2\log x) dx = \\ &= \sum_{n=0}^{\infty} \left(\frac{4}{(n+2)^3} + \frac{6}{(n+2)^4} + \frac{2}{(n+2)^2} \right) \\ &= 6\xi(4) + 4\xi(3) + 2\xi(2) - 12 \therefore M = -3\xi(4) = -\frac{\pi^4}{30} \quad (\text{Answer}) \end{aligned}$$

$$803. \sin 3^\circ = \frac{1}{4} \left(\sqrt{8 - \sqrt{10 - 2\sqrt{5}}} - \sqrt{3} - \sqrt{15} \right) = \frac{1}{4} \left(\sqrt{8 - \sqrt{\sqrt{10}(\sqrt{10} - \sqrt{2})}} - \sqrt{3} - \sqrt{15} \right)$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Ahmed Salama Hegazy-Cairo-Egypt

$$L = \sin 3 = \sin(18 - 15) = \sin(18)\cos(15) - \cos(18)\sin(15)$$

Now we find sin(18), let x = 18 \Rightarrow 5x = 90 \Rightarrow 2x + 3x = 90

$$\therefore 2x = 90 - 3x, \therefore \sin(2x) = \sin(90 - 3x) = \cos(3x)$$

$$\therefore 2 \sin x \cos x = 4 \cos^3(x) - 3 \cos x$$

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$$\therefore 4 \cos^3(x) - 2 \sin x \cos x - 3 \cos x = 0, \therefore \cos x (4 \cos^2 x - 2 \sin x - 3) = 0$$

$$\therefore 4(1 - \sin^2(x) - 2 \sin x - 3 = 0) \gg 4 \sin^2(x) + 2 \sin x - 1 = 0$$

$$\therefore \sin x = \frac{\sqrt{5}-1}{4}, \therefore \sin 18 = \frac{\sqrt{5}-1}{4}, \text{ and } \cos 18 = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$$

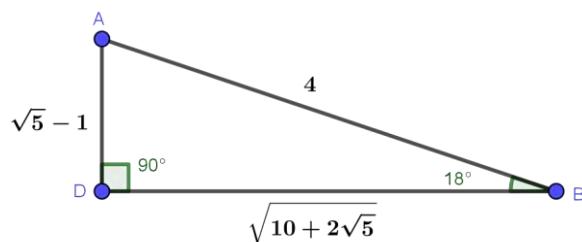
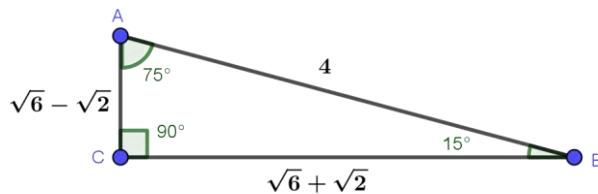
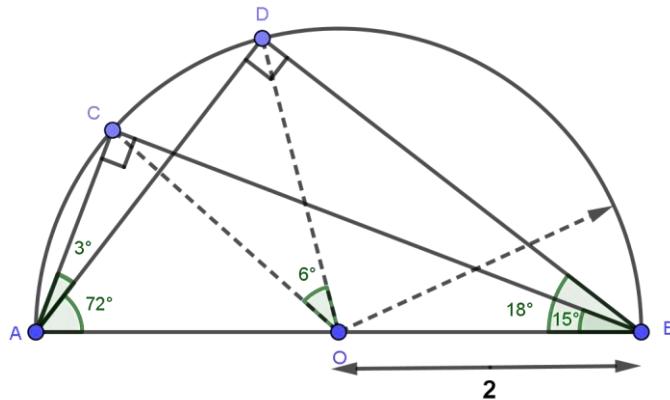
$$\text{since } \sin x = \frac{1}{\sqrt{2}} \sqrt{1 - \cos 2x}, \cos x = \frac{1}{\sqrt{2}} \sqrt{1 + \cos 2x}, \text{ put } x = 15$$

$$\therefore \sin 15 = \frac{\sqrt{3}-1}{2\sqrt{2}}, \cos 15 = \frac{1+\sqrt{3}}{2\sqrt{2}}, \text{ by substitution we have}$$

$$\sin 3 = \frac{1}{4} \sqrt{-\sqrt{3} - \sqrt{15} + 8 - \sqrt{10 - 2\sqrt{5}}}$$

Solution 2 by Sergio Esteban-Argentina

Calculo del $\sin 3^\circ$ geometricamente. Por triangulos notables sabemos



i) Por Angulo inscrito sabemos que

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$$m\angle COD = 2m\angle CBD \text{ y } m\angle ACB = m\angle ADB = 90^\circ$$

ii) Sabemos que:

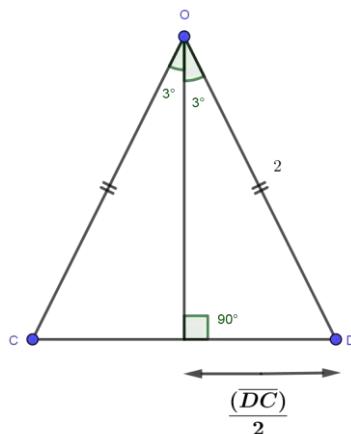
$$\overline{AC} = \sqrt{6} - \sqrt{2}, \overline{BC} = \sqrt{6} + \sqrt{2}, \overline{AD} = \sqrt{5} - 1, \overline{BD} = \sqrt{10 + 2\sqrt{5}} \text{ y } AB = 4$$

Usando el teorema de Ptolomeo $\overline{BC} \cdot \overline{AD} = \overline{AC} \cdot \overline{BD} + \overline{DC} \cdot \overline{AB}$

$$(\sqrt{6} + \sqrt{2})(\sqrt{5} - 1) = (\sqrt{6} - \sqrt{2}) \left(\sqrt{10 + 2\sqrt{5}} \right) + \overline{DC} \cdot 4$$

$$\Rightarrow \overline{DC} = \frac{(\sqrt{6} + \sqrt{2})(\sqrt{5} - 1) - (\sqrt{6} - \sqrt{2}) \left(\sqrt{10 + 2\sqrt{5}} \right)}{4}$$

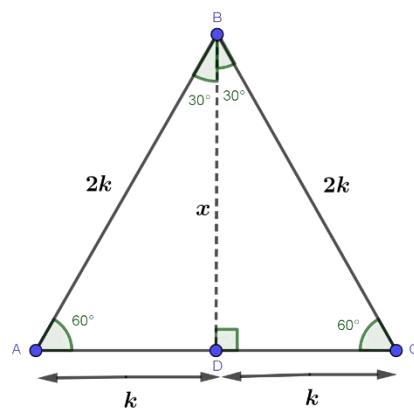
Entonces:



$$\sin 3^\circ = \frac{\overline{DC}}{2 \cdot 2} = \frac{1}{4} \sqrt{-\sqrt{3} - \sqrt{15} + 8 - \sqrt{10 - 2\sqrt{5}}}$$

Preliminares

i) Sabemos que el triangulo equilátero

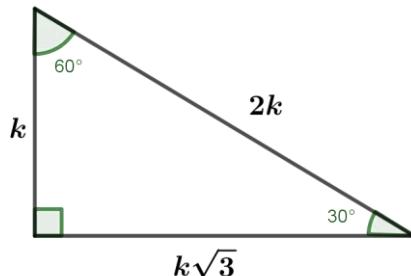


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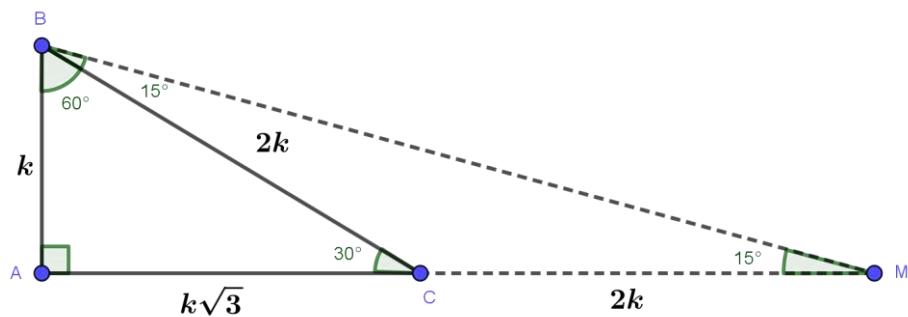
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Aplicando Pitágoras: $x^2 + k^2 = (2k)^2; x = k\sqrt{3} \therefore$

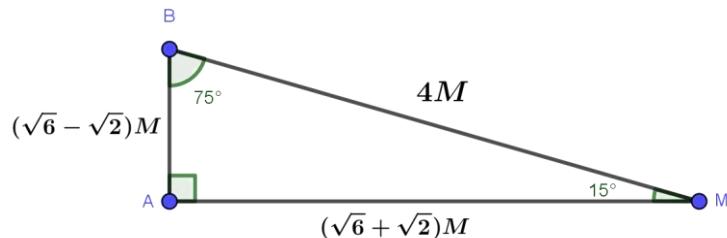


ii) Trabajando i)



Prolongamos \overline{AC} hasta M tai que $\overline{BC} = \overline{CM}$. Si damos $k = (\sqrt{6} - \sqrt{2})M$

Entonces $\overline{BM}^2 = \overline{AB}^2 + \overline{AM}^2$; $\overline{BM} = 4m$

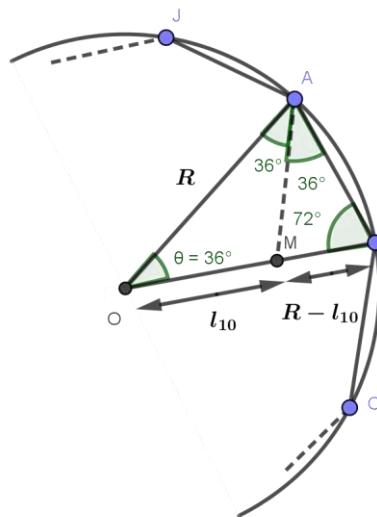


iii) en el decagon regular

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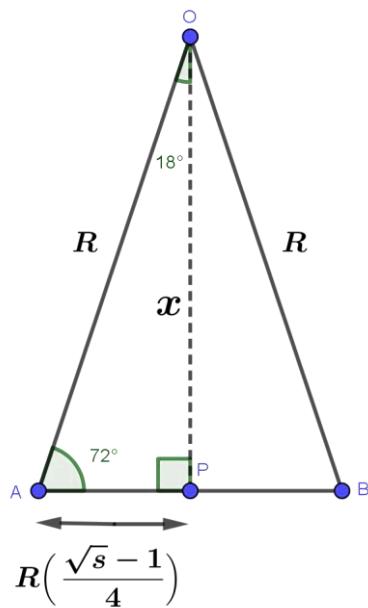
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$$\text{medida de } \theta^\circ = \frac{360^\circ}{10} = 36^\circ; \overline{AB} = l_{10}, \overline{AO} = \overline{OB} = R$$

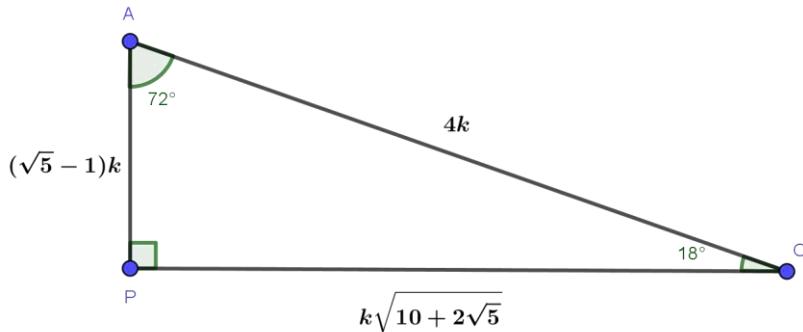
Se traza la bisectriz interior $\overline{AM} \Rightarrow \overline{AM} = \overline{AB} = l_{10}$

Por el teorema de la bisectriz interior $\frac{R}{l_{10}} = \frac{l_{10}}{R - l_{10}}$. Operando $l_{10} = R \left(\frac{\sqrt{5}-1}{2} \right)$. Entonces:



$$x^2 + \left(R \left(\frac{\sqrt{5}-1}{4} \right) \right)^2 = R^2$$

$$x = \frac{R}{4} \sqrt{10 + 2\sqrt{5}} \therefore \text{si damos } R = 4k$$



804. Find: $\Omega = \frac{1}{\sin^2 2^\circ} + \frac{1}{\sin^2 6^\circ} + \frac{1}{\sin^2 10^\circ} + \cdots + \frac{1}{\sin^2 86^\circ}$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution 1 by Gabriel Ruddy Cruz Mendez-Lima-Peru

Sabemos: $\cot^2(x) + \cot^2(60^\circ - x) + \cot^2(60^\circ + x) = 9 \cot^2(3x) + 6$

$$\left. \begin{array}{l} \cot^2(2^\circ) + \cot^2(58^\circ) + \cot^2(62^\circ) = 9 \cot^2(6^\circ) + 6 \\ \cot^2(6^\circ) + \cot^2(54^\circ) + \cot^2(66^\circ) = 9 \cot^2(18^\circ) + 6 \\ \cot^2(10^\circ) + \cot^2(50^\circ) + \cot^2(70^\circ) = 9 \cot^2(30^\circ) + 6 \\ \cot^2(14^\circ) + \cot^2(46^\circ) + \cot^2(74^\circ) = 9 \cot^2(42^\circ) + 6 \\ \cot^2(18^\circ) + \cot^2(42^\circ) + \cot^2(78^\circ) = 9 \cot^2(54^\circ) + 6 \\ \cot^2(22^\circ) + \cot^2(38^\circ) + \cot^2(82^\circ) = 9 \cot^2(66^\circ) + 6 \\ \cot^2(26^\circ) + \cot^2(34^\circ) + \cot^2(86^\circ) = 9 \cot^2(78^\circ) + 6 \end{array} \right\}$$

Sumando verticalmente: $\sum_{i=1}^{22} \cot^2(2(2i-1))^\circ = 45 + 9 \sum_{i=1}^7 \cot^2(6(2i-1))^\circ$

$$\cot^2(6^\circ) + \cot^2(54^\circ) + \cot^2(66^\circ) = 9 \cot^2(18^\circ) + 6$$

$$\cot^2(18^\circ) + \cot^2(42^\circ) + \cot^2(78^\circ) = 9 \cot^2(54^\circ) + 6$$

$$\sum_{i=1}^7 \cot^2[6(2i-1)]^\circ = 9 \left(\underbrace{\cot^2 18^\circ + \cot^2 54^\circ}_{10} \right) + 15$$

$$\sum_{i=1}^7 \cot^2[6(2i-1)]^\circ = 105 \rightarrow \sum_{i=1}^{22} \cot^2(2(2i-1))^\circ = 990$$

$$\therefore \sum_{i=1}^{22} \csc^2(2(2i-1))^\circ = 1012$$



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Solution 2 by Gerald Bernabel-Lima-Peru

$$R = \sum_{k=1}^{22} \sec^2(4k^\circ) = 22 + \sum_{k=1}^{22} \tan^2(4k^\circ) = ??$$

$$\tan(45x) = \frac{\binom{45}{1}\tan(x) - \binom{45}{3}\tan^3(x) + \binom{45}{5}\tan^5(x) - \dots - \binom{45}{43}\tan^{43}(x) + \binom{45}{45}\tan^{45}(x)}{1 - \binom{45}{2}\tan^2(x) + \binom{45}{4}\tan^4(x) - \dots + \binom{45}{44}\tan^{44}(x)} = 0$$

$$\rightarrow Si: \tan(45x) = 0 \rightarrow 45x = 180^\circ k \rightarrow x = 4k^\circ$$

$$\tan(x) \left(\binom{45}{45}\tan^{44}(x) - \binom{45}{43}\tan^{42}(x) + \binom{45}{41}\tan^{40}(x) - \dots + \binom{45}{1} \right) = 0$$

$$\binom{45}{1} - \binom{45}{3}\tan^2(x) + \binom{45}{3}\tan^4(x) - \dots - \binom{45}{43}\tan^{42}(x) + \binom{45}{45}\tan^{44}(x) = 0$$

$$Si: y = \tan^2(x) \text{ entonces la ecuacion: } \binom{45}{45}y^{22} - \binom{45}{43}y^{21} + \binom{45}{41}y^{20} \dots + \binom{45}{1} = 0$$

Presenta 22 raíces de la forma $y_k = \tan^2(4k^\circ)$ donde $k = \{1, 2, 3, \dots, 22\}$ →

$$\rightarrow \sum_{k=1}^{22} \tan^2(4k^\circ) = \frac{\binom{45}{43}}{\binom{45}{45}} = 990 \therefore R = 1012$$

805. Prove that:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!} = \\ & Ei\left(\frac{\exp(-\pi)}{2}\right) - 2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) + \\ & + 2\exp\left(\frac{\exp(-\pi)}{2+\pi}\right) - 2\exp(\pi) + 3 \end{aligned}$$

γ : Euler-Mascheroni constant; Ei : exponential integral

erfi : imaginary error function

Proposed by Mokhtar Khassani-Mostaganem-Algerie



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Solution by Dawid Bialek-Poland

$$\sum_{n=1}^{\infty} \frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!} \xrightarrow{(2n)!! = n!2^n} \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{n(n+1)(2n+1)n!} - \underbrace{\sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{nn!}}_{S_1} +$$

$$+ \underbrace{\sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)n!}}_{S_2} - 4 \underbrace{\sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(2n+1)n!}}_{S_3} \quad (1)$$

$$S_1 - \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{nn!} \xrightarrow{(I)} E_t\left(\frac{e^{-\pi}}{2}\right) - \gamma - \log\left(\frac{e^{-\pi}}{2}\right) - E_i\left(\frac{\exp(-\pi)}{2}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) \quad (2)$$

Where (1) $E_i(x) \stackrel{\text{def}}{=} \gamma + \log(x) + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$, E_i - exponential integral

$$S_2 = \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)n!} = \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)!} - 2e^{-\pi} - 1 \xrightarrow{(II)} \frac{e^{\left(\frac{e^{-\pi}}{2}\right)}}{e^{-\pi}} - 2e^{-\pi} - 1 = 2 \exp\left(\frac{\exp(-\pi)}{2} + \pi\right) - 2 \exp(\pi) - 1 \quad (3)$$

$$\text{Where (II)} \frac{e^x}{x} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \frac{1}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=-1}^{\infty} \frac{x^n}{(n+1)!}$$

$$\begin{aligned} S_3 &= 4 \sum_{n=1}^{\infty} \frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(2n+1)n!} = 4 \cdot \sum_{n=0}^{\infty} \frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n}}{(2n+1)n!} = 4 \cdot \sqrt{\frac{2}{e^{-\pi}}} \cdot \sum_{n=1}^{\infty} \frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!} = \\ &= 4\sqrt{2e^{\pi}} \left[\sum_{n=0}^{\infty} \frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!} - \sqrt{\frac{e^{-\pi}}{2}} \right] = -4 + 4\sqrt{2e^{\pi}} \cdot \sum_{n=0}^{\infty} \frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!} \end{aligned}$$

$$\xrightarrow{(III)} -4 + 4\sqrt{2e^{\pi}} \cdot \frac{\sqrt{\pi}}{2} \operatorname{erfi}\left(\sqrt{\frac{e^{-\pi}}{2}}\right) = -4 + 2e^{\left(\frac{\pi}{2}\right)} \cdot \sqrt{2\pi} \cdot \operatorname{erfi}\left(\frac{e^{\left(\frac{-\pi}{2}\right)}}{\sqrt{2}}\right) = -4 + 2 \exp\left(\frac{\pi}{2}\right) \sqrt{2\pi} \operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) \quad (4)$$

Where (III) $\operatorname{erfi}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1) \cdot n!}$, $\operatorname{erfi}(x)$ - imaginary function

Rewriting (1) with (2), (3) and (4) we get:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!} &= S_1 + S_2 - S_3 = E_t\left(\frac{\exp(-\pi)}{2}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) + \\ &+ 2 \exp\left(\frac{\exp(-\pi)}{2} + \pi\right) - 2 \exp(\pi) - 1 + 4 - 2 \exp\left(\frac{\pi}{2}\right) \sqrt{2\pi} \operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) = \end{aligned}$$



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$$\begin{aligned}
 &= E_i\left(\frac{\exp(-\pi)}{2}\right) - 2 \exp\left(\frac{\pi}{2}\right) \sqrt{2\pi} \operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) + \\
 &\quad + 2 \exp\left(\frac{\exp(-\pi)}{2} + \pi\right) - 2 \exp(\pi) + e
 \end{aligned}$$

806.

$$\int_0^1 \sqrt{1+\sqrt{x}} \times \sqrt{1+\sqrt{1+\sqrt{x}}} \times \dots \times \underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\dots+\sqrt{1+\sqrt{x}}}}}}_{\text{for "n" times number "1"}} dx = \phi_{n+1}$$

$\phi_{n+1} \cong F_{n+1}$; F_{n+1} – fibonacci number; $\varphi = 1.618033\dots$

Prove that $\frac{\phi_{n+1}}{\phi_n}$ is faster to give golden number φ then $\frac{F_{n+1}}{F_n}$

Proposed by Mohammed Bouras-Morocco

Solution by Kamel Benaicha-Algiers-Algerie

$$\varphi_{n+1} = \int_0^1 \sqrt{1+\sqrt{x}} \sqrt{1+\sqrt{1+\sqrt{x}}} \dots \sqrt{1+\sqrt{1+\dots\sqrt{1+\sqrt{x}}}} dx$$

$$\text{Put: } u_{n+1}(x) = \sqrt{1+\sqrt{1+\dots\sqrt{1+\sqrt{x}}}} \quad (1)$$

$$\text{and } v_{n+1}(x) = \sqrt{1+\sqrt{x}} \sqrt{1+\sqrt{1+\sqrt{x}}} \dots \sqrt{1+\sqrt{1+\dots\sqrt{1+\sqrt{x}}}}$$

$$\text{So, } v_{n+1}(x) = v_n(x)u_{n+1}(x) \quad (2)$$

$$\text{We have: } u_{n+1}(x) = \sqrt{1+u_n(x)} \Rightarrow u_{n+1}^2 - u_n - 1 = 0 \quad (a)$$

$$u_{n+1}(x) - u_n(x) = \sqrt{1+u_n(x)} - u_n(x) = \frac{1+u_n(x)-u_n^2(x)}{\sqrt{1+u_n(x)}+u_n(x)} = \frac{(\varphi-u_n(x))\left(\frac{1}{\varphi}+u_n(x)\right)}{\sqrt{1+u_n(x)}+u_n(x)} \quad (3)$$

$$\text{Put } f(t) = \sqrt{1+\sqrt{t}}; \quad f(\mathbb{R}_+) \subset \mathbb{R}_+$$

f increasing and we have: $\forall x \in [0, 1]: x^2 - x = x(x-1) \leq 0 \Rightarrow x^2 \leq x$



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$$\therefore x \leq \sqrt{x} \Rightarrow x \leq 1 + \sqrt{x} \Rightarrow f(x) = \sqrt{1 + \sqrt{x}} \leq \sqrt{1 + \sqrt{1 + \sqrt{x}}} = f(1 + \sqrt{x})$$

$\therefore u_2(x) \geq u_1(x)$ and $f \nearrow$ then $(u_n(x))_{n \in \mathbb{N}^*}$ is increasing, $u_{n+1}(x) \geq u_n(x)$ (4)

$u_n(x) > 0$, by using (3) and (4), then $u_n \leq \varphi$

So, $(u_n(x))_{n \in \mathbb{N}^*} \nearrow$ and $u_n \leq \varphi \Rightarrow \lim_{n \rightarrow +\infty} u_n(x) = l \in \mathbb{R}_+$

l is solution of equation $l^2 - l - 1 = 0$ (result (a)) with condition $l \geq 0$

So, $\lim_{n \rightarrow +\infty} u_n(x) = \varphi$ (b)

$$\lim_{n \rightarrow +\infty} \frac{\varphi_{n+1}}{\varphi_n} = \lim_{n \rightarrow +\infty} \frac{\int_0^1 v_n(x) u_{n+1}(x) dx}{\int_0^1 v_n(x) dx} \quad (\text{using result (2)})$$

$\forall x \in [0; 1] \varphi_n(x) > 0$ then by using second Middle value theory:

$$\exists \alpha \in]0; 1[\text{ such that } \int_0^1 v_n(x) u_{n+1}(x) dx = u_{n+1}(\alpha) \int_0^1 v_n(x) dx$$

$$\therefore \lim_{n \rightarrow +\infty} \frac{\varphi_{n+1}}{\varphi_n} = \lim_{n \rightarrow +\infty} \frac{u_{n+1}(\alpha) \int_0^1 v_n(x) dx}{\int_0^1 v_n(x) dx} = \lim_{n \rightarrow +\infty} u_{n+1}(\alpha) = \varphi \quad (\text{using result (b)})$$

$$\therefore \lim_{n \rightarrow +\infty} \frac{\varphi_{n+1}}{\varphi_n} = \varphi. \text{ We have } \frac{\varphi_{n+1}}{\varphi_n} = u_{n+1}(\alpha)/\alpha \in]0; 1[$$

Put $w_n = \frac{F_{n+1}}{F_n} \Rightarrow w_{n+1} = 1 + \frac{1}{w_n}$ where $(F_n)_{n \in \mathbb{N}^*}$ is the Fibonacci sequence.

Put $g(x) = 1 + \frac{1}{x}$, g is decreasing $\therefore (w_n)_{n \in \mathbb{N}^*}$ is fluctuate sequence.

By comparison of fluctuate sequence $(w_n)_{n \in \mathbb{N}^*}$ and increasing sequence $(u_n(\alpha))_{n \in \mathbb{N}^*}$

then $\frac{\varphi_{n+1}}{\varphi_n}$ is faster than $\frac{F_{n+1}}{F_n}$ to φ . φ denote the golden ratio: $\varphi = \frac{1+\sqrt{5}}{2}$

807.

$$\frac{\pi}{4} = \int_0^\infty \frac{2x^8 + x^7 + 3x^6 + x^5 - x^4 + x^3 - x^2 + x + 1}{x^{12} + x^{11} + 2x^{10} + 3x^9 + 3x^8 + 4x^7 + 4x^6 + 4x^5 + 3x^4 + 3x^3 + 2x^2 + x + 1} dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Pedro Nagasava-Brazil

$$\Omega = \int_0^\infty \frac{2x^8 + x^7 + 3x^6 + x^5 - x^4 + x^3 - x^2 + x + 1}{x^{12} + x^{11} + 2x^{10} + 3x^9 + 3x^8 + 4x^7 + 4x^6 + 4x^5 + 3x^4 + 3x^3 + 2x^2 + x + 1} dx(a)$$

Notice that $(x + 1)$ is a common factor of both polynomials, therefore:



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$$\Omega = \int_0^\infty \frac{2x^7 - x^6 + 4x^5 - 3x^4 + 2x^3 - x^2 + 1}{x^{11} + 2x^9 + x^8 + 2x^7 + 2x^6 + 2x^5 + 2x^4 + x^3 + 2x^2 + 1} dx(b)$$

With (a) + (b): $2\Omega = \int_0^\infty \frac{x^9 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1}{x^{11} + 2x^9 + x^8 + 2x^7 + 2x^6 + 2x^5 + 2x^4 + x^3 + 2x^2 + 1} dx$

$$\Omega = \frac{1}{2} \int_0^\infty \frac{(x^4 + 1)(x^3 + 1)(x^2 + 1)}{(x^3 + 1)(x^8 + 2x^6 + 2x^4 + 2x^2 + 1)} dx$$

$$\Omega = \frac{1}{2} \int_0^\infty \frac{\left[\left(x - \frac{1}{2} \right)^2 + 2 \right]}{\left[\left(x - \frac{1}{x} \right)^4 + 6 \left(x - \frac{1}{x} \right)^2 + 8 \right]} \left(1 + \frac{1}{x^2} \right) dx$$

$$\text{Let } x - \frac{1}{2} \rightarrow x: \Omega = \frac{1}{2} \int_0^\infty \frac{x^2 + 2}{x^4 + 6x^2 + 8} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\frac{\arctan(\frac{x}{2})}{2} \right]_{-\infty}^\infty = \frac{\pi}{4}$$

Hence:

$$\frac{\pi}{4} = \int_0^\infty \frac{2x^8 + x^7 + 3x^6 + x^5 - x^4 + x^3 - x^2 + x + 1}{x^{12} + x^{11} + 2x^{10} + 3x^9 + 3x^8 + 4x^7 + 4x^6 + 4x^5 + 3x^4 + 3x^3 + 2x^2 + x + 1} dx$$

808. Define for all $n \geq 1$, $f(\theta) = \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + 2 \cos(2^n \theta)}}}}$

n times radicals

then prove that

$$\begin{aligned} & \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} (\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta) \frac{f(\theta)}{4 \sin \theta} d\theta \\ &= \frac{1}{2} + \frac{11\pi}{96} - \frac{\sqrt{8 - 2\sqrt{6} - 2\sqrt{2} + \sqrt{2 - \sqrt{3}}}}{4 - \sqrt{6} - \sqrt{2}} \end{aligned}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Sergio Esteban-Argentina



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- i) *Probamos que si $A \in [0, \pi]$, se cumple que*

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + 2 \cos A}}}} = 2 \cos\left(\frac{A}{2^n}\right)$$

Demonstración:

$$\sqrt{2 + 2 \cos A} = \sqrt{2(1 + \cos A)} = \sqrt{2 \cdot 2 \cos^2 \frac{A}{2}} = 2 \left| \cos\left(\frac{A}{2}\right) \right| = 2 \cos \frac{A}{2}$$

$$\text{Lado que } 0 \leq \frac{A}{2} \leq \frac{\pi}{2} \Rightarrow \cos\left(\frac{A}{2}\right) = 0$$

$$\sqrt{2 + \sqrt{2 + 2 \cos A}} = \sqrt{2 + 2 \cos\left(\frac{A}{2}\right)} = 2 \cos \frac{A}{2^2}$$

Reemplazando lo intenso:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos A}}}} = \sqrt{2 + 2 \cos\left(\frac{A}{2^2}\right)} = 2 \cos\left(\frac{A}{2^3}\right)$$

Análogamente

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + 2 \cos A}}}}}_{\text{"n" radicales}} = 2 \cos\left(\frac{A}{2^n}\right), \forall n \in \mathbb{N}, n \geq 1$$

$$\Rightarrow \text{en el problema } A = 2^n \theta \rightarrow f(\theta) = 2 \cos \theta \quad (1)$$

- ii) *Probamos que $\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta$ (2)*

Es fácil probar que $\tan \theta - \cot \theta = -2 \cot 2\theta$

Entonces: $\underbrace{\tan \theta - \cot \theta}_{=-2 \cot 2\theta} + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta + \cot \theta$

$$\underbrace{_{=-2 \cot 2\theta} + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta + \cot \theta}_{=-4 \cot 4\theta}$$

$$= -8 \cot 8\theta + \quad +8 \cot 8\theta + \cot \theta \\ = \cot \theta$$

Nota:

$$\cot \theta - \tan \theta = \frac{1}{\tan \theta} - \tan \theta = \frac{1 - \tan^2 \theta}{\tan \theta} \cdot \frac{2}{2} = \frac{2}{\tan 2\theta} = 2 \cot 2\theta$$

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Usando (1) y (2) en la integral

$$I = \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \cot \theta \cdot \frac{1}{2} \cdot \cot \theta d\theta = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \cot^2 \theta d\theta = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} (\csc^2 \theta - 1) d\theta$$

$$I = \frac{1}{2} \left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \csc^2 \theta d\theta - \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} d\theta \right) = \frac{1}{2} \left(-\cot \theta \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{240}} - \theta \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \right)$$

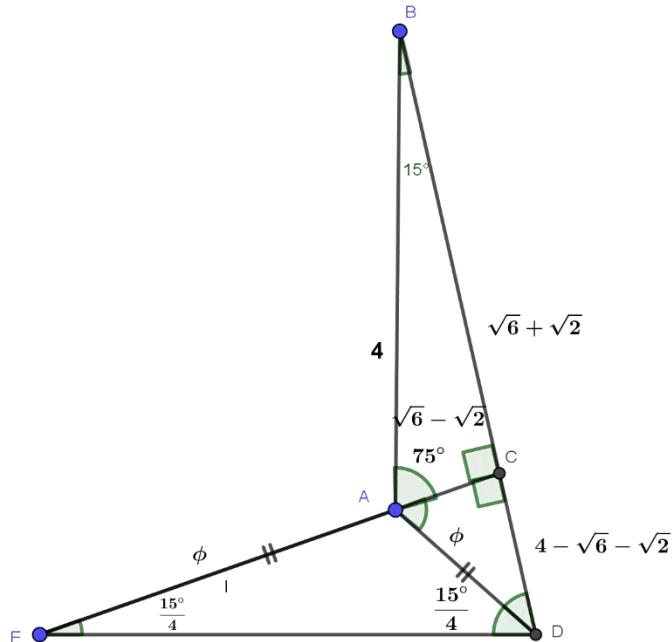
$$I = \frac{1}{2} \left(\cot \frac{\pi}{4} - \cot \left(\frac{5\pi}{240} \right) - \frac{5\pi}{240} + \frac{\pi}{4} \right)$$

$$I = \frac{1}{2} + \frac{11\pi}{96} - \frac{1}{2} \cot \left(\frac{5\pi}{240} \right)$$

Resolvemos $\cot \left(\frac{5\pi}{240} \right)$. Consideramos el triangulo notable de $75'$ y $15'$

i) $AB = BD$

ii) $EA = AD = \emptyset$



$$\phi = \sqrt{[4 - (\sqrt{6} + \sqrt{2})]^2 + (\sqrt{6} - \sqrt{2})^2}$$



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$$\theta = \sqrt{16 + (\sqrt{6} + \sqrt{2})^2 - 8(\sqrt{6} + \sqrt{2}) + (\sqrt{6} - \sqrt{2})^2} = \sqrt{32 - 8\sqrt{6} - 8\sqrt{2}}$$

y SABEMOS que $\sqrt{6} - \sqrt{2} = 2\sqrt{2 - \sqrt{3}}$ esto se deduce de $(\sqrt{6} - \sqrt{2})^2$. Entonces

$$\cot\left(\frac{5\pi}{240}\right) = \cot\left(\frac{15^\circ}{4}\right) = \frac{\emptyset + (\sqrt{6} - \sqrt{2})}{4 - \sqrt{6} - \sqrt{2}} = \frac{\sqrt{32 - 8\sqrt{6} - 8\sqrt{2}} + 2\sqrt{2 - \sqrt{3}}}{4 - \sqrt{6} - \sqrt{2}}$$

$$\text{Por lo tanto } \frac{1}{2} \cot\left(\frac{15^\circ}{4}\right) = \frac{\sqrt{8 - 2\sqrt{6} - 2\sqrt{2} + \sqrt{2 - \sqrt{3}}}}{4 - \sqrt{6} - \sqrt{2}}$$

$$\therefore I = \frac{1}{2} = \frac{11\pi}{96} - \left(\frac{\sqrt{8 - 2\sqrt{6} - 2\sqrt{2} + \sqrt{2 - \sqrt{3}}}}{4 - \sqrt{6} - \sqrt{2}} \right)$$

809. Prove that Catalan's constant:

$$G = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+n)^2} - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+n)^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)^2}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Sergio Esteban-Argentina

$$\Omega = \Omega_1 + \Omega_2; \quad \Omega_1 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m - (-1)^m (-1)^n}{(2m+n)^2}; \quad \Omega_2 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)^2}$$

We calculate Ω_1

$$\begin{aligned} \Omega_1 &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^m (-1 - (-1)^n) \int_0^1 -(\log x) (x)^{2m+n-1} dx \\ \Omega_1 &= \int_0^1 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^m (-1 - (-1)^n) x^{2n} x^n \frac{(-\log x)}{x} dx \\ &= \int_0^1 \frac{-\log(x)}{x} \sum_{m=1}^{\infty} (-x^2)^m \left[\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (-x)^n \right] dx \\ &= \int_0^1 \frac{-\log(x)}{x} \left(\frac{-x^2}{1+x^2} \right) \left[\frac{1}{1-x} - \frac{1}{1+x} \right] dx \end{aligned}$$



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$$\begin{aligned}
 &= \int_0^1 -\log(x) \frac{2x^2}{(1+x^2)(x-1)(x+1)} dx \\
 &= \int_0^1 -\log(x) \left[\frac{1}{x^2+1} - \frac{1}{2(x+1)} + \frac{1}{2(x-1)} \right] dx \\
 &= \int_0^1 \frac{-\log(x)}{x^2+1} dx + \frac{1}{2} \int_0^1 \frac{\log(x)}{x+1} dx - \frac{1}{2} \int_0^1 \frac{\log(x)}{x-1} dx
 \end{aligned}$$

Now, we calculate Ω_2

$$\begin{aligned}
 \Omega_2 &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)^2} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n \int_0^1 -\log(x) \cdot x^{m+n-1} dx \\
 &= \int_0^1 \frac{-\log(x)}{x} \sum_{m=1}^{\infty} x^m \sum_{n=0}^{\infty} (-x)^n dx = \int_0^1 \frac{-\log(x)}{x} \left(\frac{-x}{x-1} \right) \left(\frac{1}{x+1} \right) dx \\
 &= \int_0^1 \log(x) \left[\frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right] dx = \frac{1}{2} \int_0^1 \frac{\log(x)}{x-1} dx - \frac{1}{2} \int_0^1 \frac{\log(x)}{x+1} dx \\
 \Omega &= \Omega_1 + \Omega_2 = \int_0^1 \frac{-\log(x)}{x^2+1} dx = G, G \text{ is Catalan's constant};
 \end{aligned}$$

810. Prove that:

$$\Omega = \int_0^1 \text{Li}_2 \left(\frac{-x}{1-x} \right) \log(x) \frac{dx}{x} = \frac{5}{4} \zeta(4)$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Dawid Bialek-Poland

$$\Omega = \int_0^1 \text{Li}_2 \left(\frac{-x}{1-x} \right) \log(x) \frac{dx}{x} = - \int_0^1 \frac{\text{Li}_2(x)}{x} \log(x) dx - \frac{1}{2} \int_0^1 \frac{\log(x)}{x} \log^2(1-x) dx = I_1 + I_2 \dots (1)$$

$$\text{where: (I) } \text{Li}_2 \left(\frac{-x}{1-x} \right) = -\text{Li}_2(x) - \frac{1}{2} \log^2(1-x)$$



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$$\begin{aligned}
 I_1 &= -\int_0^1 \frac{Li_2(x)}{x} \log(x) dx = -\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{n-1} \log(x) dx \stackrel{I.B.P.}{=} \\
 &\quad -\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left[\frac{x^n}{n} \log(x) \right] \Big|_0^1 - \frac{1}{n} \int_0^1 x^{n-1} dx \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\frac{x^n}{n} \right] \Big|_0^1 = \sum_{n=1}^{\infty} \frac{1}{n^4} \stackrel{II}{=} \zeta(4) \dots (2) \\
 I_2 &= -\frac{1}{2} \int_0^1 \frac{\log(x)}{x} \log^2(1-x) dx \stackrel{t=1-x}{=} -\frac{1}{2} \int_0^1 \frac{\log(1-t)}{1-t} \log^2(t) dt \stackrel{(III)}{=} \frac{1}{2} \int_0^1 t^n H_n \log^2(t) dt = \\
 &\quad = \frac{1}{2} \sum_{n=1}^{\infty} H_n \int_0^1 t^n \log^2(t) dt \stackrel{I.B.P.}{=} \\
 &\quad = \frac{1}{2} \sum_{n=1}^{\infty} H_n \left(\left[\frac{t^{n+1}}{n+1} \log^2(t) \right] \Big|_0^1 - \frac{2}{n+1} \int_0^1 t^n \log(t) dt \right) \\
 &= -\sum_{n=1}^{\infty} \frac{H_n}{n+1} \int_0^1 t^n \log(t) dt \stackrel{I.B.P.}{=} -\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\left[\frac{t^{n+1}}{n+1} \log(t) \right] \Big|_0^1 - \frac{1}{n+1} \int_0^1 t^n dt \right) \\
 &= \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \left(\left[\frac{t^{n+1}}{n+1} \log(t) \right] \Big|_0^1 \right) = \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^3} \\
 &\stackrel{H_n = H_{n+1} - \frac{1}{n+1}}{\cong} \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^3} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^4} \\
 &= \sum_{n=1}^{\infty} \frac{H_n}{n^3} - 1 - \sum_{n=1}^{\infty} \frac{1}{n^4} + 1 \stackrel{(II)}{=} \frac{5}{4} \zeta(4) - \zeta(4) = \frac{1}{4} \zeta(4) \dots (3) \text{ Where:} \\
 (II) \sum_{n=1}^{\infty} \frac{H_n}{n^3} &= \frac{5}{4} \zeta(4), \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4) \\
 (III) \sum_{n=1}^{\infty} x^n H_n &= -\frac{\log(1-x)}{1-x}
 \end{aligned}$$

Rewriting (1) with (2) and (3), we get

$$\Omega = \int_0^1 Li_2 \left(\frac{-x}{1-x} \right) \log(x) \frac{dx}{x} = \frac{5}{4} \zeta(4)$$



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811. Solve for $x > 0$:

$$e^2 + \int_e^x (t^{\log t} (1 + 2 \log t)) dt = x^4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} e^2 + \int_e^x t^{\ln(t)} (1 + 2 \ln(t)) dt &= x^4 \quad (E); \text{ for } x > 0 \\ f(x) &= \int_e^x t^{\ln(t)} (1 + 2 \ln(t)) dt = \int_e^x e^{\ln^2(t)} (1 + 2 \ln(t)) dt \\ &= \int_e^x e^{\ln^2(t)+\ln(t)} (1 + 2 \ln(t)) \frac{dt}{t} = \int_1^{\ln(x)} e^{z^2+z} (2z+1) dz \\ &= e^{z^2+z} \Big|_1^{\ln(x)} = e^{\ln^2(x)+\ln(x)} - e^2 \\ \therefore (E) \Leftrightarrow e^{\ln^2(x)+\ln(x)} &= x^4 \Leftrightarrow e^{\ln^2(x)+\ln(x)} = e^{4 \ln(x)} \\ \forall t > 0: t \rightarrow e^t \text{ bijection function} \end{aligned}$$

$$So: (E) \Leftrightarrow \ln^2(x) - 3 \ln(x) = 0 \Leftrightarrow \ln(x) = 0 \vee \ln(x) = 3 \Leftrightarrow x = 1 \vee x = e^3$$

Solution 2 by Ravi Prakash-New Delhi-India

$$Let I = \int_e^x t^{\log t} (1 + 2 \log t) dt = \int_e^x t^{\log t} \left(\frac{2 \log t}{t} \right) t dt + \int_e^x t^{\log t} dt$$

$$Let t^{\log t} = y \Rightarrow \log y = (\log t)(\log t)$$

$$\frac{1}{y} \frac{dy}{dt} = \frac{2(\log t)}{t} \therefore \int t^{\log t} \frac{(2 \log t)}{t} dt = \int dy = y = t^{\log t}. Thus,$$

$$I = t \cdot t^{\log t} \Big|_e^x - \int_e^x t^{\log t} dt + \int_e^x t^{\log t} dt = x \cdot x^{\log x} - e \cdot e' = x^{\log x + 1} - e^2$$

Thus, the given equation becomes $e^2 + x^{1+\log x} - e^2 = x^4$

$$\Rightarrow x \cdot x^{\log x} = x^4 \Rightarrow x^{\log x} = x^3 \Rightarrow x = 1 \text{ or } \log x = 3 \Rightarrow x = 1 \text{ or } x = e^3$$

Solution 3 by Timson Azeez Folorunsho-Nigeria

$$e^2 + \int_e^x (t^{\ln t} (2 \ln t)) dt = x^4 \text{ put } y = \ln t; t = e^y$$



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$$\begin{aligned}
 e^2 + \int_1^{\ln x} (t^{\ln e^y} (1+2y)) e^y dy &= x^4; e^2 + \int_1^{\ln x} (t^y (1+y)) e^y dy = x^4 \\
 e^2 + \int_1^{\ln x} (e^{y^2} (1+2y) e^y dy) &= x^4; e^2 + \int_1^{\ln x} (e^{y^2+y} (1+2y) dy) = x^4 \\
 e^2 + \int_1^{\ln x} (e^{y^2+y} (1+2y) dy) &= x^4; \text{ Put } u = y^2 + y; dy = \frac{du}{2y+1} \\
 e^2 + \int_1^{\ln x} (e^u (1+2y)) \cdot \frac{du}{2y+1} &= x^4; e^2 + \int_1^{\ln x} e^u dy = x^4 \\
 e^2 + e^u|_1^{\ln x} &= x^4; e^2 + e^{y^2+y}|_1^{\ln x} = x^4; e^2 + e^{y(y+1)}|_1^{\ln x} = x^4 \\
 e^2 + x^{(\ln x+1)} - e^2 &= x^4; e^2 + x^{\ln x} + x - e^2 = x^4 \\
 x^4 - x^{\ln x} - e^2 - x + e^2 &= 0; x^4 = x^{\ln x} + x; x^4 = x^{\ln x+1} \\
 x^4 - x^{\ln x+1} &= 0; x(x^3 - x^{\ln x}) = 0; x^3 = x^{\ln x} \rightarrow x \text{ is also 1 a solution} \\
 \ln x &= 3; x = e^3
 \end{aligned}$$

812. Solve for real numbers:

$$\int_1^x \left(\frac{\log t - 1}{t^2 - \log^2 t} \right) dt = \frac{1}{2} \log \left(\frac{e-1}{e+1} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$I = \int_1^x \frac{\ln(t) - 1}{t^2 - \ln^2(t)} dt = \int_1^x \frac{\ln(t) - 1}{1 - \left(\frac{\ln(t)}{t} \right)^2} dt$$

$$\text{Put: } z = \frac{\ln(t)}{t}, \text{ so } dz = \frac{1-\ln(t)}{t^2} dt \therefore I = \int_0^{\ln(x)/x} \frac{dz}{z^2-1} = \frac{1}{2} \ln \left| \frac{\ln(x)-x}{\ln(x)+x} \right|$$

$$\text{Then, if } \int_1^x \frac{\ln(t)-1}{t^2-\ln^2(t)} dt = \frac{1}{2} \ln \left(\frac{e-1}{e+1} \right) \quad (E)$$

$$(E) \Leftrightarrow \frac{\ln(x)-x}{\ln(x)+x} = \frac{e-1}{e+1} \vee \frac{x-\ln(x)}{x+\ln(x)} = \frac{e-1}{e+1} \Leftrightarrow \ln(x) = ex \vee \ln(x) = \frac{x}{e}$$



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Put: $f(x) = \frac{\ln(x)}{x}$; $x \in]0, +\infty[$; $f'(x) = \frac{1-\ln(x)}{x^2}$. So: $f' > 0 \Rightarrow 1 > \ln(x) \Rightarrow x < e$

$$f < 0 \Rightarrow 1 - \ln(x) < 0 \Rightarrow x \geq e$$

x	0	1	0	$+\infty$
			$\frac{1}{e}$	

$$\therefore f \nearrow \text{for } x < e \text{ and } f \searrow \text{for } x \geq e. \text{ So: } \frac{\ln(x)}{x} - \frac{1}{e} \leq 0 \forall x \in]0; +\infty[$$

$y = \frac{1}{e}$ is the maximum of $x \rightarrow \frac{\ln(x)}{x}$; $f'(x) = 0 \Leftrightarrow x = e$ unique value for real numbers

$$\therefore S(E) = \{x = e\}$$

Solution 2 by Surjeet Singhania-India

$$\int_1^x \frac{\log(t) - 1}{t^2 - \log^2(t)} dt = \frac{1}{2} \log\left(\frac{e-1}{e+1}\right); \int_1^x \frac{\log(t) - 1}{t^2 - \log^2(t)} dt$$

$$\text{Put } \frac{\log(t)}{t} = y, dy = \frac{1-\log(t)}{t^2}; \int_0^{\log(x)} \frac{-dy}{1-y^2} = - \int_0^{\log x} \frac{dy}{(1+y)(1-y)}$$

$$= -\frac{1}{2} \log\left(1 + \frac{\log x}{x}\right) + \frac{1}{2} \log\left(1 - \frac{\log x}{x}\right) = \frac{1}{2} \log\left(\frac{x - \log x}{x + \log x}\right)$$

$$\text{Comparing with other side we get } \frac{x - \log x}{x + \log x} = \frac{e-1}{e+1} \Rightarrow \frac{\log x}{x} = \frac{1}{e}; x = e$$

Solution 3 by Remus Florin Stanca-Romania

$$\int \frac{\ln(t) - 1}{t^2 - \ln^2(t)} dt = \int \frac{\ln(t) - 1}{\ln^2(t) \left(\left(\frac{t}{\ln(t)} \right)^2 - 1 \right)} dt = \int \frac{\ln(t) - 1}{\ln^2(t)} \cdot \frac{1}{\left(\frac{t}{\ln(t)} \right)^2 - 1} dt$$

$$\text{Let } \frac{t}{\ln(t)} = y \Rightarrow \frac{\ln(t)-1}{\ln^2(t)} dt = dy \Rightarrow \int \frac{\ln(t)-1}{t^2-\ln^2(t)} dt = \int \frac{1}{y^2-1} dy = \frac{1}{2} \int \frac{y+1-(y-1)}{(y-1)(y+1)} dy =$$

$$= \frac{1}{2} \int \left(\frac{1}{y-1} - \frac{1}{y+1} \right) dy = \frac{1}{2} \ln\left(\left|\frac{y-1}{y+1}\right|\right) + C = \frac{1}{2} \ln\left(\left|\frac{\frac{t}{\ln(t)} - 1}{\frac{t}{\ln(t)} + 1}\right|\right)$$

As we know that $t > \ln(t) \Rightarrow$

$$\Rightarrow \int \frac{\ln(t) - 1}{t^2 - \ln^2(t)} dt = \frac{1}{2} \ln\left(\frac{t - \ln(t)}{t + \ln(t)}\right) + C \Leftrightarrow \frac{1}{2} \ln\left(\frac{x - \ln(x)}{x + \ln(x)}\right) = \frac{1}{2} \ln\left(\frac{e - 1}{e + 1}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{x - \ln(x)}{x + \ln(x)} = \frac{e - 1}{e + 1} \Leftrightarrow xe + x - e \ln(x) - \ln(x) = ex + e \ln(x) - x - \ln(x) \Leftrightarrow$$



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$$\Leftrightarrow e \ln(x) = x \Leftrightarrow \frac{\ln(x)}{x} = \frac{1}{e}$$

Let $f(x) = \frac{\ln(x)}{x} \Rightarrow f'(x) = \frac{1-\ln(x)}{x^2}$ and we have that for $x < e \Rightarrow f(x) > 0$ and for

$$x \geq e \Rightarrow f(x) \leq 0 \text{ and } f(e) = \frac{1}{e} \Rightarrow$$

\Rightarrow on $(0; e)$ the function is increasing $\Rightarrow f(x) < \frac{1}{e}$ and for $x \in [e; +\infty)$ the function is decreasing with max $= \frac{1}{e} \Rightarrow x = e$.

813. $m, n > 0, \Omega(0, \pi) = -\log 2$

Find:

$$\Omega(m, n) = \int \left(\frac{(m-n)e^{mx} \sin(nx) - (m+n)e^{mx} \cos(nx) + n}{(e^{mx} - \sin(nx))(e^{mx} - \cos(nx))} \right) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Igor Soposki-Skopje-Macedonia

$$\Omega(m, n) = \int \frac{(m-n)e^{mx} \sin nx - (m+n)e^{mx} \cdot \cos nx + n}{(e^{mx} - \sin nx)(e^{mx} - \cos nx)} dx$$

$$\frac{e^{mx} - \sin nx}{e^{mx} - \cos nx} = t \Rightarrow \frac{1}{t} = \frac{e^{mx} - \cos nx}{e^{mx} - \sin nx}$$

$$dt = \frac{(me^{mx} - n \cos nx)(e^{mx} - \cos nx) - (e^{mx} - \sin nx)(me^{mx} + n \sin nx)}{(e^{mx} - \cos nx)^2} dx$$

$$dt = \frac{me^{2mx} - me^{mx} \cdot \cos nx - ne^{mx} \cdot \cos nx + n \cos^2 nx + me^{2mx} - ne^{mx} \cdot \sin nx + me^{mx} \sin x + n \sin^2 nx}{(e^{mx} - \cos nx)^2} dx$$

$$dt = \frac{(m-n)e^{mx} \cdot \sin nx - (m+n)e^{mx} \cdot \cos nx + n(\cos^2 nx + \sin^2 nx)}{(e^{mx} - \cos nx)^2} dx$$

$$\Omega(m, n) = \int \underbrace{\frac{e^{mx} - \cos nx}{e^{mx} - \sin nx}}_{\frac{1}{t}} \cdot \underbrace{\frac{(m-n)e^{mx} \cdot \sin nx - (m+n)e^{mx} \cdot \cos nx + n}{(e^{mx} - \cos nx)^2}}_{dt} dx$$

$$\Omega(m, n) = \int \frac{dt}{t} = \ln t = \ln \left| \frac{e^{mx} - \sin nx}{e^{mx} - \cos nx} \right| + C$$

$$\Omega(0, \pi) = \ln \left| \frac{1 - \sin \pi x}{1 - \cos \pi x} \right| = -\ln 2; \ln \left| \frac{1 - \sin \pi x}{1 - \cos \pi x} \right| = \ln \frac{1}{2}$$



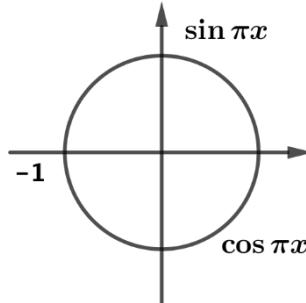
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$$1 - \cos \pi x \neq 0; \pi x \neq 2k\pi; x \neq 2k; 2(1 - \sin \pi x) = 1 - \cos \pi x$$

$$2 \sin \pi x - \cos \pi x = 1; (2 \sin \pi x - \cos \pi x)^2 + (-\sin \pi x - 2 \cos \pi x)^2 = 5$$

$$(-\sin \pi x - 2 \cos \pi x)^2 = 4$$



$$-\sin \pi x - 2 \cos \pi x = 2$$

$$\begin{cases} 2 \sin \pi x - \cos \pi x = 1 \\ -\sin \pi x - 2 \cos \pi x = 2 \end{cases} \Rightarrow \begin{cases} \sin \pi x = 0 \\ \cos \pi x = -1 \end{cases}$$

$$\tan \pi x = \frac{\sin \pi x}{\cos \pi x} = 0 \Rightarrow \pi x = (2k+1)\pi \Rightarrow x = 2k+1; k \in \mathbb{Z}$$

$$\Omega(m, n) = \ln \left| \frac{e^{(2k+1)m} - \sin(2k+1)n}{e^{(2k+1)m} - \cos(2k+1)n} \right|$$

814. Find:

$$\Omega = \int \left(\frac{1}{(2x+5)\sqrt{(x+2)(x+3)} \left(4 \tan^{-1} \left(\sqrt{\frac{x+3}{x+2}} \right) - \pi \right)^2} \right) dx, x > 0$$

Proposed by Daniel Sitaru – Romania

Solution by Igor Soposki-Skopje-Macedonia

$$\arctan \left(\sqrt{\frac{x+3}{x+2}} \right) = t; I = \int \frac{1}{(2x+5)\sqrt{(x+2)(x+3)} \left(4 \arctan \sqrt{\frac{x+3}{x+2}} - \pi \right)} dx = ?$$

$$\frac{1}{1 + \left(\sqrt{\frac{x+3}{x+2}} \right)^2} \cdot \frac{1}{2\sqrt{\frac{x+3}{x+2}}} \cdot \frac{x+2 - (x+3)}{(x+2)^2} dx = dt$$



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$$\begin{aligned}
 & -\frac{1}{1 + \frac{x+3}{x+2}} \cdot \frac{1}{2\sqrt{\frac{x+3}{x+2}}} \cdot \frac{1}{(x+2)^2} dx = dt; \quad -\frac{1}{\frac{(2x+5)}{x+2}} \cdot \frac{1}{2\sqrt{\frac{x+3}{x+2}}} \cdot \frac{1}{(x+2)^2} dx = dt \\
 & -\frac{1}{(2x+5)} \cdot \frac{1}{2\sqrt{(x+2)^2 \left(\frac{x+3}{x+2}\right)}} dx = dt; \quad -\frac{1}{(2x+5)} \cdot \frac{1}{2\sqrt{(x+2)(x+3)}} dx = dt \\
 & \frac{dx}{(2x+5)\sqrt{(x+2)(x+3)}} = -2dt \Rightarrow I = \int \frac{-2 dt}{(4t-\pi)^2} = -2 \int \frac{dt}{(4t-\pi)^2} = \\
 & \Rightarrow \begin{cases} u = 4t - \pi \\ du = 4dt \end{cases} = -2 \int \frac{1}{4} \cdot \frac{du}{u^2} = -\frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2} \int u^{-2} du = \\
 & = -\frac{1}{2} \cdot \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{2} = \frac{1}{2u} = \frac{1}{2(4t-\pi)} = \frac{1}{2 \left(4 \arctan \sqrt{\frac{x+3}{x+2}} - \pi \right)} + c
 \end{aligned}$$

815. Given that:

$$\varphi(x) = \int \left(\frac{(2-x)\sqrt{e^x}}{(1-x)\sqrt{1-x}} \right) dx$$

Prove or disprove that:

$$\omega(x) = \int \varphi(x) dx = 2i\sqrt{2\pi e} \left(\operatorname{erfi} \left(\frac{\sqrt{x-1}}{\sqrt{2}} \right) \right)$$

Proposed by Ekpo Samuel-Nigeria

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\varphi(x) = \int \frac{1+1-x}{(1-x)\sqrt{1-x}} \sqrt{e^x} dx = 2 \int \frac{\frac{\sqrt{e^x}}{2} \sqrt{1-x} + \frac{\sqrt{e^x}}{2\sqrt{1-x}}}{(\sqrt{1-x})^2} dx = 2 \frac{\sqrt{e^x}}{\sqrt{1-x}} + c$$

$$\begin{aligned}
 \omega(x) &= \int \varphi(x) dx = 2 \int \overbrace{\frac{\sqrt{e^x}}{\sqrt{1-x}} dx}^{\sqrt{1-x}=\sqrt{2}y} = -4\sqrt{2} \int e^{\frac{1}{2}-y^2} dy = -4\sqrt{e} \frac{\sqrt{\pi}}{2} \operatorname{erf}(y) + c = \\
 &= -2\sqrt{2\pi e} \operatorname{erf} \left(\frac{\sqrt{1-x}}{\sqrt{2}} \right) + c = 2i\sqrt{2\pi e} \operatorname{erfi} \left(\frac{\sqrt{x-1}}{\sqrt{2}} \right) + c
 \end{aligned}$$

Note: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $\operatorname{erf}(x) = -i \operatorname{erfi}(ix)$



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816. Prove that for: $n \geq 1$

$$\int_0^1 \frac{\log^n \left(\frac{1+x}{1-x} \right)}{\sqrt{1-x^2}} \frac{dx}{1+x} = 2^n \Gamma(n+1)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Kartick Chandra Betal-India

$$\begin{aligned} \int_0^1 \frac{\ln^n \left(\frac{1+x}{1-x} \right)}{\sqrt{1-x^2} \cdot (1+x)} dx &= (-1)^n \int_0^1 \frac{\ln^n \left(\frac{1-x}{1+x} \right)}{\sqrt{1-x^2} \cdot (1+x)} dx \\ &= (-1)^n \int_1^0 \frac{\ln^n x}{\sqrt{4x}} \cdot \frac{(1+x)^2}{2} \cdot \left(-\frac{2}{(1+x)^2} \right) dx \\ &= \frac{(-1)^n}{2} \int_0^1 \frac{\ln^n x}{\sqrt{x}} dx = \frac{(-1)^n}{2} \int_0^1 \frac{(\ln x^2)^n}{x} \cdot 2x dx \\ &= (-2)^n \int_0^1 \ln^n x dx = (-2)^n \cdot \int_{\infty}^0 (-x)^n \cdot (-e^{-x}) dx \\ &= 2^n \int_0^{\infty} x^{(n+1)-1} e^{-x} dx = 2^n \Gamma(1+n) \end{aligned}$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\begin{aligned} \underbrace{\int_0^1 \frac{\ln^n \left(\frac{1+x}{1-x} \right)}{\sqrt{1-x^2}} \frac{dx}{1+x}}_{\text{let } x=\cos 2\theta} &= \int_{\frac{\pi}{4}}^0 \frac{\ln^n \left(\frac{1+\cos 2\theta}{1-\cos 2\theta} \right)}{\sqrt{1-\cos^2 2\theta}} \cdot \frac{-2 \sin 2\theta d\theta}{1+\cos 2\theta} \\ &= \int_0^{\frac{\pi}{4}} \frac{\ln^n \left(\frac{2 \cos^2 \theta}{2 \sin^2 \theta} \right)}{\sin 2\theta} \cdot \frac{2 \sin 2\theta}{2 \cos^2 \theta} d\theta = 2^n \underbrace{\int_0^{\frac{\pi}{4}} \ln^n(\cot \theta) \cdot \sec^2 \theta d\theta}_{\text{let } y=\ln(\cot \theta) \Rightarrow \tan \theta=e^{-y}} \\ &= 2^n \int_{\infty}^0 y^n \cdot (-e^{-y}) dy = 2^n \int_0^{\infty} y^n e^{-y} dy = 2^n \Gamma(n+1) \end{aligned}$$



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817. Find:

$$\Omega(a) = \int_0^{\infty} \left(\frac{x}{(x^4 + x^2 + 1)(ax + 1)} \right) dx, a > 0$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned}
A &= \int \frac{x}{(1 + x^2 + x^4)(ax + 1)} dx = \\
&= \int \left(\frac{a(1-x)+1}{2(a^2+a+1)(x^2-x+1)} + \frac{a(1+x)-1}{2(a^2-a+1)(x^2+x+1)} - \frac{a^3}{(1+a^2+a^4)(1+ax)} \right) dx \\
&= -\frac{a^2 \log(1+ax)}{1+a^2+a^4} + \frac{M}{2(a^2+a+1)} + \frac{N}{2(a^2-a+1)} \\
M &= \int \frac{a(1-x)+1}{1-x+x^2} dx = \int \left(\frac{\frac{a}{2}+1}{1-x+x^2} - \frac{a}{2} \cdot \frac{2x-1}{1-x+x^2} \right) dx = \\
&= -\frac{a}{2} \log(1-x+x^2) + \int \frac{\frac{a}{2}+1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x-\frac{1}{2}\right) = \\
&= -\frac{a}{2} \log(1-x+x^2) + \left(\frac{a+2}{\sqrt{3}}\right) \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c \\
N &= \int \frac{a(1+x)-1}{1+x+x^2} dx = \int \left(\frac{\frac{a}{2}(2x+1)}{1+x+x^2} + \frac{\frac{a}{2}-1}{1+x+x^2} \right) dx = \\
&= \frac{a}{2} \log(1+x+x^2) + \int \frac{\frac{a}{2}-1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x+\frac{1}{2}\right) = \\
&= \frac{a}{2} \log(1+x+x^2) + \left(\frac{a-2}{\sqrt{3}}\right) \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c \\
\Omega &= \left\{ \begin{array}{l} \frac{a^2 \log(1+ax)}{1+a^2+a^4} - \frac{a}{4(a^2+a+1)} \left(\log(1-x+x^2) - \left(\frac{a+2}{\sqrt{3}}\right) \arctan\left(\frac{2x-1}{\sqrt{3}}\right) \right) + \\ + \frac{a}{4(a^2-a+1)} \left(\log(1+x+x^2) + \left(\frac{a-2}{\sqrt{3}}\right) \arctan\left(\frac{2x+1}{\sqrt{3}}\right) \right) \end{array} \right\}_0^{\infty}
\end{aligned}$$



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$$\begin{aligned}
 &= \frac{a(a+2)\pi}{8\sqrt{3}(a^2+a+1)} + \frac{a(a-2)\pi}{8\sqrt{3}(a^2-a+1)} - \frac{a^2 \log a}{1+a^2+a^4} + \\
 &\quad + \arctan\left(\frac{1}{\sqrt{3}}\right) \left(\frac{a(a+2)\pi}{8\sqrt{3}(a^2+a+1)} - \frac{a(a-2)\pi}{8\sqrt{3}(a^2-a+1)} \right)
 \end{aligned}$$

818. Show that:

$$\int_0^\pi e^{e^{\cos(x)} \cos(\sin(x))} \cos(\sin(\sin(x))) e^{\cos(x)} dx = e\pi$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned}
 M &= \int_0^\pi e^{e^{\cos x} \cos(\sin x)} \cos(\sin(\sin x)) e^{\cos x} dx = \operatorname{Re} \int_0^\pi e^{e^{\cos x} \cos(\sin x) + i \sin(\sin x) e^{\cos x}} dx = \\
 &= \operatorname{Re} \int_0^\pi e^{e^{\cos x} e^{\sin x}} dx = \operatorname{Re} \int_0^\pi e^{e^{\cos x + i \sin x}} dx \\
 &= \operatorname{Re} \int_0^\pi e^{e^{ix}} dx \text{ now let: } z = e^{ix} \Rightarrow dx = -\frac{idz}{z}
 \end{aligned}$$

$$\text{So: } M = -i \int_{|z|=1 \atop Imz \geq 0} \frac{e^{e^2}}{z} dz = -i \left(\pi i \operatorname{Res} \left(\frac{e^{e^2}}{z}, z = 0 \right) \right) = \pi \lim_{z \rightarrow 0} \left(\frac{(z-0)e^{e^2}}{z} \right) = e\pi$$

819. Find:

$$\Omega = \int \left(\frac{\operatorname{erf}(x)}{\sqrt{x}} \right) dx$$

Proposed by Ekpo Samuel-Nigeria

Solution 1 by Abdul Hafeez Ayinde-Nigeria

$$\begin{aligned}
 \Omega &= \int \frac{\operatorname{erf}(x)}{\sqrt{x}} dx; \quad \Omega = \int \operatorname{erf}(x) d(2\sqrt{x}); \quad \Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{4}{\sqrt{\pi}} \int \sqrt{x} e^{-x^2} dx \\
 \Omega &= 2\sqrt{x} \operatorname{erf}(x) - \frac{4}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int \sqrt{x} \cdot x^{2k} dx
 \end{aligned}$$



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$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{4}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int x^{2k+\frac{1}{2}} dx$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{4x\sqrt{x}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k! \left(2k + \frac{3}{2}\right)} + C$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{2x\sqrt{x}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k! \left(k + \frac{3}{4}\right)} + C$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{2x\sqrt{x}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k} \Gamma\left(k + \frac{3}{4}\right)}{k! \Gamma\left(k + \frac{7}{4}\right)} + C$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{2x\sqrt{x}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k} \left(\frac{3}{4}\right)_k}{k! \left(\frac{7}{4}\right)_k} + C$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{8x\sqrt{x}}{3\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k} \left(\frac{3}{4}\right)_k}{k! \left(\frac{7}{4}\right)_k} + C$$

$$\Omega = 2\sqrt{x} \operatorname{erf}(x) - \frac{8x\sqrt{x}}{3\sqrt{\pi}} {}_1F_1\left(\frac{3}{4}, \frac{7}{4}; -x^2\right) + C$$

Where ${}_1F_1(a, b; c; z)$ is the classic hypergeometric function.

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned} \Omega &= \overbrace{\int \frac{\operatorname{erf}(x)}{\sqrt{x}} dx}^{IBP} = \{2\sqrt{x} \operatorname{erf}(x)\} - \frac{4}{\sqrt{\pi}} \overbrace{\int \sqrt{x} e^{-x^2} dx}^{x^2=y} = \\ &= 2\sqrt{x} \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \int \frac{e^{-y}}{\sqrt[4]{y}} dy = 2\sqrt{x} \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{4}, y\right) = 2\sqrt{x} \operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{4}, x^2\right) \end{aligned}$$

820. Find a closed form:

$$\Omega = \int_0^1 \left(\frac{x(1-x^2)\sqrt{x}}{\sqrt[4]{2-x^2}} \right) dx$$

Proposed by Ekpo Samuel-Nigeria



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Solution 1 by Kamel Benaicha-Algiers-Algerie

$$I = \int_0^1 \frac{x(1-x^2)\sqrt{x}}{\sqrt[4]{2-x^2}} dx. \text{ Let } t = x^2, \text{ then: } I = \frac{1}{2} \int_0^1 \frac{(1-t)t^{\frac{1}{4}}}{(2-t)^{\frac{1}{4}}} dt = \frac{1}{2} \int_0^1 \frac{t(1-t)^{\frac{1}{4}}}{(1+t)^{\frac{1}{4}}} dt$$

$$z = \frac{t}{2-t} \Rightarrow t = \frac{2z}{1+z} \Rightarrow dt = \frac{2dz}{(1+z)^2} \therefore I = \int_0^1 \frac{z^{\frac{1}{4}}(1-z)}{(1+z)^3} dz$$

Integration by parts gives:

$$\begin{aligned} &= \frac{1}{8} \int_0^1 \frac{z^{-\frac{3}{4}} - 5z^{\frac{1}{4}}}{(1+z)^2} dz = \frac{1}{8} \int_1^{+\infty} \frac{t^{-\frac{3}{4}} - 5t^{\frac{1}{4}}}{(1+t)^2} dz \left(z \rightarrow \frac{1}{t} \right) \\ &= -\frac{1}{4} + \frac{1}{8} \int_0^{+\infty} \frac{3t^{-\frac{1}{4}} + 5t^{-\frac{5}{4}}}{1+t} dt = -\frac{1}{4} + \frac{1}{8} \int_0^1 \frac{3x^{-\frac{3}{4}} + 5x^{\frac{1}{4}}}{1+x} dx \left(x \rightarrow \frac{1}{t} \right) \\ &= -\frac{1}{4} + \frac{1}{8} \int_0^1 \frac{\left(3x^{-\frac{3}{4}} + 5x^{\frac{1}{4}}\right)(1-x)}{1-x^2} dx = -\frac{1}{4} + \frac{1}{8} \int_0^1 \frac{3x^{-\frac{3}{4}} - 3x^{\frac{1}{4}} + 5x^{\frac{1}{4}} - 5x^{\frac{5}{4}}}{1-x^2} dx \\ &= -\frac{1}{4} + \frac{1}{64} \int_0^1 \frac{3x^{-\frac{7}{8}} + 2x^{-\frac{3}{8}} - 5x^{\frac{1}{8}}}{1-x} dx \\ &= -\frac{1}{4} + \frac{1}{64} \left(3 \int_0^1 \frac{x^{\frac{1}{8}} - x^{-\frac{7}{8}}}{x-1} dx + 2 \int_0^1 \frac{x^{\frac{1}{8}} - x^{-\frac{3}{8}}}{x-1} dx \right) \\ &= \frac{5}{64} \Psi\left(\frac{9}{8}\right) - \frac{2}{64} \Psi\left(\frac{5}{8}\right) - \frac{3}{64} \Psi\left(\frac{1}{8}\right) - \frac{1}{4} = \frac{1}{32} \left(\Psi\left(\frac{1}{8}\right) - \Psi\left(\frac{5}{8}\right) \right) + \frac{3}{8} \end{aligned}$$

By using Gauss theorem:

$$\Psi\left(\frac{1}{8}\right) - \Psi\left(\frac{5}{8}\right) = \sqrt{2} \ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) - \frac{\pi}{2} \left(\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} \right)$$

$$\Psi\left(\frac{1}{8}\right) - \Psi\left(\frac{5}{8}\right) = 2\sqrt{2} \ln(\sqrt{2}-1) - \frac{\pi}{2} \left(\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} \right)$$



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$$\therefore I = \frac{\sqrt{2}}{16} \ln(\sqrt{2}-1) - \frac{\pi}{64} \left(\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} \right) + \frac{3}{8} = \frac{3}{8} - \frac{\pi}{16\sqrt{2}} + \frac{\ln(\sqrt{2}-1)}{8\sqrt{2}}$$

$$So: \int_0^1 \frac{x(1-x^2)\sqrt{x}}{\sqrt[4]{2-x^2}} dx = \frac{3}{8} - \frac{\pi}{16\sqrt{2}} + \frac{\ln(\sqrt{2}-1)}{8\sqrt{2}}$$

Solution 2 by Kartick Chandra Betal-India

$$\begin{aligned}
\int_0^1 \frac{x(1-x^2)\sqrt{x}}{\sqrt[4]{2-x^2}} dx &= \frac{1}{2} \int_0^1 \frac{(1-x)x^{\frac{1}{4}}}{\sqrt[4]{2-x}} dx = \frac{1}{2} \int_0^1 \frac{x(1-x)^{\frac{1}{4}}}{(1+x)^{\frac{1}{4}}} dx = \frac{1}{2} \int_0^1 x \left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} dx \\
&= \frac{1}{2} \int_1^0 \left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} \cdot \left\{ \frac{-2dx}{(1+x)^2} \right\} = \int_0^1 \frac{x^{\frac{1}{4}}(1-x)}{(1+x)^3} dx \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (-1)^{n-1} \int_0^1 \left(x^{n+\frac{1}{4}-1} - x^{n+\frac{1}{4}} \right) dx \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} (-1)^{n-1} \left[\frac{1}{n+\frac{1}{4}} - \frac{1}{n+\frac{1}{4}+1} \right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} n(n+1) \cdot \frac{1}{(n+\frac{1}{4})(n+\frac{1}{4}+1)} \\
&= 8 \sum_{n=1}^{\infty} (-1)^{n-1} (n^2+n) \cdot \frac{1}{(4n+1)^2 + 4(4n+1)} \\
&= 8 \sum_{n=1}^{\infty} (-1)^{n-1} (n^2+n) \cdot \frac{1}{16n^2 + 24n + 5} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{16n^2 + 24n + 5 - 8n - 5}{16n^2 + 24n + 5} \right\} = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{8n+5}{16n^2 + 24n + 5} \\
&= \frac{1}{4} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{8n+5}{16n^2 + 20n + 4n + 5} = \frac{1}{4} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{8n+5}{(4n+5)(4n+1)} \\
&= \frac{1}{4} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{\frac{5}{4}}{4n+5} + \frac{\frac{3}{4}}{4n+1} \right\} = \frac{1}{4} + \frac{5}{8} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{4n+1} - \frac{3}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n+1}
\end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{4} + \frac{1}{4} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{4n+1} - \frac{3}{8 \times 5} = \frac{1}{4} - \frac{3}{40} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n+1} - \frac{1}{20} \\
 &= \frac{1}{4} - \frac{1}{8} + \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 x^{4n} dx = \frac{1}{8} + \frac{1}{4} \int_0^1 x^4 \sum_{n=1}^{\infty} (-x^4)^{n-1} dx \\
 &= \frac{1}{8} + \frac{1}{4} \int_0^1 \frac{x^4 dx}{(1+x^4)} = \frac{1}{8} + \frac{1}{4} \cdot 1 - \frac{1}{4} \int_0^1 \frac{dx}{1+x^4} \\
 &= \frac{3}{8} - \frac{1}{8} \int_0^1 \frac{x^2}{x^2 + \frac{1}{x^2}} dx = \frac{3}{8} - \frac{1}{8} \int_0^1 \left\{ \frac{1 + \frac{1}{x^2} - \left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} \right\} dx \\
 &= \frac{3}{8} - \frac{1}{8} \int_0^1 \frac{\frac{\partial}{\partial x} \left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} + \frac{1}{8} \int_0^1 \frac{d \left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - (\sqrt{2})^2} \\
 &= \frac{3}{8} - \frac{1}{8} \cdot \frac{1}{\sqrt{2}} \cdot \left\{ 0 + \frac{\pi}{2} \right\} + \frac{1}{8} \cdot \frac{1}{2\sqrt{2}} \left[\ln \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| \right]_0^1 \\
 &= \frac{3}{8} - \frac{\pi}{16\sqrt{2}} + \frac{1}{16\sqrt{2}} \ln \left| \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right| = \frac{3}{8} - \frac{\pi}{16\sqrt{2}} + \frac{1}{8\sqrt{2}} \ln(\sqrt{2} - 1)
 \end{aligned}$$

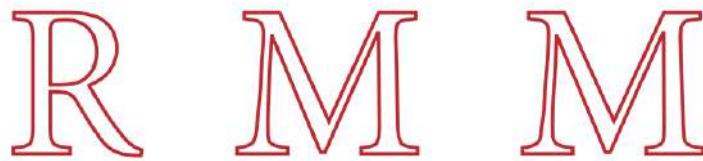
821. Find without softs:

$$\Omega = \int_0^1 \int_0^x \frac{xy^y}{y^y + (x-y)^{x-y}} dy dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\begin{aligned}
 \Omega &= \int_0^x \left(\frac{y^y}{y^y + (x-y)^{x-y}} \right) dy \stackrel{x-y=t}{=} - \int_x^0 \left(\frac{(x-t)^{x-t}}{(x-t)^{x-t} + t^t} \right) dt = \\
 &= \int_0^x \left(\frac{(x-t)^{x-t}}{(x-t)^{x-t} + t^t} \right) dt = \int_0^x \left(\frac{(x-y)^{x-y}}{(x-y)^{x-y} + y^y} \right) dt
 \end{aligned}$$



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$$2\Omega = \int_0^x \left(\frac{y^y + (x-y)^{x-y}}{(x-y)^{x-y} + y^y} \right) dy = \int_0^x dy \Rightarrow \Omega = \frac{x}{2}$$

$$\int_0^1 \left(\int_0^x \left(\frac{y^y}{y^y + (x-y)^{x-y}} \right) dy \right) dx = \int_0^1 \left(x \cdot \frac{x}{2} \right) dx = \frac{1}{6}$$

822. For $n \geq 1$, we have:

$$\int_{-\pi}^{\pi} \frac{x^{2n}}{1 + \sin(x) + \sqrt{\sin^2(x) + 1}} dx = \frac{\pi^{2n+1}}{2n+1}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Florentin Vișescu-Romania

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \frac{x^{2n}}{1 + \sin x + \sqrt{\sin^2 x + 1}} dx; \quad y = -x \\ I &= \int_{-\pi}^{\pi} \frac{y^{2n}}{1 - \sin y + \sqrt{\sin^2 y + 1}} (-dy) = \int_{-\pi}^{\pi} \frac{y^{2n}}{1 - \sin y + \sqrt{\sin^2 y + 1}} dy \\ &= \int_{-\pi}^{\pi} \frac{x^{2n}}{1 - \sin x + \sqrt{\sin^2 x + 1}} dx \\ 2I &= \int_{-\pi}^{\pi} \frac{x^{2n}}{1 + \sin x + \sqrt{\sin^2 x + 1}} + \frac{x^{2n}}{1 - \sin x + \sqrt{\sin^2 x + 1}} dx \\ &= \int_{-\pi}^{\pi} \frac{x^{2n}(2 + 2\sqrt{\sin^2 x + 1})}{(1 + \sqrt{\sin^2 x + 1})^2 - \sin^2 x} dx = \int_{-\pi}^{\pi} \frac{x^{2n} \cdot 2(1 + \sqrt{\sin^2 x + 1})}{1 + \sin^2 x + 1 + 2\sqrt{\sin^2 x + 1} - \sin^2 x} dx \\ &= \int_{-\pi}^{\pi} x^{2n} dx = \frac{x^{2n+1}}{2n+1} \Big|_{-\pi}^{\pi} = \frac{\pi^{2n+1}}{2n+1} + \frac{\pi^{2n+1}}{2n+1}; \quad I = \frac{\pi^{2n+1}}{2n+1} \end{aligned}$$

Solution 2 by Tobi Joshua-Nigeria

$$I = \int_{-\pi}^{\pi} \frac{x^{2n}}{1 + \sin x + \sqrt{\sin^2 x + 1}} dx; \quad n \geq 1$$



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$$I = \int_{-\pi}^{\pi} \frac{x^{2n}(1 + \sin x - \sqrt{\sin^2 x + 1})}{(1 + \sin x + \sqrt{\sin^2 x + 1})(1 + \sin x - \sqrt{\sin^2 x + 1})} dx$$

$$I = \int_{-\pi}^{\pi} \frac{x^{2n}(1 + \sin x - \sqrt{\sin^2 x + 1})}{2 \sin x} dx$$

$$I = \int_{-\pi}^{\pi} \frac{x^{2n}}{2 \sin x} dx + \int_{-\pi}^{\pi} \frac{x^{2n}}{2} dx - \int_{-\pi}^{\pi} x^{2n} \frac{\sqrt{1 + \sin^2 x}}{2 \sin x} dx$$

$$I = 0 \text{ (odd ffn)} + \int_{-\pi}^{\pi} \frac{x^{2n}}{2} dx - 0 \text{ (odd ffn)}$$

$$I = \int_0^{\pi} x^{2n} dx \text{ (even function); } I = \left[\frac{x^{2n}}{2n+1} \right]_0^{\pi} = \frac{\pi^{2n+1}}{2n+1}$$

823. Prove that:

$$\int_0^1 \int_0^1 \frac{((1+x)\log(x) - (1+y)\log(y))}{x-y} (1 + \log(xy)) dy dx = 3 - \frac{\pi^2}{3} - 6\zeta(3)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 I &= \int_0^1 \int_0^1 \frac{(1+x)\ln(x) - (1+y)\ln(y)}{x-y} (1 + \ln(xy)) dy dx \\
 I &= \int_0^1 \int_0^{\frac{1}{x}} \frac{(1+x)\ln(x) - (1+xt)(\ln(x) + \ln(t))}{1-t} (1 + 2\ln(x) + \ln(t)) dt dx \\
 &\quad = \int_0^1 \int_0^{\frac{1}{x}} x \ln(x) (1 + 2\ln(x) + \ln(t)) dt dx - \\
 &\quad - \int_0^1 \int_0^{\frac{1}{x}} \frac{(1+xt)\ln(t)}{(1-t)} (1 + 2\ln(x) + \ln(t)) dt dx \\
 &= \int_0^1 x \ln(x) \left(\frac{1 + 2\ln(x) - \ln(x) - 1}{x} \right) dx - \int_0^1 \int_0^1 \frac{(1+xt)\ln(t)}{(1-t)} (1 + 2\ln(x) \ln(t)) dx dt +
 \end{aligned}$$



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$$-\int_0^{+\infty} \int_0^{\frac{1}{t}} \frac{(1+xt)\ln(t)}{(1-t)}(1+2\ln(x)+\ln(t))dxdt$$

$$I_1 = \int_0^1 x \ln(x) \left(\frac{1+2\ln(x)-\ln(x)-1}{x} \right) dx -$$

$$-\int_0^1 \int_0^1 \frac{(1+xt)\ln(t)}{(1-t)}(1+2\ln(x)+\ln(t))dxdt +$$

$$I_2 = -\int_0^{+\infty} \int_0^{\frac{1}{t}} \frac{(1+xt)\ln(t)}{(1-t)}(1+2\ln(x)+\ln(t))dxdt$$

$$I = I_1 + I_2 \quad (1)$$

$$\begin{aligned} D &= \left\{ (x, t) / 0 < x < 1, 0 < t < \frac{1}{x} \right\} = \\ &= \left\{ (t, x) / 0 < t < 1, 0 < x < 1 \right\} \cup \left\{ (t, x) / 1 < t < +\infty, 0 < x < \frac{1}{t} \right\} \end{aligned}$$

$$I_1 = \int_0^1 \ln^2(x) dx - \int_0^1 \frac{\ln(t)}{1-t} [x + 2x\ln(x) - 2x + x\ln(t)]_0^1 dt -$$

$$-\int_0^1 \frac{t\ln(t)}{1-t} \left[\frac{1}{2}x^2 + x^2\ln(x) - \frac{1}{2}x^2 + \frac{1}{2}x^2\ln(t) \right]_0^1 dt$$

$$= \int_0^1 \ln^2(t) dt + \int_0^1 \frac{\ln(t)}{1-t} dt - \int_0^1 \frac{\ln^2(t)}{1-t} dt - \frac{1}{2} \int_0^1 \frac{t\ln^2(t)}{1-t} dt$$

$$= \frac{3}{2} \int_0^1 \ln^2(t) dt + \int_0^1 \frac{\ln(t)}{1-t} dt - \frac{3}{2} \int_0^1 \frac{\ln^2(t)}{1-t} dt = 3 - \frac{\pi^2}{6} - 3\zeta(3) \quad (2)$$

$$I_2 = -\int_0^{+\infty} \int_0^{\frac{1}{t}} \frac{(1+xt)\ln(t)}{(1-t)}(1+2\ln(x)+\ln(t)) dx dt$$

$$I_2 = -\int_0^{+\infty} \int_0^1 \frac{\ln(t)}{(1-t)}(1+2\ln(x)+\ln(t)) dx dt -$$



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$$\begin{aligned}
 & - \int_1^{+\infty} \frac{t \ln(t)}{1-t} \int_0^{\frac{1}{t}} (2x \ln(x) + x(1 + \ln(t))) dx dt \\
 I_2 &= - \left[\int_0^{+\infty} \frac{\ln(t)}{(1-t)} (x + 2(x \ln(x) - x) + x \ln(t)) \right]_0^{\frac{1}{t}} dt - \\
 & - \int_0^{+\infty} \frac{t \ln(t)}{1-t} \left[x^2 \ln(x) - \frac{1}{2} x^2 + \frac{1}{2} x^2 (1 + \ln(t)) \right]_0^{\frac{1}{t}} dt \\
 &= - \int_0^{+\infty} \left(-\frac{\ln(t)}{t(1-t)} - \frac{\ln^2(t)}{t(1-t)} dt + \frac{1}{2} \int_0^{+\infty} \frac{\ln^2(t)}{t(1-t)} dt \right) \\
 &= \int_0^{+\infty} \frac{\ln(t)}{t(1-t)} dt + \frac{3}{2} \int_0^{+\infty} \frac{\ln^2(t)}{t(1-t)} dt \quad (t \rightarrow \frac{1}{t}) = \int_0^1 \frac{\ln(t)}{1-t} dt - \frac{3}{2} \int_0^1 \frac{\ln^2(t)}{1-t} dt \\
 &= -\frac{\pi^2}{6} - 3\zeta(3) \quad (3) \\
 (1), (2) \text{ and } (3) &\Rightarrow I = 3 - \frac{\pi^2}{3} - 6\zeta(3)
 \end{aligned}$$

$$\therefore \int_0^1 \int_0^1 \frac{(1+x) \ln(x) - (1+y) \ln(y)}{x-y} (1 + \ln(xy)) dy dx = 3 - \frac{\pi^2}{3} - 6\zeta(3)$$

824. Evaluate the integral \aleph

$$\text{If } \aleph = \int_0^\infty \int_0^\infty \frac{y^{\frac{1}{2}} e^{-4y} \left(e^{-(x+\frac{9}{y})} - e^{-(xz+\frac{9}{y})} \right)}{x} dx dy$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution by Pedro Nagasava-Brazil

1. Separating variables:

$$\aleph = \int_0^\infty y^{\frac{1}{2}} e^{-(4y+\frac{9}{y})} dy \int_0^\infty \frac{e^{-x} - e^{-xz}}{x} dx$$

2. For the x variable, let's apply Frullani integral:



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$$N_1 = \int_0^\infty \frac{e^{-x} - e^{-xz}}{x} dx = \left(e^0 - \lim_{n \rightarrow \infty} e^{-n} \right) \log\left(\frac{z}{1}\right) = \log(z)$$

3. For the y variable, let $y = t^2$:

$$N_2 = 2 \int_0^\infty t^2 e^{-(4t^2 + \frac{9}{t^2})} dt$$

Now, let's apply Differentiation Under the Integral Sign:

$$N_2(a, b) = \int_0^\infty e^{-(a^2 t^2 + \frac{b^2}{t^2})} dt \rightarrow a N_2(a, b) = e^{-2ab} \int_0^\infty a e^{-(at - \frac{b}{t})^2} dt \quad (1)$$

$$\text{Let } t \rightarrow \frac{b}{at};$$

$$N_2(a, b) = \frac{be^{-2ab}}{a} \int_0^\infty \frac{1}{t^2} e^{-(\frac{b}{t} - at)^2} dt \rightarrow a N_2(a, b) = e^{-2ab} \int_0^\infty \frac{b}{t^2} e^{-(at - \frac{b}{t})^2} dt \quad (2)$$

With (1)+(2):

$$2aN_2(a, b) = e^{-2ab} \int_0^\infty \left(a + \frac{b}{t^2} \right) e^{-(at - \frac{b}{t})^2} dt = e^{-2ab} \sqrt{\pi} \rightarrow N_2(a, b) = \frac{e^{-2ab} \sqrt{\pi}}{2a}$$

$$\frac{\partial N_2(a, b)}{\partial a} = -2a \int_0^\infty t^2 e^{-(a^2 t^2 + \frac{b^2}{t^2})} dt = \sqrt{\frac{\pi}{2}} \left[-\frac{2be^{-2ab}}{a} - \frac{e^{-2ab}}{a^2} \right]$$

$$2 \int_0^\infty t^2 e^{-(a^2 t^2 + \frac{b^2}{t^2})} dt = \frac{\sqrt{\pi}}{2} \left[\frac{2be^{-2ab}}{a^2} + \frac{e^{-2ab}}{a^3} \right]$$

$$\text{Hence: } N_2 = 2 \int_0^\infty t^2 e^{-(4t^2 + \frac{9}{t^2})} dt = \frac{\sqrt{\pi}}{2} \left[\frac{6e^{-12}}{4} + \frac{e^{-12}}{8} \right] = \frac{13\sqrt{\pi}}{16e^{12}}$$

Finally:

$$N = \int_0^\infty \int_0^\infty \frac{y^{\frac{1}{2}} e^{-4y} \left(e^{-(x+\frac{9}{y})} - e^{-(xz+\frac{9}{y})} \right)}{x} dx dy = \frac{13\sqrt{\pi}}{16e^{12}} \log(z)$$

825. If $a, b > 0$, then $\exists c \in [0, \frac{\pi}{4}]$ such that:



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$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2\cos^2 x + b^2\sin^2 x)} = \frac{1}{ab(\pi+4)} \left(\pi \tan^{-1} \frac{btanc}{a} + 4 \tan^{-1} \frac{b}{a} \right)$$

Proposed by Florică Anastase-Romania

Solution 1 by George Florin Șerban-Romania

From Osiann-Bonet formula we have:

f continuous and decreasing on [a, b], g monotone on [a, b], then $\exists c \in [a, b]$ with:

$$\int_a^b f(x)g(x)dx = f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx$$

$$\text{Let } \tan x = t \rightarrow x = \tan^{-1} t, dx = \frac{1}{t^2 + 1} dt \rightarrow$$

$$I = \int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2\cos^2 x + b^2\sin^2 x)} = \int_0^1 \frac{dt}{(1+\tan^{-1} t)(a^2 + b^2 t^2)}$$

Let $f: [0, 1] \rightarrow R, g(t) = \frac{1}{1+tan^{-1}t}$ derivable with

$$f'(t) = \frac{-1}{(1+\tan^{-1}t)^2} \cdot \frac{1}{1+t^2} < 0 \rightarrow f \text{ decreasing on } [0, 1], \text{ and}$$

$$g: [0, 1] \rightarrow R, g(t) = \frac{1}{a^2 + b^2 t^2} \text{ continue} \rightarrow \exists c_1 \in [0, 1] \text{ then:}$$

$$I = f(0) \int_0^{c_1} g(t)dt + f(1) \int_{c_1}^1 g(t)dt$$

$$\int f(t)dt = \int \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int \frac{dt}{t^2 + \left(\frac{a}{b}\right)^2} = \frac{1}{ab} \tan^{-1} \left(\frac{tb}{a} \right) + C$$



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$$\begin{aligned}
 I &= 1 \cdot \int_0^{c_1} g(t) dt + \frac{1}{\pi/4 + 1} \cdot \int_{c_1}^1 g(t) dt = \\
 &= \frac{1}{ab} \tan^{-1} \frac{bt \operatorname{anc}}{a} + \frac{1}{\pi/4 + 1} \cdot \frac{1}{ab} \left(\tan^{-1} \frac{b}{a} - \tan^{-1} \frac{bt \operatorname{anc}}{a} \right) \\
 &= \frac{1}{ab(\pi/4 + 1)} \left(\pi \tan^{-1} \frac{bt \operatorname{anc}}{a} + 4 \tan^{-1} \frac{b}{a} \right)
 \end{aligned}$$

Solution 2 by proposer

Theorem (Bonnet-Weierstrass):

If $f: [a, b] \rightarrow \mathbb{R}$ decreasing function of C^1 class and $g: [a, b] \rightarrow \mathbb{R}$ continuous function,

then $\exists c \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx$$

Demonstration:

Let $h: [a, b] \rightarrow \mathbb{R}$, $h(x) = f(x) - f(b)$ decreasing and $h(x) \geq 0, \forall x \in [a, b]$.

From theorem 2 of means $\exists c \in [a, b]$ such that:

$$\begin{aligned}
 \int_a^b g(x)h(x)dx &= h(a) \int_a^c g(x)dx \\
 \int_a^b g(x)(f(x) - f(b))dx &= (f(a) - f(b)) \int_a^c g(x)dx \\
 \int_a^b f(x)g(x)dx &= f(b) \int_a^b g(x)dx + (f(b) - f(a)) \int_a^c g(x)dx = f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx
 \end{aligned}$$

q.e.d. Let $f, g: [0, \frac{\pi}{4}] \rightarrow \mathbb{R}$, $g(x) = \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$, $f(x) = \frac{1}{x+1}$,

$$f'(x) = -\frac{1}{(x+1)^2} < 0 \text{ then } f \text{ is decreasing}$$

$$G(x) = \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int \frac{1}{a^2 + b^2 \tan^2 x} \cdot \frac{dx}{\cos^2 x} = \frac{1}{b^2} \int \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{1}{ab} \tan^{-1} \frac{bt \operatorname{anc}}{a} + C$$

Then $\exists c \in [0, \frac{\pi}{4}]$ for which:



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$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2 \cos^2 x + b^2 \sin^2 x)} &= f(\mathbf{0})(G(c) - G(\mathbf{0})) + f\left(\frac{\pi}{4}\right)(G(b) - G(c)) \\
 &= \frac{1}{ab} \tan^{-1} \frac{btanc}{a} + \frac{1}{\frac{\pi}{4} + 1} \cdot \frac{1}{ab} \left(\tan^{-1} \frac{b}{a} - \tan^{-1} \frac{btanc}{a} \right) \\
 &= \frac{1}{ab(\pi + 4)} \left(\pi \tan^{-1} \frac{btanc}{a} + 4 \tan^{-1} \frac{b}{a} \right)
 \end{aligned}$$

826. Find:

$$\int_0^{\frac{\pi}{4}} x^4 \log(\sin(2x)) dx$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Avishek Mitra-West Bengal-India

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{4}} x^4 \log(\sin 2x) dx = \int_0^{\frac{\pi}{4}} x^4 \left[-\log 2 - \sum_n^{\infty} \frac{\cos(4nx)}{n} \right] dx \\
 &= -\log \int_0^{\frac{\pi}{4}} x^4 dx - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{4}} x^4 \cos(4nx) dx \\
 I_1 &= \int_0^{\frac{\pi}{4}} x^4 \cos(4nx) dx = \left[x^4 \cdot \frac{\sin(4nx)}{4n} \right]_0^{\frac{\pi}{4}} - 4 \int_0^{\frac{\pi}{4}} x^3 \frac{\sin(4nx)}{4n} dx \\
 &= -\frac{1}{n} \int_0^{\frac{\pi}{4}} x^3 \sin(4nx) dx \\
 &= -\frac{1}{n} \left[\left\{ -x^3 \cdot \frac{\cos(4nx)}{4n} \right\}_0^{\frac{\pi}{4}} + 3 \int_0^{\frac{\pi}{4}} x^2 \cdot \frac{\cos(4nx)}{4n} dx \right]
 \end{aligned}$$



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$$\begin{aligned}
 &= -\frac{1}{n} \left[-\frac{\pi^3}{64} \cdot \frac{\cos(nx)}{4n} + \frac{3}{4n} \int_0^{\frac{\pi}{4}} x^2 \cos(4nx) dx \right] \\
 &= \frac{\pi^3(-1)^n}{256n^2} - \frac{3}{4n^2} \left[\left\{ x^2 \cdot \frac{\sin(4nx)}{4n} \right\}_0^{\frac{\pi}{4}} - 2 \int_0^{\frac{\pi}{4}} x \cdot \frac{\sin(4nx)}{4n} dx \right] \\
 &= \frac{\pi^3(-1)^n}{256 \cdot n^2} - \frac{3}{4n^2} \left[-\frac{1}{2n} \int_0^{\frac{\pi}{4}} x \sin(4nx) dx \right] \\
 &= \frac{\pi^3(-1)^n}{256 \cdot n^2} + \frac{3}{8n^3} \left[\left\{ -x \cdot \frac{\cos(4nx)}{4n} \right\}_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{\cos(4nx)}{4n} dx \right] \\
 &= \frac{\pi^3}{256} \cdot \frac{(-1)^n}{n^2} + \frac{3}{8n^3} \left[-\frac{\pi}{4} \cdot \frac{(-1)^n}{4n} + \frac{1}{26n^2} \cdot \{\sin(4nx)\}_0^{\frac{\pi}{4}} \right] \\
 &\quad = -\frac{\pi^3}{256} \cdot \frac{(-1)^{n-1}}{n^2} + \frac{3\pi}{128} \cdot \frac{(-1)^{n-1}}{n^4} \\
 I &= -\log 2 \cdot \left[\frac{x^5}{5} \right]_0^{\frac{\pi}{4}} + \frac{\pi^3}{256} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} - \frac{3\pi}{128} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5} \\
 &= -\log 2 \cdot \frac{\pi^5}{5 \cdot 4^5} + \frac{\pi^3}{256} \eta(3) - \frac{3\pi}{128} \eta(5) \\
 &= -\log 2 \cdot \frac{\pi^5}{5 \times 1024} + \frac{\pi^3}{256} (1 - 2^{1-3}) \zeta(3) - \frac{3\pi}{128} \cdot (1 - 2^{1-5}) \zeta(5) \\
 &= -\log 2 \cdot \frac{\pi^5}{5120} + \frac{\pi^3}{256} \cdot \frac{3}{4} \zeta(3) - \frac{3\pi}{128} \cdot \frac{15}{16} \zeta(5) \\
 &= -\log 2 \cdot \frac{\pi^5}{5120} + \frac{3\pi^3}{1024} \zeta(3) - \frac{45\pi}{2048} \zeta(5)
 \end{aligned}$$

827. Evaluate the following integral:



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$$I = \int_0^1 \frac{\ln^2(1-x^2) \ln^2(x)}{(1-x^2)} \cdot \frac{dx}{x}$$

Proposed by Ahmed Hegazi-Cairo-Egypt

Solution by Avishek Mitra-West Bengal-India

$$\begin{aligned} & \Leftrightarrow \Omega = \int_0^1 \frac{\ln^2(1-x^2) \ln^2 x}{(1-x^2)} \cdot \frac{dx}{x} \\ & \quad \left[x^2 = z \Rightarrow 2x dx = dz \Rightarrow \frac{2dx}{x} = \frac{dz}{z} \right] \\ & = \frac{1}{8} \int_0^1 \frac{\ln^2(1-z) \cdot \ln^2 z}{(1-z)} \cdot \frac{dz}{z} = \frac{1}{8} \left[\int_0^1 \frac{\ln^2(1-z) \ln^2 z}{(1-z)} dz + \int_0^1 \frac{\ln^2(1-z) \ln^2 z}{z} dz \right] \\ & = \frac{1}{4} \int_0^1 \frac{\ln^2(1-z) \cdot \ln^2 z}{z} dz = P \\ & = \frac{1}{4} \cdot 4[2\zeta(5) - \zeta(2)\zeta(3)] = 2\zeta(5) - \zeta(2)\zeta(3) \quad (\text{Answer}) \end{aligned}$$

828. Prove that:

$$\int_0^1 \ln \left((1-x^n)^{\frac{1}{x}-\zeta(2)x^{n-1}} \right) dx = 0$$

Proposed by Mohammed Bouras-Morocco

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} I &= \int_0^1 \ln \left((1-x^n)^{\frac{1}{x}-\zeta(2)x^{n-1}} \right) dx = \int_0^1 \frac{\ln(1-x^n)}{x} dx - \zeta(2) \int_0^1 x^{n-1} \ln(1-x^n) dx \\ &\text{Put: } t = 1-x^n \Rightarrow dt = -nx^{n-1} dx \Rightarrow x^{n-1} dx = -\frac{dt}{n} \end{aligned}$$

$$\therefore I = - \sum_{p=1}^{+\infty} \frac{1}{p} \int_0^1 x^{np-1} dx - \frac{\zeta(2)}{n} \ln(t) dt = - \frac{1}{n} \sum_{p=1}^{+\infty} \frac{1}{n^p} - \frac{\zeta(2)}{n} (t \ln(t) - t) \Big|_0^1$$



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$$= -\frac{\zeta(2)}{n} + \frac{\zeta(2)}{n} = 0 \therefore \int_0^1 \ln((1-x^n)^{\frac{1}{x}-\zeta(2)x^{n-1}}) dx = 0$$

829. Let: $\phi(x) = \sum_{n=2}^{\infty} \frac{n}{x^{n-1}}$

Prove that:

$$\int_0^a \tan^{-1} \left(\frac{a}{2} \sqrt{(\phi(1+ax) - \phi(1-ax))} \right) dx = a, a > 0$$

Proposed by Mohammed Bouras-Morocco

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Phi = \sum_{n=2}^{\infty} \frac{n}{x^{n-1}} = -x^2 \sum_{n=2}^{\infty} \frac{d\left(\frac{1}{x^n}\right)}{dx} = -x^2 \frac{d}{dx} \left(\frac{x}{x-1} - 1 - \frac{1}{x} \right) = \frac{2x-1}{(x-1)^2}, |x| > 1$$

$$\text{Now: } M = \int_0^a \arctan \left(\frac{a}{2} \sqrt{\Phi(1+ax) - \Phi(1-ax)} \right) dx =$$

$$\begin{aligned} &= \overbrace{\int_0^a \arctan \left(\sqrt{\frac{a}{x}} \right) dx}^{IBP} = \left\{ x \arctan \left(\sqrt{\frac{a}{x}} \right) \right\}_0^a + \frac{\sqrt{a}}{2} \int_0^a \frac{x \, dx}{\sqrt{x}(a+x)} \\ &= a \cdot \frac{\pi}{4} + \frac{\sqrt{a}}{2} \left(\int_0^a \frac{dx}{\sqrt{x}} - \int_0^a \frac{dx}{\sqrt{x}(a+x)} \right) = \frac{\pi}{4} a + \left\{ \sqrt{ax} - a \arctan \left(\sqrt{\frac{x}{a}} \right) \right\}_0^a = a \end{aligned}$$

830. Find:

$$\Omega = \int_0^1 \frac{\log^2(1-x) \cdot \log^2(x)}{x} dx$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \Omega = \int_0^1 \frac{\log^2(1-x) \cdot \log^2 x}{x} dx$$



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$$\begin{aligned}
 &= \left[\log^2(1-x) \cdot \int \frac{\log^2 x}{x} dx \right]_0^1 + 2 \int_0^1 \frac{\log(1-x)}{(1-x)} \left[\int \frac{\log^2 x}{x} dx \right] dx \\
 &= \left[\frac{1}{3} \log^2(1-x) \cdot \log^3 x \right]_0^1 + \frac{2}{3} \int_0^1 \frac{\log(1-x) \log^3 x}{(1-x)} dx \\
 &= -\frac{2}{3} \int_0^1 \frac{Li_1(x) \log^3 x}{(1-x)} dx = -\frac{2}{3} \sum_{n=1}^{\infty} H_n \int_0^1 x^n \log^3 x dx \\
 &= -\frac{2}{3} \sum_{n=1}^{\infty} H_n \cdot \frac{(-1)^3 (3!)}{(n+1)^{3+1}} = \frac{2}{3} \cdot 6 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} \\
 &= 4 \sum_{n=1}^{\infty} \frac{H_{(n+1)} - \frac{1}{(n+1)}}{(n+1)^4} = 4 \left[\sum_{n=1}^{\infty} \frac{H_{(n+1)}}{(n+1)^4} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^5} \right] \\
 &= 4 \left[\sum_{n=1}^{\infty} \frac{H_n}{n^4} - \sum_{n=1}^{\infty} \frac{1}{n^5} \right] = 4[3\zeta(5) - \zeta(2)\zeta(3) - \zeta(5)] = 8\zeta(5) - 4\zeta(2)\zeta(3) \quad (\text{Answer})
 \end{aligned}$$

Solution 2 by Ovwie Edafe-Nigeria

$$\begin{aligned}
 \Omega &= \int_0^1 \frac{\log^2(1-x) \cdot \log^2 x}{x} dx \\
 \Omega &\stackrel{IBP}{=} \left[\log^2(1-x) \cdot \frac{\log^3 x}{3} \right]_0^1 + \frac{2}{3} \int_0^1 \frac{\log(1-x)}{1-x} \cdot \log^3 x dx \\
 \Omega &= -\frac{2}{3} \sum_{n=1}^{\infty} H_n \int_0^1 x^n \log^3 x dx; \quad \int_0^1 x^n \log^a x = \frac{(-1)^a}{(n+1)^{a+1}} a! \\
 \Omega &= 4 \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4}; \quad \Omega = 4 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^4}; \quad \Omega = 4 \sum_{n=1}^{\infty} \frac{H_n}{n^4} - 4 \sum_{n=1}^{\infty} \frac{1}{n^5} \\
 \Omega &= 4[3\zeta(5) - \zeta(2)\zeta(3)] - 4\zeta(5); \quad \Omega = 12\zeta(5) - 4\zeta(2)\zeta(3) - 4\zeta(5) \\
 \Omega &= 8\zeta(5) - 4\zeta(2)\zeta(3) \\
 \sum_{n=1}^{\infty} \frac{H_n}{n^p} &= \left(1 + \frac{p}{2}\right) \zeta(p+1) - \frac{1}{2} \sum_{k=1}^{p-2} \zeta(k+1) \cdot \zeta(p-k)
 \end{aligned}$$



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831. Prove that:

$$\int_0^1 \frac{x \log^2(1+x^2) \log(1-x^2)}{1+x^2} dx = Li_4\left(\frac{1}{2}\right) + \zeta(3) \log(2) - \zeta(4) - \frac{\zeta(2)}{2} \log^2 2 + \frac{\log^4 2}{6}$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Dawid Bialek-Poland

$$I = \int_0^1 \frac{x \log^2(1+x^2) \log(1-x^2)}{1+x^2} dx \stackrel{t=1+x^2}{=} \frac{1}{2} \int_1^2 \frac{\log^2(t) \log(2-t)}{t} dt = \\ = \underbrace{\frac{1}{2} \log(2) \int_1^2 \frac{\log^2(t)}{t} dt}_{I_1} + \underbrace{\frac{1}{2} \int_1^2 \frac{\log^2(t) \log(1-\frac{t}{2})}{t} dt}_{I_2} \quad (1)$$

$$I_1 = \frac{1}{2} \log(2) \int_1^2 \frac{\log^2(t)}{t} dt \stackrel{I.B.P}{\Rightarrow} \frac{1}{6} \log^4(2) \quad (2)$$

$$I_2 = \frac{1}{2} \int_1^2 \frac{\log^2(t) \log\left(1-\frac{t}{2}\right)}{t} dt = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n 2^n} \int_1^2 \log^2(t) t^{n-1} dt \stackrel{I.B.P}{\Rightarrow} \\ \Rightarrow -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n 2^n} \left[\frac{\log^2(2) 2^n}{n} - \frac{2}{n} \int_1^2 \log(t) t^{n-1} dt \right] = \\ = -\frac{1}{2} \log^2(2) \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \int_1^2 \log(t) t^{n-1} dt \stackrel{I.B.P}{\Rightarrow} \\ \Rightarrow -\frac{1}{2} \zeta(2) \log^2(2) + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \left[\frac{\log(2) 2^n}{n} - \frac{1}{n} \int_1^2 t^{n-1} dt \right] = \\ = -\frac{1}{2} \zeta(2) \log^2(2) + \log(2) \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^3 2^n} \left[\frac{2^n}{n} - \frac{1}{n} \right] = \\ = -\frac{1}{2} \zeta(2) \log^2(2) + \log(2) \zeta(3) - \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{1}{n^4 2^n} = \\ = -\frac{1}{2} \zeta(2) \log^2(2) + \log(2) \zeta(3) - \zeta(4) + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n^4} = -\frac{1}{2} \zeta(2) \log^2(2) + \log(2) \zeta(3) - \zeta(4) Li_4\left(\frac{1}{2}\right) \quad (3)$$

Rewriting (1) with (2) and (3), we get:



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$$\begin{aligned}
 I &= \frac{1}{2} \log(2) \int_1^2 \frac{\log^2(t)}{t} dt + \frac{1}{2} \int_1^2 \frac{\log^2(t) \log\left(1 - \frac{t}{2}\right)}{t} dt = I_1 + I_2 = \\
 &= \frac{1}{6} \log^4(2) - \frac{1}{2} \zeta(2) \log^2(2) + \log(2) \zeta(3) - \zeta(4) + Li_4\left(\frac{1}{2}\right)
 \end{aligned}$$

$$\text{Note: } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, Li_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

$$832. A(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \prod_{k=1}^n \left(\sqrt[2^k]{x} + \frac{1}{\sqrt[2^k]{x}} \right)$$

Prove that:

$$\int_0^1 \frac{1}{2 \cdot A(x)} \left(1 + \frac{1}{3x} \right) dx = \frac{\pi^2}{12}$$

Proposed by Mohammed Bouras-Morocco

Solution by Samir HajAli-Damascus-Syria

$$A(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \prod_{k=1}^n \left(\sqrt[2^k]{x} + \frac{1}{\sqrt[2^k]{x}} \right) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(\frac{e^{\frac{\ln(x)}{2^k}} + e^{-\frac{\ln(x)}{2^k}}}{2} \right)$$

$$= \lim_{n \rightarrow \infty} \prod_{k=1}^n \cosh\left(\frac{\ln(x)}{2^k}\right) = \frac{\sinh(\ln(x))}{\ln(x)} = \frac{x^2 - 1}{2x \cdot \ln(x)}$$

$$\Omega = \int_0^1 \frac{1}{2 \cdot A(x)} \left(1 + \frac{1}{3x} \right) dx = \int_0^1 \frac{x \ln(x)}{x^2 - 1} \left(1 + \frac{1}{3x} \right) dx$$

$$\Omega = \int_0^1 \frac{x \ln(x)}{x^2 - 1} dx + \frac{1}{3} \int_0^1 \frac{\ln(x)}{x^2 - 1} dx = I_1 + \frac{1}{3} \cdot I_2$$

$$I_1 = \int_0^1 \frac{x \ln(x)}{x^2 - 1} dx = - \sum_{n=0}^{\infty} \int_0^1 x^{2n+1} \cdot \ln(x) dx = \sum_{n=0}^{\infty} \frac{1}{(2n+2)^2} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{24}$$

$$I_2 = \int_0^1 \frac{\ln(x)}{x^2 - 1} dx = - \sum_{n=0}^{\infty} \int_0^1 x^{2n} \cdot \ln(x) dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$\Omega = \frac{\pi^2}{24} + \frac{1}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{12}$$



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833. For any complex numbers a, b with $\operatorname{Re}(a), \operatorname{Re}(b) > 0$

If

$$f_m(a, b) = \int_0^\infty (e^{-ax} - e^{-bx}) \frac{\sin(mx)}{x} dx$$

then show that

$$\frac{\partial}{\partial a} \left(\int_0^1 f_m(a, b) dm \right) + \frac{\partial}{\partial b} \left(\int_0^1 f_m(a, b) dm \right) = \frac{1}{2} \log \left(\frac{a^2(1+b^2)}{b^2(1+a^2)} \right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Nassim Nicholas Taleb-USA

$$f = \int_0^\infty \left(e^{-\frac{au}{m}} - e^{-\frac{bu}{m}} \right) \frac{\sin[mx]}{x} dx$$

$$\text{We substitute } u = mx, \text{ so } f = \int_0^\infty \frac{\left(e^{-\frac{au}{m}} - e^{-\frac{bu}{m}} \right) \sin[u]}{u} du$$

We have $\frac{\partial f}{\partial a} = -\frac{1}{m} \int_0^\infty \operatorname{sign}[m] e^{-\frac{au}{m}} \sin[u] du = -\frac{1}{m} \mathcal{L}_u[\sin(u)] \left(\frac{a}{m} \right)$, where \mathcal{L}_u is the Laplace transform.

From tables, $\mathcal{L}_u[\sin(u)](t) = \frac{1}{1+t^2}$ so $\frac{\partial f}{\partial a} = -\frac{m}{a^2+m^2}$, and reversing the order of

integration - deriv: $-\int_0^1 \frac{m}{a^2+m^2} da =$

$$-\frac{1}{2} \log[a^2+m^2] \Big|_{m=1} + \frac{1}{2} \log[a^2+m^2] \Big|_{m=0} = -\frac{1}{2} \log[a^2] + \frac{1}{2} \log[1+a^2] = \frac{1}{2} \log \left[\frac{1+a^2}{a^2} \right]$$

Repeating for b and summing gets the required result.

834. Find:

$$\Omega = \int_0^a \left(\frac{\tan^{-1} x}{(a-1)x^2 + a(x+1)} \right) dx, a > \frac{4}{3}$$

Proposed by Radu Diaconu-Romania



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Solution by Zaharia Burgheloa-Romania

$$\begin{aligned}
 & \text{Let } \frac{a-x}{1+at} = t \Rightarrow x = \frac{a-t}{1+at} \Rightarrow dx = -\frac{1+a^2}{(1+at)^2} dt \\
 & \Rightarrow \Omega = \int_0^a \frac{\arctan x}{(a-1)x^2 + a(x+1)} dx = \\
 & = \int_0^a \frac{\arctan\left(\frac{a-t}{1+at}\right)}{(a-1)\left(\frac{a-t}{1+at}\right)^2 + a\left(\frac{a-t}{1+at} + 1\right)} \frac{1+a^2}{(1+at)^2} dt \\
 & = \int_0^a \frac{(1+a^2) \arctan\left(\frac{a-t}{1+at}\right)}{[1+a^2][(a-1)t^2 + a(t+1)]} dt \stackrel{t=x}{=} \int_0^a \frac{\arctan a - \arctan x}{(a-1)x^2 + a(x+1)} dx \\
 2\Omega &= \int_0^a \frac{\arctan x + \arctan a - \arctan x}{(a-1)x^2 + a(x+1)} dx \Rightarrow \Omega = \frac{\arctan a}{2} \int_0^a \frac{dx}{(a-1)x^2 + a(x+1)} \\
 &= \frac{\arctan a}{\sqrt{a(3a-4)}} \arctan\left(\frac{2(a-x)+a}{\sqrt{a(3a-4)}}\right) \Big|_0^a = \frac{\arctan a}{\sqrt{a(3a-4)}} \arctan\left(\frac{\sqrt{a(3a-4)}}{a+2}\right), a > \frac{4}{3}
 \end{aligned}$$

835. Prove that:

$$\int_0^\infty \frac{e^{-x}}{e^{2x}+1} \log\left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx = \frac{\pi}{2} - \frac{(4+\pi)}{4} \log(2)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Zaharia Burgheloa-Romania

$$\begin{aligned}
 I &= \int_0^\infty \frac{e^{-x}}{e^{2x}+1} \log\left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx \stackrel{e^{-x}=t}{\cong} \int_0^1 \frac{t^2 \log\left(\frac{1+t^2}{1-t^2}\right)}{t^2+1} dt = \int_0^1 \log\left(\frac{1+t^2}{1-t^2}\right) dt - \int_0^1 \frac{\log\left(\frac{1+t^2}{1-t^2}\right)}{1+t^2} dt \\
 I_1 &= \int_0^1 \log\left(\frac{1+t^2}{1-t^2}\right) dt \stackrel{IBP}{\cong} 2 \int_0^1 \frac{1}{t^2+1} dt + 2 \int_0^1 \frac{t}{t+1} dt - 2 \int_0^1 \frac{1}{t^2+1} dt = \frac{\pi}{2} - \log(2) \\
 I_2 &= \int_0^1 \frac{\log\left(\frac{1+t^2}{1-t^2}\right)}{1+t^2} dt \stackrel{\frac{1+t^2}{1-t^2}=x}{\cong} -\frac{1}{2} \int_0^1 \frac{\log(x)}{\sqrt{1-x^2}} dx \stackrel{x=\sin t}{\cong} -\frac{1}{2} \int_0^{\frac{\pi}{2}} \log(\sin t) dt = \frac{\pi}{4} \log(2)
 \end{aligned}$$



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$$\Rightarrow I = I_1 - I_2 = \frac{\pi}{2} - \log(2) - \frac{\pi}{4} \log(2)$$

836. Find:

$$\Omega = \int \sec x \cdot (\sec x + \tan x)^n \cdot n^{(\sec x + \tan x)^n} dx, n \in \mathbb{N}^*$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Avishek Mitra-West Bengal-India

$$\sec x + \tan x = z \Rightarrow \sec x (\sec x + \tan x) dx = dz \Rightarrow \sec x dx = \frac{dz}{z}$$

$$\Omega = \int z^n \cdot n^{z^n} \cdot \frac{dz}{z} = \int z^{n-1} \cdot n^{z^n} \cdot dz$$

$$z^n = u \Rightarrow nz^{n-1} dz = du \Rightarrow z^{n-1} dz = \frac{du}{n}$$

$$\Omega = \int \frac{n^u}{n} du = \int n^{u-1} du = \frac{u^{n-1}}{\log n} + C = \frac{n^{z^n}}{n \log n} + C = \frac{n^{(\sec x + \tan x)^n}}{n \log n} + C$$

837.

$$\int \frac{1}{e^x + e^{-3x} + e^{-7x}} dx$$

Proposed by Igor Soposki-Skopje-Macedonia

Solution by Tobi Joshua-Nigeria

$$\begin{aligned} & \int \frac{dx}{e^x + e^{-3x} + e^{-7x}}; \quad \int \frac{e^{7x} dx}{1 + e^{4x} + e^{8x}} \text{ set } x \Leftrightarrow e^x \\ & \int \frac{e^{7x} dx}{1 + e^{4x} + e^{8x}} \text{ set } x \Leftrightarrow e^x; \quad \int \frac{x^6 dx}{1 + x^4 + x^8} \\ & \int \frac{x^{-2} dx}{1 + x^{-4} + x^{-8}}; \quad - \int \frac{d(x^{-1})}{1 + (x^{-1})^4 + (x^{-1})^8} \text{ set } y = x^{-1} \\ & \quad - \int \frac{d(y)}{1 + (y)^4 + (y)^8} \\ & - \left[\frac{1}{4} \int \frac{d(y)}{y^2 - y + 1} + \frac{1}{4} \int \frac{dy}{y^2 + y + 1} - \frac{1}{2} \int \frac{(y^2 - 1) dy}{(y^4 - y^2 + 1)} \right] \end{aligned}$$



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$$\begin{aligned}
 & - \left[\frac{1}{4} \int \frac{d(y)}{\left(y - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{4} \int \frac{dy}{\left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2} \int \frac{\left(1 - \frac{1}{y^2}\right) dy}{\left(y^2 + \frac{1}{y^2} - 1\right)} \right] \\
 & - \left[\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2y - 1}{\sqrt{3}} \right) + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2y + 1}{\sqrt{3}} \right) - \frac{1}{2} \int \frac{d\left(y + \frac{1}{y}\right)}{\left(y + \frac{1}{y}\right)^2 - (\sqrt{3})^2} \right] \\
 & - \left[\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2y - 1}{\sqrt{3}} \right) + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2y + 1}{\sqrt{3}} \right) - \frac{1}{4\sqrt{3}} \log \left(\frac{y^2 - \sqrt{3} + 1}{y^2 + \sqrt{3} + 1} \right) + C \right] \\
 & - \left[\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2-x}{x\sqrt{3}} \right) - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2+x}{x\sqrt{3}} \right) + \frac{1}{4\sqrt{3}} \log \left(\frac{y^2 - \sqrt{3} + 1}{y^2 + \sqrt{3} + 1} \right) + C \right] \\
 & \left[\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2-e^x}{e^x\sqrt{3}} \right) - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2+e^x}{e^x\sqrt{3}} \right) + \frac{1}{4\sqrt{3}} \log \left(\frac{e^{2x} - \sqrt{3}e^x + 1}{e^{2x} + \sqrt{3}e^x + 1} \right) + C \right]
 \end{aligned}$$

838. Find:

$$\Omega(a) = \int_0^\infty \left(\frac{x}{(1+x^4)(1+a^2x^2)} \right) dx, a \in \mathbb{R}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Kamel Benaicha-Algeirs-Algerie

$$\begin{aligned}
 \Omega(a) &= \int_0^\infty \left(\frac{x}{(1+x^4)(1+a^2x^2)} \right) dx = \frac{1}{2} \int_0^\infty \left(\frac{x}{(1+x^2)(1+a^2x^2)} \right) dx \\
 &= \frac{1}{2(1+a^4)} \left(\int_0^\infty \frac{dx}{1+x^2} + a^2 \int_0^\infty \left(\frac{a^2}{1+a^2x} - \frac{x}{1+x^2} \right) dx \right) \\
 &= \frac{1}{2(1+a^4)} \left(\frac{\pi}{2} + a^2 \log \left(\frac{1+a^2x}{\sqrt{1+x^2}} \right) \Big|_0^\infty \right) = \frac{a^2 \log |a|}{1+a^4} + \frac{\pi}{4(1+a^4)}; a \neq 0 \\
 \Omega(0) &= \int_0^\infty \frac{x dx}{1+x^4} = \frac{1}{2} \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}; \Omega(a) = \begin{cases} \frac{a^2 \log |a|}{1+a^4} + \frac{\pi}{4(1+a^4)}; a \neq 0 \\ \frac{\pi}{4} \end{cases}
 \end{aligned}$$



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839. Let f be positive and continuous when $a \leq x \leq b$. Prove:

$$3 \left(\int_a^b f(x) dx \right)^2 \leq 1 + 2 \int_a^b (f(x))^3 dx$$

Proposed by Jalil Hajimir-Canada

Solution by Soumitra Mandal-Chandar Nagore-India

f is a positive continuous function on $[a, b]$

$\therefore f(x) \geq 0$ for all $x \in [a, b]$ hence $\int_a^b f(x) dx \geq 0$

Now $\int_a^b f^3(x) dx \stackrel{\text{Cauchy}}{\geq} \left(\int_a^b f(x) dx \right)^3$. Let $t = \int_a^b f(x) dx$

We have to prove, $2t^3 - 3t^2 + 1 \geq 0 \Leftrightarrow 2t^2(t-1) - (t^2 - 1) \geq 0$

$\Leftrightarrow (t-1)(2t^2 - t - 1) \geq 0 \Leftrightarrow (t-1)^2(2t+1) \geq 0$

Hence $2 \left(\int_a^b f(x) dx \right)^3 + 1 \geq 3 \left(\int_a^b f(x) dx \right)^2$

$\Leftrightarrow 2 \int_a^b f^3(x) dx + 1 \geq 2 \left(\int_a^b f(x) dx \right)^3 + 1 \geq 3 \left(\int_a^b f(x) dx \right)^2$

$\therefore 2 \int_a^b f^3(x) dx + 1 \geq 3 \left(\int_a^b f(x) dx \right)^2$

840. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \left(\frac{\tan x + \tan y - \tan(x+y)}{xy(\pi - x - y)} \right) dx dy \geq \frac{81\sqrt{3}}{\pi^3} (b-a)^2$$

Proposed by Jalil Hajimir-Canada

Solution by Daniel Sitaru-Romania

Denote $z = \pi - x - y \rightarrow x + y = \pi - z$

$$\frac{\tan x + \tan y - \tan(x+y)}{xy(\pi - x - y)} = \frac{\tan x + \tan y - \tan(\pi - z)}{xyz} \geq$$



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$$\begin{aligned}
 & \stackrel{AM-GM}{\geq} \frac{\tan x + \tan y + \tan z}{\left(\frac{x+y+z}{3}\right)^3} \stackrel{JENSEN}{\geq} \frac{3\tan\left(\frac{x+y+z}{3}\right)}{\left(\frac{\pi}{3}\right)^3} = \frac{3\tan\left(\frac{\pi}{3}\right)}{\frac{\pi^3}{27}} = \frac{81\sqrt{3}}{\pi^3} \\
 & \int_a^b \int_a^b \left(\frac{\tan x + \tan y - \tan(x+y)}{xy(\pi-x-y)} \right) dx dy \geq \int_a^b \int_a^b \left(\frac{81\sqrt{3}}{\pi^3} \right) dx dy = \\
 & = \frac{81\sqrt{3}}{\pi^3} \left(\int_a^b dx \right)^2 = \frac{81\sqrt{3}}{\pi^3} (b-a)^2
 \end{aligned}$$

Equality holds for $a = b$.

841. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \left(\frac{\sin x \cdot \sin y \cdot \sin(x+y)}{xy(\pi-x-y)} \right) dx dy \leq \frac{81\sqrt{3}(b-a)^2}{8\pi^3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jalil Hajimir-Canada

$$\frac{\sin A}{A} \cdot \frac{\sin B}{B} \cdot \frac{\sin C}{C} \leq \frac{\sin A \sin B \sin C}{\frac{(A+B+C)^3}{27}} \leq \frac{\frac{3\sqrt{3}}{8}}{\frac{\pi^3}{27}}$$

$$\therefore \frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin z}{z} \leq \frac{81\sqrt{3}}{8\pi^3} \text{ if } x+y+z=\pi$$

$$\int_1^b \int_b^b \frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin(x+y)}{\pi-x-y} dx dy dz \leq \frac{81\sqrt{3}}{8\pi^3} \int_a^b \int_a^b dx dy = \frac{81\sqrt{3}}{8\pi^3} (b-a)^2$$

$$\sin(x+y) = \sin(\pi-x-y)$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y > 0$ and $0 < a \leq b < \frac{\pi}{2}$, we have as following:

1. since $x+y+\pi-(x+y)=\pi$

Hence $\sin x + \sin y + \sin(\pi-(x+y)) \leq \frac{3\sqrt{3}}{2}$ and since $\frac{\sin x}{x} - \frac{\sin y}{y} = \frac{\sin(\pi-(x+y))}{\pi-(x+y)}$

Hence $\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin(\pi-(x+y))}{\pi-(x+y)} \leq \frac{3(3\sqrt{3})}{2\pi}$; $x=y=\pi-(x+y)=\frac{\pi}{3}$



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$$\text{Hence } 3 \sqrt[3]{\frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin(\pi-(x+y))}{\pi-(x+y)}} < \frac{3(3\sqrt{3})}{2\pi}$$

$$\text{Hence } \frac{\sin x \sin y \sin(x+y)}{xy(\pi-x-y)} \leq \frac{81\sqrt{3}}{8\pi^3}$$

$$\begin{aligned} \text{Hence } \int_a^b \int_a^b \frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin(x+y)}{\pi-(x+y)} dx dy &\leq \int_a^b \int_a^b \frac{81\sqrt{3}}{8\pi^3} dx dy \\ &= \frac{81\sqrt{3}}{8\pi^3} xy \Big|_a^b = \frac{81\sqrt{3}}{8\pi^3} (b-a)(b-a). \text{ Therefore, it is true.} \end{aligned}$$

Solution 3 by Ali Jaffal-Lebanon

$$\text{Let } f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \in [0, \frac{\pi}{2}] \\ 1 & \text{if } x = 0 \end{cases} \text{ we have } f(x) > 0, \forall x \in [0, \frac{\pi}{2}]$$

$$f''(x) = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}$$

$$\text{Let } \varphi(x) = -x^2 \sin x - 2x \cos x + 2 \sin x; \forall x \in [0, \frac{\pi}{2}]; \varphi'(x) = -x^2 \cos x$$

$$\text{then } \lim_{x \rightarrow 0} f''(x) = \lim_{x \rightarrow 0} \frac{-x^2 \cos x}{3x^2} = -\frac{1}{3} < 0 \text{ and}$$

x	0	$\frac{\pi}{2}$
$\varphi'(x)$		-----
$\varphi(x)$	0	$-\frac{\pi^2}{4} + 2$

So, $\varphi(x) \leq 0; \forall x \in [0, \frac{\pi}{2}]$ then $f''(x) < 0, \forall x \in [0, \frac{\pi}{2}]$ so, f is concave on $[0, \frac{\pi}{2}]$

$$\text{Let } x, y, z \in [0, \frac{\pi}{2}] \text{ where } x + y + z = \pi \text{ we have } f\left(\frac{x+y+z}{3}\right) \geq \frac{f(x)+f(y)+f(z)}{3}$$

$$\text{so, } \frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{z} \leq \frac{3 \sin(\frac{\pi}{3})}{\frac{\pi}{3}} \leq \frac{9\sqrt{3}}{2\pi} \text{ by GM-AM inequality we have:}$$

$$\left(\frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin z}{z}\right) \leq \frac{1}{27} \left(\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{z}\right)^3 \leq \frac{1}{27} \times \left(\frac{9\sqrt{3}}{2\pi}\right)^3 \leq \frac{81\sqrt{3}}{8\pi^3}$$

$$\text{So, } \int_a^b \int_a^b \frac{\sin x}{x} \cdot \frac{\sin y}{y} \cdot \frac{\sin(\pi-x-y)}{\pi-x-y} dx dy \leq \int_a^b \int_a^b \frac{81\sqrt{3}}{8\pi^3} dx dy \leq \frac{(b-a)^2 81\sqrt{3}}{8\pi^3}$$



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842. If $0 < a \leq b$ then:

$$\int_a^b \left(\int_a^b \left(\int_a^b \left(\frac{\sqrt[3]{(x^3 + y^3)(y^3 + z^3)(z^3 + x^3)}}{(x^2 + y^2)(y^2 + z^2)(z^2 + x^2)} dx \right) dy \right) dz \leq \frac{1}{4} \left(\log \frac{b}{a} \right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\begin{aligned} \frac{\sqrt[3]{\prod(x^3 + y^3)}}{\prod(x^2 + y^2)} &\stackrel{\text{Holder}}{\leq} \frac{xyz + xyz}{\prod(x^2 + y^2)} \stackrel{x^2 + y^2 \geq 2xy}{\leq} \frac{2xyz}{8(xyz)^2} = \frac{1}{4xyz} \\ \text{so, } \int_a^b \int_a^b \int_a^b &\frac{\sqrt[3]{\prod(x^3 + y^3)}}{\prod(x^2 + y^2)} dx dy dz \leq \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{4xyz} \\ &= \int_a^b \int_a^b \frac{dx dy}{4xy} \cdot (\ln z)_a^b = \frac{1}{4} (\ln x)_a^b \cdot (\ln y)_a^b \cdot (\ln z)_a^b = \frac{1}{4} (\ln b - \ln a)^3 = \frac{(\ln(\frac{b}{a}))^3}{4} \end{aligned}$$

Solution 2 by Sanong Hauyrerai-Nakon Pathom-Thailand

$$\begin{aligned} \text{For } a, b > 0, \text{ we have } 4^2(a^3 + b^3)^2(ab)^3 &= 16(a^9b^3 + a^3b^9 + 2a^6b^6) \\ \text{and } (a^2 + b^2)^6 &= a^{12} + b^{12} + 6a^{10}b^2 + 6a^2b^{10} + 10a^8b^4 + 15a^4b^8 + 20a^6b^6 \\ \text{and since } b(a^8b^2 + a^{10}b^2) &\geq 12a^9b^3 \\ 6(a^4b^8 + a^2b^{12}) &\geq 12a^3b^9; a^8b^4 + a^8b^4 + a^8b^4 + a^{12} &\geq 4a^9b^3 \\ a^4b^8 + a^4b^8 + a^4b^8 + 10^{12} &\geq 4a^3b^9; 6(a^8b^4 + a^4b^8) &\geq 12a^6b^6 \end{aligned}$$

$$\text{Hence } 4^2(a^3 + b^3)^2(ab)^3 \leq (a^2 + b^2)6 \Rightarrow \frac{(a^3 + b^3)^2}{(a^2 + b^2)^6} \leq \frac{1}{4^2(ab)^3} \Rightarrow \frac{(a^3 + b^3)}{(a^2 + b^2)^3} \leq \frac{1}{4(ab)^3}$$

$$\Rightarrow \frac{\sqrt[3]{(a^3 + b^3)}}{(a^2 + b^2)} \leq \frac{1}{\sqrt[3]{4}} (ab)^{\frac{1}{2}}. \text{ Hence for } x, y, z > 0$$

$$\frac{\sqrt[3]{(x^3 + y^3)(y^3 + z^3)(z^3 + x^3)}}{(x^2 + y^2)(y^2 + z^2)(z^2 + x^2)} \leq \frac{1}{\sqrt[3]{4 \times 4 \times 4}} \cdot \frac{1}{((xyz)^2)^{\frac{1}{2}}} = \frac{1}{4xyz}$$

Hence for $0 < a \leq b$

$$\int_a^b \left(\int_a^b \left(\int_a^b \frac{\sqrt[3]{(x^3 + y^3)(y^3 + z^3)(z^3 + x^3)}}{(x^2 + y^2)^2(y^2 + z^2)(z^2 + x^2)} dx \right) dy \right) dz \leq$$



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$$\begin{aligned}
 & \leq \int_a^b \left(\int_a^b \left(\int_a^b \frac{1}{4xyz} dx \right) dy \right) dz = \frac{1}{4} \ln x \ln y \ln z \Big|_a^b \\
 & = \frac{1}{4} (\ln b - \ln a)(\ln b - \ln a)(\ln b - \ln a) = \frac{1}{4} \left(\ln \frac{b}{a} \right)^3. \text{ Therefore, it is true.}
 \end{aligned}$$

Solution 3 by Jalil Hajimir-Canada

$$\text{Let } A = \frac{\sqrt{(x^3+y^3)(y^3+z^3)(z^3+x^3)}}{(x^2+y^2)(y^2+z^2)(z^2+x^2)} \text{ since } \sqrt[3]{\frac{A^3+B^3}{2}} \leq \sqrt{\frac{A^2+B^2}{2}}$$

$$A \leq \frac{\frac{1}{\sqrt{2}}}{[(x^2+y^2)(y^2+z^2)(z^2+x^2)]^{\frac{1}{2}}} \leq \frac{1}{4xyz}$$

$$\Omega = \int_a^b \int_a^b \int_a^b A \, dx \, dy \, dz \leq \int_a^b \int_a^b \int_a^b \frac{dx \, dy \, dz}{4xyz} \therefore \Omega \leq \frac{1}{4} (\log_a b)^3$$

843. If $f \in C^1([0, 1])$, $m \leq |f'(x)| \leq M, \forall x \in [0, 1]$ then:

$$\frac{m^2}{12} \leq \int_0^1 f^2(x) dx - \left(\int_0^1 f(x) dx \right)^2 \leq \frac{M^2}{12}$$

Proposed by Jalil Hajimir-Canada, Dinu Șerbănescu-Romania

Solution by Daniel Sitaru-Romania

$$f(x) - f(y) \stackrel{\text{LAGRANGE}}{\cong} f'(c)(x - y), 0 \leq x \leq c \leq y \leq 1$$

$$m^2 \leq (f'(c))^2 \leq M^2 \rightarrow m^2(x - y)^2 \leq (f'(c))^2(x - y)^2 \leq M^2(x - y)^2$$

$$m^2(x - y)^2 \leq (f(x) - f(y))^2 \leq M^2(x - y)^2$$

$$\int_0^1 \int_0^1 (x - y)^2 dx dy = \frac{1}{6} \rightarrow m^2 \cdot \frac{1}{6} \leq \int_0^1 \int_0^1 (f(x) - f(y))^2 dx dy \leq M^2 \cdot \frac{1}{6}$$

$$\frac{m^2}{6} \leq \int_0^1 \int_0^1 f^2(x) dx dy - 2 \int_0^1 \int_0^1 f(x)f(y) dx dy + \int_0^1 \int_0^1 f^2(y) dx dy \leq M^2 \cdot \frac{1}{6}$$

$$\frac{m^2}{6} \leq \int_0^1 f^2(x) dx - 2 \int_0^1 f(x) dx \int_0^1 f(y) dy + \int_0^1 f^2(y) dy \leq \frac{M^2}{6}$$



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$$\frac{m^2}{6} \leq 2 \int_0^1 f^2(x) dx - 2 \left(\int_0^1 f(x) dx \right)^2 \leq \frac{M^2}{6}$$

$$\frac{m^2}{12} \leq \int_0^1 f^2(x) dx - \left(\int_0^1 f(x) dx \right)^2 \leq \frac{M^2}{12}$$

844. If $0 < a \leq b$ then:

$$\begin{aligned} & \int_a^b \int_a^b \tan^{-1} \left(\frac{ax + by}{a+b} \right) dx dy + (b-a) \int_a^b \log x dx \leq \\ & \leq \int_a^b \int_a^b \log \left(\frac{ax + by}{a+b} \right) dx dy + (b-a) \int_a^b \tan^{-1} x dx \end{aligned}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

Let $\varphi(x) = \arctan x - \log x$, where $x \in]0, +\infty[$

$$\varphi''(x) = -\frac{2x}{(1+x^2)^2} + \frac{1}{x^2} = \frac{x^4 - 2x^3 + 2x^2 + 1}{x^2(1+x^2)^2}$$

$$\text{Let } \psi(x) = x^4 - 2x^3 + 2x^2 + 1, x \in \mathbb{R}$$

$\psi'(x) = 2x(2x^2 - 3x + 2)$ has the same sign as x since $2x^2 - 3x + 2 > 0$ for all

$x \in \mathbb{R}$. We have $2x^2 - 3x + 2 = 2\left[x^2 - \frac{3x}{2} + 1\right] = 2\left[\left(x - \frac{3}{4}\right)^2 + \frac{7}{16}\right] > 0$. So,

x	$-\infty$	0	$+\infty$
$\psi'(x)$	-----	0	+++ + + + + + + + + +
$\psi(x)$		1	

then $\psi(x) > 0$ for all $x \in \mathbb{R}$ then $\varphi''(x) > 0$ and φ is convex.

By Jensen's inequality: $\varphi\left(\frac{ax+by}{a+b}\right) \leq \frac{a\varphi(x)+b\varphi(y)}{a+b}$; $x > 0, y > 0$

$$\int_a^b \int_a^b \varphi\left(\frac{ax+by}{a+b}\right) dx dy \leq \frac{a(b-a)}{a+b} \int_a^b (\arctan x - \log x) dx + \frac{a(b-a)}{a+b} \int_a^b \varphi(y) dy$$



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$$\leq \left(\frac{a(b-a)}{a+b} + \frac{a(b-a)}{a+b} \right) \left(\int_a^b (\arctan x - \log x) dx \right)$$

$$\leq (b-a) \int_a^b \arctan x dx - (b-a) \int_a^b \log x dx$$

$$\text{Therefore: } \int_a^b \int_a^b \arctan \left(\frac{ax+by}{a+b} \right) + (b-a) \int_a^b \log x dx \leq$$

$$\leq (b-a) \int_a^b \arctan x dx + \int_a^b \int_a^b \log \left(\frac{ax+by}{a+b} \right) dx dy$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = \tan^{-1} x - \ln x$ for all $x > 0$

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{x}, f''(x) = \frac{1+x^4+2x^2-2x^3}{(x+x^3)^2}$$

Let $g(x) = 1+x^4+2x^2-2x^3$ for all $x > 0$.

Then $g'(x) = 4x \left\{ \left(x - \frac{3}{4} \right)^2 + \frac{7}{16} \right\} > 0$ for all $x > 0$. Hence g is an increasing function

$\therefore g(x) \geq g(0) = 1 > 0$ then $f''(x) > 0$ for all $x > 0$.

Hence f is a convex function, so by definition: $\frac{a}{a+b}f(x) + \frac{b}{a+b}f(y) \geq f\left(\frac{ax+by}{a+b}\right)$

$$\frac{a}{a+b}(\tan^{-1} x - \ln x) + \frac{b}{a+b}(\tan^{-1} y - \ln y) \geq \tan^{-1} \left(\frac{ax+by}{a+b} \right) - \ln \left(\frac{ax+by}{a+b} \right)$$

$$\Rightarrow \tan^{-1} x - \ln x \geq \tan^{-1} \left(\frac{ax+by}{a+b} \right) - \ln \left(\frac{ax+by}{a+b} \right) [\text{generalizing } y \text{ by } x]$$

$$\Rightarrow \tan^{-1} x + \ln \left(\frac{ax+by}{a+b} \right) \geq \tan^{-1} \left(\frac{ax+by}{a+b} \right) + \ln x$$

$$(b-a) \int_a^b \ln x dx + \int_a^b \int_a^b \tan^{-1} \left(\frac{ax+by}{a+b} \right) dx dy \leq$$

$$\leq (b-a) \int_a^b \tan^{-1} x dx + \int_a^b \int_a^b \ln \left(\frac{ax+by}{a+b} \right) dx dy. \text{ Proved.}$$

845. If $0 < a < b < c < d$ then:



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$$\frac{1}{b-a} \int_a^b e^{x^2} dx + \frac{1}{c-b} \int_b^c e^{x^2} dx + \frac{1}{d-c} \int_c^d e^{x^2} dx > 3^{\sqrt[36]{e^{(a+2b+2c+d)^2}}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$f(x) = e^{x^2}, x > 0, f''(x) = e^{x^2} \cdot (4x^2 + 2) > 0, \forall x > 0 \Rightarrow f \text{ convex, so:}$

$$\frac{1}{b-a} \int_a^b f(x) dx \stackrel{\substack{\text{Hermite} \\ \text{Hadamard}}}{\geq} f\left(\frac{a+b}{2}\right) = e^{\frac{(a+b)^2}{4}}$$

$$\text{Thus LHS} > \sum e^{\frac{(a+b)^2}{4}} \stackrel{\text{AM-GM}}{>} 3 \cdot \sqrt[3]{e^{\frac{(a+b)^2}{4}} \cdot e^{\frac{(b+c)^2}{4}} \cdot e^{\frac{(c+d)^2}{4}}}$$

$$= 3 \cdot \sqrt[3]{e^{\frac{(a+b)^2}{4} + \frac{(b+c)^2}{4} + \frac{(c+d)^2}{4}}} \stackrel{\text{Andreeescu}}{>} 3 \sqrt[3]{e^{\frac{(a+b+b+c+c+d)^2}{4+4+4}}} = 3 \cdot \sqrt[36]{e^{(a+2b+2c+d)^2}} = \text{RHS}$$

Solution 2 by Ali Jaffal-Lebanon

Let $f(x) = e^{x^2}, x \in [0, +\infty[, f'(x) = 2xe^{x^2}$

$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = (2 + 4x^2)e^{x^2}$ then f is convex on $[0, +\infty[$ so,

$$\int_m^n f(x) dx \geq (n-m)f\left(\frac{n+m}{2}\right) \text{ for all } 0 < n \leq m \text{ then}$$

$$\frac{1}{b-a} \int_a^b f(x) dx + \frac{1}{c-b} \int_b^c f(x) dx + \frac{1}{d-c} \int_c^d f(x) dx \geq$$

$$f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{d+c}{2}\right) \geq$$

$$3 \sqrt[3]{f\left(\frac{a+b}{2}\right) \cdot f\left(\frac{b+c}{2}\right) \cdot f\left(\frac{d+c}{2}\right)} \text{ by GM-AM inequality}$$

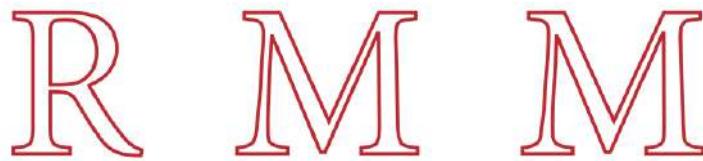
$$\geq 3 \sqrt[3]{e^{\frac{(a+b)^2 + (b+c)^2 + (d+c)^2}{4}}} \geq 3 \sqrt[12]{e^{(a+b)^2 + (b+c)^2 + (d+c)^2}}$$

but $(a+b)^2 + (b+c)^2 + (d+c)^2 \geq \frac{1}{3}((a+b) + (b+c) + (d+c))^2$

since $g(x) = x^2$ is convex on $[0, +\infty[$ then

$$3 \sqrt[12]{e^{(a+b)^2 + (b+c)^2 + (d+c)^2}} \geq 3 \sqrt[36]{e^{(a+b+b+c+d+c)^2}}$$

Therefore: $\frac{1}{b-a} \int_a^b e^{x^2} dx + \frac{1}{c-b} \int_b^c e^{x^2} dx + \frac{1}{d-c} \int_c^d e^{x^2} dx \geq 3 \sqrt[36]{e^{(a+2b+2c+d)^2}}$



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846. If $a, b, c > 0, n \in \mathbb{N}, n \geq 2$ then:

$$\int_a^{2a} \int_b^{2b} \int_c^{2c} \left(\sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}} \right) dx dy dz > 15abc$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\begin{aligned}
 & \sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}} = n \cdot \frac{\sqrt[n]{x+1}}{n} + (n+2) \cdot \frac{\sqrt[n+2]{y+1}}{n+2} + (n+4) \cdot \frac{\sqrt[n+4]{z+1}}{n+4} \geq \\
 & \stackrel{AM-GM}{\geq} (3n+6) \sqrt[3n+6]{\left(\frac{\sqrt[n]{x+1}}{n}\right)^n \cdot \left(\frac{\sqrt[n+2]{y+1}}{n+2}\right)^{n+2} \cdot \left(\frac{\sqrt[n+4]{z+1}}{n+4}\right)^{n+4}} = \\
 & = (3n+6) \sqrt[3n+6]{\frac{\frac{x+1}{y+1} \cdot \frac{y+1}{z+1} \cdot \frac{z+1}{x+1}}{n^n \cdot (n+2)^{n+2} \cdot (n+4)^{n+4}}} = (3n+6) \sqrt[3n+6]{\frac{1}{n^n \cdot (n+2)^{n+2} \cdot (n+4)^{n+4}}} \geq \\
 & \stackrel{AM-GM}{\geq} \frac{3n+6}{n \cdot n + (n+2) \cdot (n+2) + (n+4) \cdot (n+4)} = \frac{9(n^2 + 4n + 4)}{3n^2 + 12n + 20} > \frac{5}{2} \leftrightarrow \\
 & \leftrightarrow 18n^2 + 72n + 72 > 15n^2 + 60n + 100 \leftrightarrow 3n^2 + 12n > 28 \quad (n \geq 2)
 \end{aligned}$$

$$\begin{aligned}
 & \sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}} > \frac{5}{2} \rightarrow \int_a^{2a} \int_b^{3b} \int_c^{4c} \left(\sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}} \right) dx dy dz > \\
 & > \int_a^{2a} \int_b^{3b} \int_c^{4c} \frac{5}{2} dx dy dz = \frac{5}{2} (2a - a)(3b - b)(4c - c) = 15abc
 \end{aligned}$$

847. If $a, b, c \geq 0$ then:

$$\int_a^{2a} \int_b^{2b} \int_c^{2c} \left(\sqrt{\frac{e^x + e^y + e^z + 3\sqrt[3]{e^{x+y+z}}}{\sqrt{e^{x+y}} + \sqrt{e^{y+z}} + \sqrt{e^{z+x}}}} \right) dx dy dz \geq \sqrt{2}abc$$

Proposed by Jalil Hajimir-Toronto-Canada



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Solution by Daniel Sitaru-Romania

$f(x) = e^x, f'(x) = e^x, f''(x) = e^x > 0$, f – convex. By Popoviciu's inequality:

$$\frac{1}{3} \sum_{cyc} f(x) + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3} \sum_{cyc} f\left(\frac{x+y}{2}\right)$$

$$\frac{1}{3} \sum_{cyc} e^x + e^{\frac{x+y+z}{3}} \geq \frac{2}{3} \sum_{cyc} e^{\frac{x+y}{2}}$$

$$e^x + e^y + e^z + 3\sqrt[3]{e^{x+y+z}} \geq 2(\sqrt{e^{x+y}} + \sqrt{e^{y+z}} + \sqrt{e^{z+x}})$$

$$\frac{e^x + e^y + e^z + 3\sqrt[3]{e^{x+y+z}}}{\sqrt{e^{x+y}} + \sqrt{e^{y+z}} + \sqrt{e^{z+x}}} \geq 2 \rightarrow \sqrt{\frac{e^x + e^y + e^z + 3\sqrt[3]{e^{x+y+z}}}{\sqrt{e^{x+y}} + \sqrt{e^{y+z}} + \sqrt{e^{z+x}}}} \geq \sqrt{2}$$

$$\int_a^{2a} \int_b^{2b} \int_c^{2c} \left(\sqrt{\frac{e^x + e^y + e^z + 3\sqrt[3]{e^{x+y+z}}}{\sqrt{e^{x+y}} + \sqrt{e^{y+z}} + \sqrt{e^{z+x}}}} \right) dx dy dz \geq \int_a^{2a} \int_b^{2b} \int_c^{2c} \sqrt{2} dx dy dz = \sqrt{2} abc$$

Equality holds for $a = b = c = 0$.

848. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$(b-a)^2 + (b-a) \left(\int_a^b \frac{x \, dx}{\sin x} \right) \cdot e^{\frac{1}{b-a} \int_a^b \log\left(\frac{\sin x}{x}\right) dx} \leq 2 \left(\int_a^b \frac{\sin x \, dx}{x} \right) \left(\int_a^b \frac{x \, dx}{\sin x} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

We have $\varphi(t) = e^t$ is continuous and convex function on \mathbb{R} . Since $\varphi''(t) = e^t > 0$ for

all $x \in \mathbb{R}$ then

$$\varphi \left(\frac{1}{b-a} \int_a^b \log\left(\frac{\sin x}{x}\right) dx \right) \leq \frac{1}{b-a} \int_a^b \varphi \left(\log\left(\frac{\sin x}{x}\right) \right) dx$$

$$\text{But } \varphi \left(\log\left(\frac{\sin x}{x}\right) \right) = \frac{\sin x}{x}. \text{ So, } \varphi \left(\frac{1}{b-a} \int_a^b \log\left(\frac{\sin x}{x}\right) dx \right) \leq \frac{1}{b-a} \int_a^b \frac{\sin x}{x} dx$$

$$\text{We have } (b-a)^2 + (b-a) \left(\int_a^b \frac{x}{\sin x} dx \right) e^{\frac{1}{b-a} \int_a^b \log\left(\frac{\sin x}{x}\right) dx} \leq$$

$$\leq (b-a)^2 + (b-a) \left(\int_a^b \frac{x}{\sin x} dx \right) \left(\frac{1}{b-a} \right) \int_a^b \frac{\sin x}{x} dx \leq$$



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$$\leq (b-a)^2 + \left(\int_a^b \frac{x}{\sin x} dx \right) \left(\int_a^b \frac{\sin x}{x} dx \right)$$

$$\text{But } (b-a)^2 = \left(\int_a^b 1 dx \right)^2 = \left(\int_a^b \left(\frac{x}{\sin x} \right)^{\frac{1}{2}} \times \left(\frac{\sin x}{x} \right)^{\frac{1}{2}} dx \right)^2$$

$$\text{By Cauchy-Schwarz} \leq \left(\int_a^b \frac{x}{\sin x} dx \right) \left(\int_a^b \frac{\sin x}{x} dx \right)$$

$$\text{Therefore: } (b-a)^2 + \left(\int_a^b \frac{x}{\sin x} dx \right) \left(\int_a^b \frac{\sin x}{x} dx \right) \leq 2 \left(\int_a^b \frac{\sin x}{x} dx \right) \left(\int_a^b \frac{x}{\sin x} dx \right)$$

$$\text{Therefore } (b-a)^2 + (b-a) \left(\int_a^b \frac{x}{\sin x} dx \right) e^{\frac{1}{b-a} \int_a^b \log(\frac{\sin x}{x}) dx} \leq 2 \int_a^b \frac{\sin x}{x} dx \int_a^b \frac{x}{\sin x} dx$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(b-a)^2 + (b-a) \left(\int_a^b \frac{x}{\sin x} dx \right) \cdot e^{\frac{1}{b-a} \int_a^b \log(\frac{\sin x}{x}) dx} \stackrel{(a)}{\leq} 2 \left(\int_a^b \frac{\sin x}{x} dx \right) \left(\int_a^b \frac{x}{\sin x} dx \right)$$

The continuous analogue of AM-GM inequality:

$$\Rightarrow \int_a^b \frac{\sin x}{x} dx \stackrel{(1)}{\geq} (b-a) \cdot e^{\frac{1}{b-a} \int_a^b \log(\frac{\sin x}{x}) dx}$$

$$\text{and } \frac{x}{\sin x} > 1 \quad (\because 0 < x < \frac{\pi}{2}) \quad \therefore \int_a^b \frac{x}{\sin x} dx > \int_a^b dx = b-a \stackrel{(2)}{\geq} 0$$

$$(1).(2) \Rightarrow \left(\int_a^b \frac{\sin x dx}{x} \right) \left(\int_a^b \frac{x dx}{\sin x} \right) \stackrel{(3)}{\geq} (b-a) \left(\int_a^b \frac{x dx}{\sin x} \right) \cdot e^{\frac{1}{b-a} \int_a^b \log(\frac{\sin x}{x}) dx}$$

Again, continuous analogue of reverse CBS inequality \Rightarrow

$$\left(\int_a^b \frac{\sin x}{x} dx \right) \left(\int_a^b \frac{x}{\sin x} dx \right) \stackrel{(4)}{\geq} \left(\int_a^b \sqrt{\frac{\sin x}{x}} \sqrt{\frac{x}{\sin x}} dx \right)^2 = (b-a)^2$$

(3)+(4) \Rightarrow RHS of (a) \geq LHS of (a) \Rightarrow (a) is true (Proved)

849. If $f: [a, b] \rightarrow (0, \infty)$, continuous, $a \leq b$ then:

$$\int_a^b \int_a^b \left(\sqrt{\frac{f^2(x) + f^2(y)}{2}} + \sqrt{f(x)f(y)} \right) dx dy \leq 2(b-a) \int_a^b f(x) dx$$

Proposed by Daniel Sitaru – Romania



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Solution 1 by Yuval Peres-USA

It suffices to verify that $\forall x, y \in [a, b]$

$$\sqrt{\frac{f^2(x) + f^2(y)}{2}} + \sqrt{f(x)f(y)} \stackrel{?}{\leq} f(x) + f(y) \quad (*)$$

and then integrate both sides. Squaring, () is equivalent to:*

$$\frac{f^2(x) + f^2(y)}{2} + f(x)f(y) + 2\sqrt{\frac{f^2(x) + f^2(y)}{2}f(x)f(y)} \stackrel{?}{\leq} f^2(x) + f^2(y) + 2f(x)f(y)$$

$$\text{that is: } \sqrt{[f^2(x) + f^2(y)] \cdot 2f(x)f(y)} \stackrel{?}{\leq} \frac{[f^2(x) + f^2(y)] + 2f(x)f(y)}{2}$$

This is just the AM-GM inequality $\sqrt{zw} \leq \frac{z+w}{2}$

Solution 2 by Chris Kyriazis-Athens-Greece

It holds that $\sqrt{\frac{f^2(x) + f^2(y)}{2}} \leq \frac{f(x) + f(y)}{2}, \forall x, y \in \mathbb{R}$ and $\sqrt{f(x)f(y)} \leq \frac{f(x) + f(y)}{2}, \forall x, y \in \mathbb{R}$

So, adding those inequalities, we have:

$$\sqrt{\frac{f^2(x) + f^2(y)}{2}} + \sqrt{f(x)f(y)} \leq f(x) + f(y), \forall x, y \in \mathbb{R}$$

Integrating from a to b, we have:

$$\begin{aligned} \int_a^b \int_a^b \left[\sqrt{\frac{f^2(x) + f^2(y)}{2}} + \sqrt{f(x)f(y)} \right] dx dy &\leq \int_a^b \int_a^b [f(x) + f(y)] dx dy = \\ &= 2(b-a) \int_a^b f(x) dx \quad (*) \end{aligned}$$

$$\begin{aligned} (*) \int_a^b \left\{ \int_a^b [f(x) + f(y)] dx \right\} dy &= \int_a^b \left[\int_a^b f(x) dx + f(y)(b-a) \right] dy = \\ &= \int_a^b f(x) dx (b-a) + \int_a^b f(y) dy (b-a) = 2(b-a) \int_a^b f(x) dx \end{aligned}$$

Solution 3 by Jalil Hajimir-Canada

We know for any two positive real numbers m and n

$$G(m, n) + Q(m, n) \leq m + n$$



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$$\begin{aligned} \therefore \int_a^b \int_a^b [Q(f(x), f(y)) + G(f(x), f(y)) dx dy] dx dy &\leq \int_a^b \int_a^b (f(x) + f(y)) dx dy \\ \int_a^b \int_a^b \sqrt{\frac{f^2(x) + f^2(y)}{2}} + \sqrt{f(x)f(y)} dx dy &\leq 2(b-a) \int_a^b f(x) dx \end{aligned}$$

850. If $p, q, r > 0, n \in \mathbb{N}, n \geq 2, 0 < a \leq b < \pi$ then:

$$\int_a^b \int_a^b \int_a^b \sqrt[n]{\sin\left(\frac{px+qy+rz}{p+q+r}\right)} dx dy dz \geq (b-a)^2 \int_a^b \sqrt[n]{\sin x} dx$$

Proposed by Daniel Sitaru – Romania

Solution by Ali Jaffal-Lebanon

$$\text{Let } f_n(x) = (\sin x)^{\frac{1}{n}}, n \geq 2, f'_n(x) = \frac{1}{n} \cos x (\sin x)^{\frac{1}{n}-1}$$

$$f''_n(x) = -\frac{1}{n} (\sin x)^{\frac{1}{n}} + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \cos^2 x (\sin x)^{\frac{1}{n}-2}$$

we have $f''_n(x) \leq 0$ for all $x \in [0, \pi]$ then f_n is concave on $[0, \pi]$

Let $x, y, z \in [0, \pi]$, so, by Jensen's inequality we have:

$$f_n\left(\frac{px+qy+rz}{p+q+r}\right) \geq \frac{pf(x)+qf(y)+rf(z)}{p+q+r}$$

$$\int_a^b \int_a^b \int_a^b f_n\left(\frac{px+qy+rz}{p+q+r}\right) dx dy dz \geq \int_a^b \int_a^b \int_a^b \frac{pf(x)+qf(y)+rf(z)}{p+q+r} dx dy dz$$

$$\int_a^b \int_a^b \int_a^b \frac{pf_n(x)+qf_n(y)+rf_n(z)}{p+q+r} dx dy dz =$$

$$\frac{p}{p+q+r} \int_a^b dy \int_a^b dy \int_a^b f_n(x) dx + \frac{q}{p+q+r} \int_a^b dz \int_a^b dx \int_a^b f_n(y) dy +$$

$$+ \int_a^b dy \int_a^b dx \int_a^b \frac{rf_n(z)}{p+q+r} dz =$$



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$$\frac{p}{p+q+r}(b-a)^2 \int_a^b f_n(x) dx + \frac{q}{p+q+r}(b-a)^2 \int_a^b f_n(x) dx + \frac{(b-a)^2 r}{p+q+r} \int_a^b f_n(x) dx =$$

$$\frac{(p+q+r)(b-a)^2}{p+q+r} \int_a^b f_n(x) dx = (b-a)^2 \int_a^b f_n(x) dx. \text{ So,}$$

$$\int_a^b \int_a^b \int_a^b f_n\left(\frac{px+qy+rz}{p+q+r}\right) dx dy dz \geq (b-a)^2 \int_a^b f_n(x) dx$$

$$\text{Therefore: } \int_a^b \int_a^b \int_a^b \sqrt[n]{\sin\left(\frac{px+qy+rz}{p+q+r}\right)} dx dy dz \geq (b-a)^2 \int_a^b \sqrt[n]{\sin x} dx$$

851. If $f: \mathbb{R} \rightarrow \mathbb{R}$, f – continuous, $f(x) + f(y) \geq 3f(x+y)$, $\forall x, y \in \mathbb{R}$ then:

$$3 \int_0^1 \int_0^1 \int_0^1 f(x+y+z) dx dy dz \leq 5 \int_0^1 f(x) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

$$f(x) + f(y) \geq 3f(x+y) \Rightarrow f(x+y) \leq \frac{f(x) + f(y)}{3} \Rightarrow$$

$$\Rightarrow f(x+y+z) \leq \frac{f(x+y) + f(z)}{3} \leq \frac{\frac{f(x) + f(y)}{3} + f(z)}{3} = \frac{f(x) + f(y) + 3f(z)}{9}$$

$$\text{Similarly: } f(x+y+z) \leq \frac{f(x)+f(z)+3f(y)}{9}; f(x+y+z) \leq \frac{f(y)+f(z)+3f(x)}{9}$$

$$\text{So, } 3f(x+y+z) \leq \frac{5(f(x)+f(y)+f(z))}{9}$$

$$3 \int_0^1 \int_0^1 \int_0^1 f(x+y+z) dx dy dz \leq \frac{5}{9} \int_0^1 \int_0^1 \int_0^1 (f(x) + f(y) + f(z)) dx dy dz =$$

$$= \frac{5}{9} \cdot 3 \int_0^1 f(x) dx = \frac{5}{3} \int_0^1 f(x) dx \leq 5 \int_0^1 f(x) dx$$

852. $\Omega_1(t) = \int_0^1 \left(\frac{x^{\sin^2 t}}{1+x^{\cos^2 t}} \right) dt$, $\Omega_2(t) = \int_0^1 \left(\frac{x^{\cos^2 t}}{1+x^{\sin^2 t}} \right) dt$



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Prove that:

$$\Omega_1(t) + \Omega_2(t) \geq \frac{4 - \sin 2t}{8 + \sin 2t}, t \in \left[0, \frac{\pi}{2}\right]$$

Proposed by Daniel Sitaru – Romania

Solution by Șerban George Florin – Romania

$$\begin{aligned} \frac{x^{\sin^2 t}}{1+x^{\cos^2 t}} + \frac{x^{\cos^2 t}}{1+x^{\sin^2 t}} &\geq 2 \sqrt{\frac{x^{\sin^2 t} + \cos^2 t}{(1+x^{\cos^2 t})(1+x^{\sin^2 t})}} = \\ &= 2 \sqrt{\frac{x}{(1+x^{\cos^2 t})(1+x^{\sin^2 t})}} \geq 2 \sqrt{\frac{x}{(1+1)(1+1)}} = \frac{2\sqrt{x}}{2} = \sqrt{x} \end{aligned}$$

because: $x^{\cos^2 t} \leq 1 = x^0, (\forall)x \in [0, 1], 0 \leq \cos^2 t \leq 1$ and

$$x^{\sin^2 t} \leq 1 = x^0, (\forall)x \in [0, 1], 0 \leq \sin^2 t \leq 1$$

$$\begin{aligned} \Rightarrow \Omega_1(t) + \Omega_2(t) &= \int_0^1 \left(\frac{x^{\sin^2 t}}{1+x^{\cos^2 t}} + \frac{x^{\cos^2 t}}{1+x^{\sin^2 t}} \right) dt \geq \int_0^1 \sqrt{t} dt = \\ &= \int_0^1 t^{\frac{1}{2}} dt = \frac{\frac{1}{2}+1}{\frac{1}{2}+1} \Big|_0^1 = \frac{\frac{3}{2}}{\frac{3}{2}} \Big|_0^1 = \frac{2}{3} \geq \frac{4 - \sin 2t}{8 + \sin 2t} \end{aligned}$$

$$\sin 2t = y \geq 0, t \in \left[0, \frac{\pi}{2}\right] \Rightarrow 2t \in [0, \pi] \Rightarrow 8 + y > 0 \text{ and}$$

$$4 - y > 0 \text{ because } y \in [0, 1] \Rightarrow 2(8 + y) \geq 3 \cdot (4 - y), 16 + 2y \geq 12 - 3y$$

$$\Rightarrow 4 + 5y \geq 0, \text{ true, because } y \geq 0$$

853. Prove without softs:

$$\int_0^1 \int_0^1 \int_0^1 \frac{z\sqrt{xy}}{(x^2+1)(y+1)} dx dy dz < \frac{1}{6}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Soumava Chakraborty-Kolkata-India

Proof : Let $\sqrt{x} = \theta \therefore x = \theta^2 \Rightarrow dx = 2\theta d\theta$ and let $\sqrt{y} = \varphi \therefore y = \varphi^2 \Rightarrow dy = 2\varphi d\varphi$



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$$\therefore \int_0^1 \int_0^1 \int_0^1 \frac{z\sqrt{xy}}{(x^2+1)(y+1)} dx dy dz \stackrel{(1)}{\cong} 4 \int_0^1 \int_0^1 \int_0^1 \frac{z\theta^2\varphi^2}{(\theta^4+1)(\varphi^2+1)} d\theta d\varphi dz$$

$$\text{Let } \Omega = \int_0^1 \frac{\theta^2 d\theta}{\theta^4 + 1} \quad \text{Let } \theta^2 = \tan\omega \left(0 \leq \omega \leq \frac{\pi}{4}\right) \therefore 2\theta d\theta = \sec^2 \omega d\omega \Rightarrow d\theta = \frac{\sec^2 \omega d\omega}{2\sqrt{\tan\omega}}$$

$$\therefore \Omega = \int_0^{\frac{\pi}{4}} \frac{\tan\omega \sec^2 \omega d\omega}{2\sec^2 \omega \sqrt{\tan\omega}} = \frac{1}{2} \int_0^{\frac{\pi}{4}} \sqrt{\tan\omega} d\omega \stackrel{(2)}{\cong} \frac{1}{2} \int_0^{\frac{\pi}{4}} \sqrt{\cot\omega} d\omega$$

$$\text{Let } I = \int \sqrt{\tan\omega} d\omega \text{ and let } J = \int \sqrt{\cot\omega} d\omega$$

$$\begin{aligned} I + J &= \int (\sqrt{\tan\omega} + \sqrt{\cot\omega}) d\omega = \sqrt{2} \int \frac{(\sin\omega + \cos\omega)}{\sqrt{2\sin\omega\cos\omega}} d\omega = \\ &= -\sqrt{2} \int \frac{-(\sin\omega + \cos\omega) d\omega}{\sqrt{1 - (\cos\omega - \sin\omega)^2}} = -\sqrt{2} \int \frac{dt}{\sqrt{1 - t^2}} (t = \cos\omega - \sin\omega) \\ &\stackrel{(a)}{\cong} -\sqrt{2} \sin^{-1}(\cos\omega - \sin\omega) + C \end{aligned}$$

$$\begin{aligned} I - J &= \int (\sqrt{\tan\omega} - \sqrt{\cot\omega}) d\omega = \sqrt{2} \int \frac{(\sin\omega - \cos\omega)}{\sqrt{2\sin\omega\cos\omega}} d\omega = \\ &= -\sqrt{2} \int \frac{(\cos\omega - \sin\omega) d\omega}{\sqrt{(\sin\omega + \cos\omega)^2 - 1}} = -\sqrt{2} \int \frac{dm}{\sqrt{m^2 - 1}} (t = \sin\omega + \cos\omega) \\ &\stackrel{(b)}{\cong} -\sqrt{2} \ln|\sin\omega + \cos\omega + \sqrt{2\sin\omega\cos\omega}| + K \end{aligned}$$

$$\begin{aligned} (a) + (b) &\Rightarrow 2I = 2 \int \sqrt{\tan\omega} d\omega = \\ &= -\sqrt{2} \sin^{-1}(\cos\omega - \sin\omega) - \sqrt{2} \ln|\sin\omega + \cos\omega + \sqrt{2\sin\omega\cos\omega}| + C + K \end{aligned}$$

$$\begin{aligned} \therefore 2 \int_0^{\frac{\pi}{4}} \sqrt{\tan\omega} d\omega &= -\sqrt{2} \sin^{-1}\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) - \\ &- \sqrt{2} \ln\left|\sin\frac{\pi}{4} + \cos\frac{\pi}{4} + \sqrt{2\sin\frac{\pi}{4}\cos\frac{\pi}{4}}\right| + \sqrt{2} \sin^{-1}(\cos 0 - \sin 0) + \sqrt{2} \ln|\sin 0 + \cos 0 + \sqrt{2\sin 0\cos 0}| \end{aligned}$$

$$= -\sqrt{2} \ln(\sqrt{2} + 1) + \sqrt{2} \left(\frac{\pi}{2}\right) \Rightarrow \frac{1}{2} \int_0^{\frac{\pi}{4}} \sqrt{\tan\omega} d\omega \stackrel{(3)}{\cong} \frac{\sqrt{2}}{4} \left(\frac{\pi}{2} - \ln(\sqrt{2} + 1)\right)$$



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$$So, (2), (3) \Rightarrow \int_0^1 \frac{\theta^2 d\theta}{\theta^4 + 1} \stackrel{(i)}{=} \frac{\sqrt{2}}{4} \left(\frac{\pi}{2} - \ln(\sqrt{2} + 1) \right)$$

$$Again, \int_0^1 \frac{\varphi^2 d\varphi}{\varphi^2 + 1} = \int_0^1 \frac{(\varphi^2 + 1)d\varphi}{\varphi^2 + 1} - \int_0^1 \frac{d\varphi}{\varphi^2 + 1} = 1 - \tan^{-1} 1 = \frac{4 - \pi}{4} \Rightarrow \int_0^1 \frac{\varphi^2 d\varphi}{\varphi^2 + 1} \stackrel{(ii)}{=} \frac{4 - \pi}{4}$$

$$(1), (i) \Rightarrow \int_0^1 \int_0^1 \int_0^1 \frac{z\sqrt{xy}}{(x^2 + 1)(y + 1)} dx dy dz = 4 \left(\frac{\sqrt{2}}{4} \right) \left(\frac{\pi}{2} - \ln(\sqrt{2} + 1) \right) \int_0^1 \int_0^1 \frac{z\varphi^2 d\varphi dz}{\varphi^2 + 1}$$

$$= \sqrt{2} \left(\frac{\pi}{2} - \ln(\sqrt{2} + 1) \right) \left(\frac{4 - \pi}{4} \right) \int_0^1 z dz$$

$$= \boxed{\frac{\sqrt{2}}{8} \left(\frac{\pi}{2} - \ln(\sqrt{2} + 1) \right) (4 - \pi)} \approx 1046 < \frac{1}{6} \quad (Proved)$$

854. Let f be continuous on $[0, 1]$. If $af(b) + bf(a) \leq 2; \forall b \in [0, 1]$, prove:

$$\int_0^1 f(x) dx \leq \frac{\pi}{2}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Khaled Abd Imouti-Damascus-Syria

$$Suppose: r = \sqrt{a^2 + b^2}. So: \left(\frac{a}{\sqrt{a^2 + b^2}} \right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}} \right)^2 = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

$$So: \exists \theta \in \mathbb{R}: \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin \theta = \frac{b}{\sqrt{a^2 + b^2}}; \cos \theta f(b) + \sin \theta f(a) \leq \frac{2}{\sqrt{a^2 + b^2}}$$

For: $b = \sin \theta, a = \cos \theta \in [0, 1]$. Then: $\cos \theta f(\sin \theta) + \sin \theta f(\cos \theta) \leq 2$

$$I_1 = \int_0^1 f(x) dx = \int_0^{\frac{\pi}{2}} f(\sin \theta) \cdot \cos \theta d\theta \begin{cases} x = \sin \theta \\ dx = \cos \theta d\theta \end{cases} \begin{cases} x = 0 \Rightarrow \theta = 0 \\ x = 1 \Rightarrow \theta = \frac{\pi}{2} \end{cases}$$

$$I_2 = \int_0^1 f(x) dx = \int_{\frac{\pi}{2}}^0 -f(\cos \theta) \sin \theta d\theta \begin{cases} y = \cos \theta \\ dy = -\sin \theta d\theta \end{cases}$$

$$\begin{cases} y = 0 \Rightarrow \theta = \frac{\pi}{2} \\ y = 1 \Rightarrow \theta = 0 \end{cases}$$



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$$2 \int_0^1 f(x) dx = \int_0^{\frac{\pi}{2}} f(\sin \theta) \cos \theta d\theta + \int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta d\theta \leq 2 \int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}} = \pi$$

$$\int_0^1 f(x) dx \leq \frac{\pi}{2}$$

855. If $f: \mathbb{R} \rightarrow (0, \infty)$, $a, b \in \mathbb{R}$, $a \leq b$ then:

$$\int_a^b \int_a^b \left((f(x) + f(y)) \tan^{-1} \left(\frac{f(x) + f(y)}{2} \right) \right) dx dy \leq 2(b-a) \int_a^b (f(x) \tan^{-1}(f(x))) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Florentin Vișescu-Romania

$$f(t) = \arctan t; f'(t) = \arctan t + \frac{t}{1+t^2}$$

$$f''(t) = \frac{1}{1+t^2} + \frac{1+t^2 - 2t^2}{(1+t^2)^2} = \frac{1}{1+t^2} + \frac{1-t^2}{(1+t^2)^2} = \frac{1+t^2 - t^2}{(1+t^2)^2}$$

$$= \frac{2}{(1+t^2)^2} > 0 \Rightarrow f \text{ convexe} \Rightarrow f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

$$\frac{x+y}{2} \arctan \frac{x+y}{2} \leq \frac{1}{2}(x \arctan x + y \arctan y)$$

$$(x+y) \arctan \frac{x+y}{2} \leq x \arctan x + y \arctan y; \forall x, y \in \mathbb{R}$$

$$x \rightarrow f(x); y \rightarrow f(y)$$

$$\int_a^b \int_a^b [f(x) + f(y)] \arctan \frac{f(x) + f(y)}{2} dx dy \leq \int_a^b \int_a^b f(x) \arctan(x) + f(y) \arctan(y) dx dy$$

$$\int_a^b \int_a^b (f(x) + f(y)) \arctan \frac{f(x) + f(y)}{2} dx dy \leq 2(b-a) \int_a^b f(x) \arctan(x) dx$$

856. Prove without softs:

$$\int_0^1 x^x (1-x)^x dx < \frac{3}{4}$$



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Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$x \in [0, 1] \Rightarrow x \geq 0, 1-x \geq 0; x(1-x) \stackrel{AM-GM}{\leq} \left(\frac{x+1-x}{2}\right)^2 = \frac{1}{4}$$

$$(x(1-x))^x \leq \left(\frac{1}{4}\right)^x; \int_0^1 x^x (1-x)^x dx < \int_0^1 \left(\frac{1}{4}\right)^x dx = \frac{1}{\log \frac{1}{4}} \left(\left(\frac{1}{4}\right)^1 - \left(\frac{1}{4}\right)^0\right)$$

$$\int_0^1 x^x (1-x)^x dx < \frac{1}{-\log 4} \left(\frac{1}{4} - 1\right) = \frac{3}{4 \log 4} < \frac{3}{4}$$

857. If $a, b \in \mathbb{R}, a \leq b, f: \mathbb{R} \rightarrow \left[0, \frac{\pi}{2}\right], f$ – continuous then:

$$4 \int_a^b \csc(2f(x)) dx + \int_a^b \cos\left(\frac{\pi}{4} - f(x)\right) dx \geq 5(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution by Khaled Abd Imouti-Damascus-Syria

$$\int_a^b (4 \csc(2f(x)) + \cos\left(\frac{\pi}{4} - f(x)\right) dx \stackrel{?}{\geq} 5(b-a)$$

Let be the function: $g(x) = 4 \csc(2x) + \cos\left(\frac{\pi}{4} - x\right), x \in \left[0, \frac{\pi}{2}\right]$

$$g(x) = \frac{4}{\sin 2x} + \cos\left(\frac{\pi}{4} - x\right), x \in \left[0, \frac{\pi}{2}\right]; \lim_{x \rightarrow 0^+} [g(x)] = +\infty, \lim_{x \rightarrow \frac{\pi}{2}^-} [g(x)] = +\infty$$

$$g'(x) = \frac{-8 \cos 2x}{\sin^2 2x} + \sin\left(\frac{\pi}{4} - x\right); g'(x) = 0 \Rightarrow x = \frac{\pi}{4}, g\left(\frac{\pi}{4}\right) = 5$$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$g'(x)$	—	-0 + + + + + + +	
$g(x)$	$+\infty$	5	$+\infty$

So: $\int_a^b (4 \csc(2f(x)) + \cos\left(\frac{\pi}{4} - f(x)\right) dx \geq \int_a^b 5 dx = 5(b-a)$



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858. Prove without softs:

$$\Omega = \int_0^1 \sqrt{\frac{x^x \sin(\pi x)}{x^x + (1-x)^{1-x}}} dx < \frac{1}{\sqrt{\pi}}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\begin{aligned} \Omega &= \int_0^1 \left(\frac{x^x \sin(\pi x)}{x^x + (1-x)^{1-x}} \right) dx \stackrel{1-x=y}{=} \int_1^0 \left(\frac{(1-y)^{1-y} \sin(\pi - \pi y)}{y^y + (1-y)^{1-y}} \right) (-dy) = \\ &= \int_0^1 \left(\frac{(1-y)^{1-y} \sin(\pi y)}{y^y + (1-y)^{1-y}} \right) dy = \int_0^1 \left(\frac{(1-x)^{1-x} \sin(\pi x)}{x^x + (1-x)^{1-x}} \right) dx \\ 2\Omega &= \int_0^1 \left(\frac{(x^x + (1-x)^{1-x}) \sin(\pi x)}{x^x + (1-x)^{1-x}} \right) dx = \int_0^1 (\sin(\pi x)) dx = \frac{2}{\pi}; \quad \Omega = \frac{1}{\pi} \\ \int_0^1 \left(\sqrt{\frac{x^x \sin(\pi x)}{x^x + (1-x)^{1-x}}} \right) dx &\stackrel{CBS}{<} \sqrt{\Omega \cdot \int_0^1 dx} = \frac{1}{\sqrt{\pi}} \end{aligned}$$

859. Prove without softs:

$$\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} ((\sin x + \cos x)^{\sec^2 x - 1} + (\sin x + \cos x)^{\csc^2 x - 1}) dx > \frac{3}{2}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$x \in \left[\frac{\pi}{8}, \frac{3\pi}{8} \right] \Rightarrow 2x \in \left[\frac{\pi}{4}, \frac{3\pi}{4} \right] \Rightarrow \sin 2x \geq \frac{\sqrt{2}}{2}$$

$$\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} ((\sin x + \cos x)^{\sec^2 x - 1} + (\sin x + \cos x)^{\csc^2 x - 1}) dx \stackrel{AM-GM}{\leq}$$



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$$\begin{aligned}
 & \geq \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left((2\sqrt{\sin x \cos x})^{\tan^2 x} + (2\sqrt{\sin x \cos x})^{\cot^2 x} \right) dx \stackrel{AM-GM}{\geq} \\
 & \geq 2 \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\sqrt{(2\sqrt{\sin x \cos x})^{\tan^2 x + \cot^2 x}} \right) dx \stackrel{AM-GM}{\geq} 2 \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\sqrt{(2\sqrt{\sin x \cos x})^{2\tan x \cot x}} \right) dx = \\
 & = 2 \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\sqrt{(\sqrt{2\sin 2x})^2} \right) dx = 2 \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (\sqrt{\sin 2x}) dx > 2 \sqrt{2 \cdot \frac{\sqrt{2}}{2}} \left(\frac{3\pi}{8} - \frac{\pi}{8} \right) = \\
 & = 2^{\frac{5}{4}} \sqrt{2} \cdot \frac{\pi}{4} > \frac{3}{2} \Leftrightarrow \pi^{\frac{5}{4}} \sqrt{2} > 3 \Leftrightarrow 2\pi^4 > 81
 \end{aligned}$$

860. If $0 < a \leq b$ then:

$$\left(\int_a^b e^{-13x^2} dx \right) \left(\int_a^b e^{-8x^2} dx \right) \geq \left(\int_a^b e^{-10x^2} dx \right) \left(\int_a^b e^{-11x^2} dx \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$\begin{aligned}
 & \left(\int_a^b e^{-13x^2} dx \right) \left(\int_a^b e^{-8x^2} dx \right) = \left(\int_a^b (e^{-2x^2})^5 \cdot e^{-3x^2} dx \right) \left(\int_a^b (e^{-2x^2})^4 dx \right) \geq \\
 & \stackrel{CEBYSHEV}{\geq} \left(\int_a^b (e^{-2x^2})^5 dx \right) \left(\int_a^b (e^{-2x^2})^4 \cdot e^{-3x^2} dx \right) = \left(\int_a^b e^{-10x^2} dx \right) \left(\int_a^b e^{-11x^2} dx \right)
 \end{aligned}$$

Equality holds for $a = b$.

Solution 2 by Sergio Esteban-Argentina

Let $f(x) = e^{-18x^2}$, $x, y \in [a, b]$ since f is positive and nonincreasing, it follows that

$$\varphi = \int_a^b \left[\int_a^b f(x) f(y) (e^{-5x^2} - e^{-(2x^2+3y^2)} - e^{-(2y^2+3x^2)} + e^{-5y^2}) dx \right] dy \geq 0$$

we would just have to prove that: $e^{-5y^2} + e^{-5x^2} \geq e^{-(2x^2+3y^2)} + e^{-(2y^2+3x^2)}$



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The function $9(\tau) = e^\tau$ is a convex on \mathbb{R} , so according to Karamata's inequality it suffices to prove that without loss of generality $x \geq y > 0$ such that $(-5y^2, -5x^2)$

majorizes $(-2x^2 - 3y^2, -2y^2 - 3x^2)$

i) $-5y^2 \geq -5x^2$ and $-(2x^2 + 3y^2) \geq -(2y^2 + 3x^2)$

ii) $-5y^2 - 5x^2 \geq -(2x^2 + 3y^2) - (2y^2 + 3x^2)$ true! Expanding φ , we obtain

$$\begin{aligned} \varphi &= \int_a^b f(x) e^{-5x^2} dx \cdot \int_a^b f(y) dy - \int_a^b f(x) \cdot e^{-2x^2} dx \int_a^b f(y) \cdot e^{-3y^2} dy - \\ &\quad - \int_a^b f(y) \cdot e^{-2y^2} dy \cdot \int_a^b f(x) \cdot e^{-3x^2} dx + \int_a^b f(y) \cdot e^{-5y^2} dy \int_a^b f(x) dx \\ &= 2 \int_a^b f(x) e^{-5x^2} dx \cdot \int_a^b f(x) dx - 2 \int_a^b f(x) \cdot e^{-2x^2} dx \int_a^b f(x) \cdot e^{-3x^2} dx \geq 0 \end{aligned}$$

the proof is now complete.

861. Let $f: [a, b] \rightarrow \mathbb{R}$, be continuos and increasing. Prove that:

$$\int_a^b \int_a^b \int_a^b \left(\frac{xe^{f(x)} + ye^{f(y)} + ze^{f(z)}}{x+y+z} \right) dx dy dz \geq (b-a)^2 \int_a^b e^{f(x)} dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

Let $x, y, z \in [a, b]$. WLOG: $x \leq y \leq z$.

f –increasing $\Rightarrow f(x) \leq f(y) \leq f(z) \Rightarrow e^{f(x)} \leq e^{f(y)} \leq e^{f(z)}$

By Cebyshev's inequality: $xe^{f(x)} + ye^{f(y)} + ze^{f(z)} \geq \frac{1}{3}(x+y+z)(e^{f(x)} + e^{f(y)} + e^{f(z)})$

$$\frac{xe^{f(x)} + ye^{f(y)} + ze^{f(z)}}{x+y+z} \geq \frac{1}{3}(e^{f(x)} + e^{f(y)} + e^{f(z)})$$

$$\int_a^b \int_a^b \int_a^b \left(\frac{xe^{f(x)} + ye^{f(y)} + ze^{f(z)}}{x+y+z} \right) dx dy dz \geq \int_a^b \int_a^b \int_a^b \left(\frac{1}{3}(e^{f(x)} + e^{f(y)} + e^{f(z)}) \right) dx dy dz =$$



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$$= \frac{1}{3} \cdot 3 \int_a^b \int_a^b \int_a^b f(x) dx dy dz = (b-a)^2 \int_a^b e^{f(x)} dx$$

862. If $f: \mathbb{R} \rightarrow (0, \infty)$, f – continuous, $a, b \in \mathbb{R}$, $a \leq b$ then:

$$\int_a^b \int_a^b \left(\log \left(\frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} \right) \right) dx dy \leq (b-a) \int_a^b f^2(x) dx - \left(\int_a^b f(x) dx \right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$(1+f(x))(1+f(y)) \stackrel{MG \leq MA}{\leq} \left(\frac{1+f(x)+1+f(y)}{2} \right)^2 = \left(1 + \frac{f(x)+f(y)}{2} \right)^2 \Rightarrow$$

$$\Rightarrow \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} \leq 1 \Rightarrow \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} < 0$$

$$a < b \Rightarrow \int_a^b \int_a^b \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} dx dy < 0 \quad (1)$$

$$\left(\int_a^b (f(x) \cdot 1) dx \right)^2 \stackrel{C.B.S.}{\leq} \int_a^b f^2(x) dx \cdot \int_a^b 1 dx = x \Big|_a^b \cdot \int_a^b f^2(x) dx =$$

$$= (b-a) \int_a^b f^2(x) dx \Rightarrow (b-a) \int_a^b f^2(x) dx - \left(\int_a^b f(x) dx \right)^2 \geq 0 \quad (2)$$

$$\text{From (1) and (2)} \Rightarrow \int_a^b \int_a^b \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} dx dy \leq (b-a) \int_a^b f^2(x) dx - \left(\int_a^b f(x) dx \right)^2$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $F(x) = x^2 - \ln(1+x)$ for all $x \geq 0$ then $F'(x) = 2x - \frac{1}{1+x}$, $F''(x) = 2 + \frac{1}{(1+x)^2} > 0$

hence f is convex function $\therefore \frac{f^2(x)-\ln(1+f(x))+f^2(y)-\ln(1+f(y))}{2} \geq$

$$\geq \left(\frac{f(x)+f(y)}{2} \right)^2 - \ln \left(1 + \frac{f(x)+f(y)}{2} \right)$$

$$\Rightarrow \frac{f^2(x)+f^2(y)}{2} - \left(\frac{f(x)+f(y)}{2} \right)^2 \geq \frac{\ln(1+f(x))(1+f(y))}{2} - \ln \left(1 + \frac{f(x)+f(y)}{2} \right)$$



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$$\begin{aligned}
 & \Rightarrow \frac{f^2(x) + f^2(y) - 2f(x)f(y)}{2} \geq \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} \\
 & \Rightarrow \frac{1}{2} \int_a^b \int_a^b f^2(x) dx dy + \frac{1}{2} \int_a^b \int_a^b f^2(y) dy dx - \left(\int_a^b f(x) dx \right) \left(\int_a^b f(y) dy \right) \\
 & \quad \geq \int_a^b \int_a^b \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} dx dy \\
 & \therefore (b-a) \int_a^b f^2(x) dx - \left(\int_a^b f(x) dx \right)^2 \geq \int_a^b \int_a^b \ln \frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2} dx dy
 \end{aligned}$$

863. Prove:

$$\frac{\pi}{16} < \int_0^1 \sqrt{\frac{x(1-x)}{\sin \pi x + \cos \pi x + 2}} dx < \frac{\pi}{8}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned}
 & \int_0^1 \sqrt{x(1-x)} dx = \int_0^{\frac{1}{2}} \sqrt{\frac{1}{4} - \left(\frac{1}{2}-x\right)^2} dx + \int_{\frac{1}{2}}^1 \sqrt{\frac{1}{4} - \left(x-\frac{1}{4}\right)^2} dx \\
 & = \left[\frac{1}{8} \sin^{-1} \frac{\frac{1}{2}-x}{\frac{1}{2}} + \frac{1}{2} \sqrt{x(1-x)} \right]_{x=0}^{x=\frac{1}{2}} + \left[\frac{1}{8} \sin^{-1} \frac{x-\frac{1}{2}}{\frac{1}{2}} + \frac{x-\frac{1}{2}}{2} \sqrt{x(x-1)} \right]_{x=\frac{1}{2}}^{x=1} \\
 & = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}. \text{ Again, } \frac{1}{2} \geq \sqrt{x(1-x)} \text{ for all } x \geq 0
 \end{aligned}$$

Let $f(x) = \sin \pi x + \cos \pi x + 2$ for all $x \geq 0$

$$f'(x) = \pi(\cos \pi x - \sin \pi x), f''(x) = -\pi^2(\sin \pi x + \cos \pi x)$$

$$\text{for } f'(x) = 0 \Rightarrow \tan \pi x = 1 \Rightarrow x = \frac{1}{4} (0 \leq x \leq 1)$$



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$$\therefore f''\left(\frac{1}{4}\right) = -\sqrt{2}\pi^2 < 0 \text{ hence } f \text{ has local maximum at } x = \frac{1}{4}$$

$$\text{So, } \sin \frac{\pi}{4} + \cos \frac{\pi}{4} + 2 \geq f(x) \geq \sin 0 + \cos 0 + 2 = 3$$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{3}} &\geq \frac{1}{\sqrt{\sin \pi x + \cos \pi x + 2}} \geq \frac{1}{\sqrt{2 + \sqrt{2}}} \Rightarrow \frac{1}{2\sqrt{3}} \geq \sqrt{\frac{x(1-x)}{\sin \pi x + \cos \pi x + 2}} \geq \frac{\sqrt{x(1-x)}}{\sqrt{2 + \sqrt{2}}} \\ \Rightarrow \frac{1}{2\sqrt{3}} \int_0^1 dx &\geq \int_0^1 \sqrt{\frac{x(1-x)}{\sin \pi x + \cos \pi x + 2}} dx \geq \frac{1}{\sqrt{2 + \sqrt{2}}} \int_0^1 \sqrt{x(1-x)} dx \\ \Rightarrow \frac{\pi}{8} &> \frac{1}{2\sqrt{3}} \int_0^1 \sqrt{\frac{x(1-x)}{\sin \pi x + \cos \pi x + 2}} dx \geq \frac{\pi}{8\sqrt{2+\sqrt{2}}} > \frac{\pi}{16} \text{ Hence proved} \end{aligned}$$

864. $f: [0, 1] \rightarrow (0, \infty)$, f – continuous. Find: $\Omega = \min_{k \in \mathbb{R}} K$ such that:

$$\int_0^1 \int_0^1 \int_0^1 ((f(x) + f(y))^{f(z)} (f(y) + f(z))^{f(x)} (f(z) + f(x))^{f(y)}) dx dy dz \leq K \int_0^1 f(x) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } f(x) = u; f(y) = v; f(z) = t \Rightarrow u, v, t > 0$$

We choose $u, v, t > 0$ such that $u + v + t = 1$. Using Jensen's Inequality:

$$\begin{aligned} u \ln(t+v) + v \ln(u+t) + t \cdot \ln(u+v) &\leq \ln(2[uv+vt+tu]) \\ \Rightarrow (u+v)^t \cdot (u+t)^u \cdot (t+v)^v &\leq e^{\ln(2[uv+vt+tu])} = 2(uv+vt+tu) \leq \\ &\leq 2 \cdot \frac{(u+v+t)^2}{3} = \frac{2}{3}(u+v+t) \\ \Rightarrow \int_0^1 \int_0^1 \int_0^1 &(f(x) + f(y))^{f(z)} (f(y) + f(z))^{f(x)} (f(z) + f(x))^{f(y)} dx dy dz \leq \\ &\leq \frac{2}{3} \int_0^1 \int_0^1 \int_0^1 [f(x) + f(y) + f(z)] dx dy dz \\ &= \frac{2}{3} \left[\left\{ \int_0^1 f(x) dx \int_0^1 dy \int_0^1 dz \right\} + \left\{ \int_0^1 f(y) dy \int_0^1 dz \int_0^1 dx \right\} + \left\{ \int_0^1 f(z) dz \int_0^1 dx \int_0^1 dy \right\} \right] \end{aligned}$$



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$$\begin{aligned}
 &= \frac{2}{3} \left[\int_0^1 f(x) dx + \int_0^1 f(y) dy + \int_0^1 f(z) dz \right] \\
 &= \frac{2}{3} \cdot 3 \cdot \int_0^1 f(x) dx = 2 \int_0^1 f(x) dx \Rightarrow K = 2 \Rightarrow \Omega = 2
 \end{aligned}$$

865. If $0 < a \leq b, n \in \mathbb{N} - \{0\}$ then:

$$(b-a)^{n-1} \int_a^b \left(\prod_{k=1}^n \operatorname{erf}(kx) \right) dx \geq \prod_{k=1}^n \int_a^b \operatorname{erf}(kx) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Avishek Mitra-West Bengal-India

$$\begin{aligned}
 \Leftrightarrow \operatorname{erf}(kx) &= \frac{2}{\sqrt{\pi}} \int_0^{kx} e^{-t^2} dt \Rightarrow \operatorname{erf}'(kx) = \frac{2k}{\sqrt{\pi}} e^{-k^2 x^2} > 0 \quad [\text{for all } x \in \mathbb{R}] \\
 \Leftrightarrow \operatorname{erf}'(kx) &> 0 \text{ in } x \in [a, b] - \{0\}
 \end{aligned}$$

\Leftrightarrow Also, we know for $x_1 < x_2 \Rightarrow \operatorname{erf}(kx_1) < \operatorname{erf}(kx_2) \Rightarrow$ for any a_1, b_1 in the interval

$$[a, b] - \{0\} \text{ if } a_1 \leq b_1, \operatorname{erf}(ka_1) < \operatorname{erf}(kb_1); [k = \overline{1, n}]$$

$\Leftrightarrow f(x) = \operatorname{erf}(kx)$ is monotonically increasing in $[a, b] - \{0\}$

$$\begin{aligned}
 \Leftrightarrow \text{also } I &= \int_a^b \operatorname{erf}(kx) dx = \frac{1}{k} \int_a^b \operatorname{erf}(z) dz = \frac{1}{k} \left[z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}} \right]_a^b \\
 &= \frac{1}{k} \left[b \operatorname{erf}(b) - a \operatorname{erf}(a) + \frac{e^{-b^2} - e^{-a^2}}{\sqrt{\pi}} \right]
 \end{aligned}$$

$\Leftrightarrow f(x) = \operatorname{erf}(kx)$ is the integrable between $[a, b] - \{0\}$

$$\Leftrightarrow (b-a)^{n-1} \int_a^b f_1(x) \cdot f_2(x) \cdot \dots \cdot f_n(x) dx \stackrel{\text{chebyshev}}{\geq} \int_a^b f_1(x) dx \cdot \int_a^b f_2(x) dx \dots \int_a^b f_n(x) dx$$

[Let us denote $f_1(x) = \operatorname{erf}(x), f_2(x) = \operatorname{erf}(2x) \dots f_n(x) = \operatorname{erf}_n(x)$]

$$\Leftrightarrow (b-a)^{n-1} \int_a^b \operatorname{erf}(x) \operatorname{erf}(2x) \dots \operatorname{erf}(nx) dx \geq \int_a^b \operatorname{erf}(x) dx \int_a^b \operatorname{erf}(2x) dx \dots \int_a^b \operatorname{erf}(nx) dx$$



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$$\Leftrightarrow (b-a)^{n-1} \int_a^b \prod_{k=1}^n \operatorname{erf}(kx) dx \geq \prod_{k=1}^n \int_a^b \operatorname{erf}(kx) dx$$

866. If $f: [0, 1] \rightarrow (0, \infty)$, f – continuous then:

$$2 \left(\int_0^1 \sqrt{f(x)} dx \right)^2 \leq \int_0^1 f(x) dx + \left(\int_0^1 \sqrt[3]{f(x)} dx \right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

Using Cauchy – Schwarz inequality:

$$\begin{aligned} \left(\int_0^1 \sqrt{f(x)} dx \right)^2 &= \left(\int_0^1 \sqrt{f(x)} \cdot 1 dx \right)^2 \leq \int_0^1 1^2 dx \cdot \int_0^1 f(x) dx = \int_0^1 f(x) dx \\ \left(\int_0^1 \sqrt{f(x)} dx \right)^2 &= \left(\int_0^1 \sqrt[6]{f(x)} \cdot \sqrt[6]{f(x)} \cdot \sqrt[6]{f(x)} dx \right)^2 \\ &\leq \int_0^1 \left[\sqrt[8]{f(x)} \right]^2 dx \cdot \int_0^1 \left[\sqrt[6]{f(x)} \right]^2 dx \cdot \int_0^1 \left[\sqrt[6]{f(x)} \right]^2 dx = \left(\int_0^1 \sqrt[3]{f(x)} dx \right)^3 \\ &\Rightarrow 2 \left(\int_0^1 \sqrt{f(x)} dx \right)^2 \leq \int_0^1 f(x) dx + \left(\int_0^1 \sqrt[3]{f(x)} dx \right)^3 \end{aligned}$$

867. If $0 < a \leq b$ then:

$$\int_a^b \frac{\tan^{-1}(e^{-x^2})}{e^{x^2}} dx \geq \int_a^b \frac{1}{e^{x^2}} dx \cdot \tan^{-1} \left(\int_a^b \frac{1}{e^{x^2}} dx \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Sergio Esteban-Argentina

We can notice that: $f(x) = \tan^{-1}(x) \cdot x$ and $g(x) = e^{-x^2}$



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as $f''(x) = \frac{2}{(x^2+1)^2} > 0 \rightarrow f$ is strictly convex. By integral form of Jensen's inequality

$$\int_a^b f \circ g \, dx \geq f\left(\int_a^b g(x) \, dx\right) = \int_a^b \frac{1}{e^{x^2}} \, dx \cdot \tan^{-1}\left(\int_a^b \frac{1}{e^{x^2}} \, dx\right)$$

868. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{4}{3}(b^3 - a^3) + \pi^3 \int_a^b \frac{\sin x}{x} \, dx \geq 3\pi^2(b - a)$$

Proposed by Daniel Sitaru – Romania

Solution by Florentin Vișescu-Romania

$$f: \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, f(x) = \sin x - \frac{3x}{\pi} + \frac{4x^3}{\pi^3}$$

$$f'(x) = \cos x - \frac{3}{\pi} + \frac{12x^2}{\pi^3}, f''(x) = -\sin x + \frac{24x}{\pi^3}$$

$$f'''(x) = -\cos x + \frac{24}{\pi^3}; f'''(x) = \sin x > 0, x \in \left(0, \frac{\pi}{2}\right)$$

x	0	r_3	r_1	r_2	$\frac{\pi}{2}$
$f'''(x)$	+++ + + + + + + + + + + + + + + +				
$f''(x)$	-----0 + + + + + + + + + + + + + + + +				
$f'(x)$	-----f''(r_1) + 0 + + + + + + + + + +				
$f(x)$	+ + + + 0 -----f'(r_2) - - - -				

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} f'''(x) = \frac{24}{\pi^3} - 1 < 0; \lim_{\substack{x \rightarrow \frac{\pi}{2} \\ x < \frac{\pi}{2}}} f'''(x) = \frac{2}{\pi^3} > 0. \text{ So, } f''' \text{ has a root } r \text{ in } \left(0, \frac{\pi}{2}\right)$$

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} f''(x) = 0; \lim_{\substack{x \rightarrow 0 \\ x < 0}} f''(x) = \frac{12}{\pi^2} - 1 > 0. \text{ So } f''(r_1) < 0 \text{ and } f'' \text{ has a root } r_2 \text{ in}$$

$$\left(r_1, \frac{\pi}{2}\right)$$



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$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} f'(x) = 1 - \frac{3}{\pi} > 0; \lim_{\substack{x \rightarrow \frac{\pi}{2} \\ x < \frac{\pi}{2}}} f'(x) = 0$$

So, $f'(r_2) < 0$ and f' has a root r_3 in $(0; r_2)$. It doesn't matter where is towards r_1

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} f(x) = 0; \lim_{\substack{x \rightarrow \frac{\pi}{2} \\ x < \frac{\pi}{2}}} f(x) = 0$$

So, $f(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$. Then: $\sin x > \frac{3x}{\pi} - \frac{4x^3}{\pi^3}; \frac{\sin x}{x} > \frac{3}{\pi} - \frac{4x^2}{\pi^3}$

$$\int_a^b \frac{\sin x}{x} dx > \frac{3}{\pi}(b-a) - \frac{4}{3\pi^3}(b^3 - a^3); \frac{4}{3}(b^3 - a^3) + \pi^3 \int_a^b \frac{\sin x}{x} dx > 3\pi^2(b-a)$$

869. If $f: [0, 1] \rightarrow \mathbb{R}$ has a continuous second derivative and $f(0) = f(1)$

then:

$$3(f'(1))^2 \leq \int_0^1 (f''(x))^2 dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\begin{aligned} & \left(\int_0^1 (f''(x))^2 dx \right) \left(\int_0^1 x^2 dx \right) \stackrel{CBS}{\geq} \left(\int_0^1 x^2 f''(x) dx \right)^2 \\ & \left(\int_0^1 (f''(x))^2 dx \right) \cdot \frac{1}{3} \geq \left(1^2 \cdot f'(1) - 0 \cdot f'(0) - \int_0^1 2x f'(x) dx \right)^2 = \\ & = \left(f'(1) - 2 \left(1 \cdot f(1) - 0 \cdot f(0) - \int_0^1 f(x) dx \right) \right)^2 = \\ & = \left(f'(1) - 2f(1) + 2 \int_0^1 x' f(x) dx \right)^2 = \\ & = (f'(1) - 2f(1) + 2f(1) - 2(f(1) - f(0)))^2 = (f'(1))^2 \end{aligned}$$



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$$\left(\int_0^1 (f''(x))^2 dx \right) \geq 3(f'(1))^2$$

870. If $0 < a \leq b$ then:

$$\frac{1}{2} \int_a^b \int_a^b \frac{x+y}{\sqrt{xy}} dx dy + 2 \int_a^b \int_a^b \frac{\sqrt{xy}}{x+y} dx dy \leq \log \left(\frac{b}{a} \right)^{b^2-a^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \frac{x+y}{2\sqrt{xy}} + \frac{2\sqrt{xy}}{(x+y)} = \frac{(x+y)^2 + 4xy}{2\sqrt{xy}(x+y)} AM - GM \leq \frac{(x+y)^2 + 4xy}{4xy}$$

⇒ Need to show

$$\Leftrightarrow \frac{(x+y)^2 + 4xy}{4xy} \leq \frac{x^2 + y^2}{xy} \Rightarrow x^2 + y^2 + 6xy \leq 4x^2 + 4y^2$$

$$\Rightarrow 3(x-y)^2 \geq 0 (* \text{ true}) \Leftrightarrow \frac{x+y}{2\sqrt{xy}} + \frac{2\sqrt{xy}}{x+y} \leq \frac{x}{y} + \frac{y}{x}$$

$$\Rightarrow \frac{1}{2} \int_a^b \int_a^b \frac{(x+y)}{\sqrt{xy}} dx dy + 2 \int_a^b \int_a^b \frac{\sqrt{xy}}{(x+y)} dx dy \leq \int_a^b \int_a^b \left(\frac{x}{y} + \frac{y}{x} \right) dx dy$$

$$\Leftrightarrow \Omega \leq \int_a^b \int_a^b \frac{x}{y} dx dy + \int_a^b \int_a^b \frac{y}{x} dx dy = \left[\frac{x^2}{2} \right]_a^b [\log y]_a^b + \left[\frac{y^2}{2} \right]_a^b [\log x]_a^b$$

$$= 2 \times \frac{1}{2} \times (b^2 - a^2)(\log b - \log a) = \log \left(\frac{b}{a} \right)^{(b^2-a^2)}$$

$$\Leftrightarrow \Omega \leq \log \left(\frac{b}{a} \right)^{(b^2-a^2)} \quad (\text{proved})$$

871. If $0 < a \leq b < \frac{\pi}{2}$ then:



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$$8\pi^3 \int_a^b \int_a^b \left(\frac{\sin x \sin y \sin(x+y)}{xy(\pi-x-y)} \right) dx dy \leq 81\sqrt{3}(b-a)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } \varphi(t) = 2\pi \sin t - 3t\sqrt{3}, \forall t \in (0, \pi)$$

$$\varphi'(t) = 2\pi \cos t - 3\sqrt{3} \Rightarrow \varphi'(t) = 0 \Leftrightarrow \cos t = \frac{3\sqrt{3}}{2\pi} \Rightarrow t = \alpha = \cos^{-1}\left(\frac{3\sqrt{3}}{2\pi}\right) \in (0, \pi)$$

$$\Rightarrow \varphi'(t) < 0, \forall t \in (0, \alpha); \varphi'(t) > 0, \forall t \in (\alpha, \pi)$$

$$\Rightarrow \varphi(t) < \varphi(0) = 0; \varphi(t) < \varphi(\pi) = -3\pi\sqrt{3} < 0$$

$$\text{Hence for } x, y \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{\sin x}{x} \leq \frac{3\sqrt{3}}{2\pi} \text{ and } \frac{\sin y}{y} \leq \frac{3\sqrt{3}}{2\pi}$$

$$\Rightarrow \frac{\sin(x+y)}{\pi - (x+y)} = \frac{\sin[\pi - (x+y)]}{\pi - (x+y)} \leq \frac{3\sqrt{3}}{2\pi};$$

$$\left(\therefore x, y \in \left(0, \frac{\pi}{2}\right) \Rightarrow x + y < \pi \Rightarrow \pi > \pi - (x+y) > 0 \right). \text{ Hence}$$

$$8\pi^3 \int_a^b \int_a^b \left(\frac{\sin x \sin y \sin(x+y)}{xy(\pi-x-y)} \right) dx dy \leq 8\pi^3 \int_a^b \int_a^b \left(\frac{3\sqrt{3}}{2\pi} \right)^3 dx dy = 81\sqrt{3} \cdot (b-a)^2$$

Proved. Equality for a=b.

872. Prove without softs:

$$\frac{\pi}{2} < \int_0^2 (\cot^{-1}(\sin x)) dx < 2$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } f(x) = \cot^{-1}(\sin x), (0 < x < 2)$$

$$f'(x) = -\frac{\cos x}{1 + \sin^2 x} \Rightarrow f'(x) = 0 \Leftrightarrow -\cos x = 0 \stackrel{0 < x < 2}{\Leftrightarrow} x = \frac{\pi}{2}$$

$$\text{We compute: } \therefore f\left(\frac{\pi}{2}\right) = \cot^{-1}\left(\sin\frac{\pi}{2}\right) = \cot^{-1}(1) = \frac{\pi}{4};$$



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$$\therefore f(0) = \cot^{-1}(\sin 0) = \cot^{-1}(0) = \frac{\pi}{2};$$

$$\begin{aligned}\therefore f(2) &= \cot^{-1}(\sin 2) \approx 0.8328 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \Rightarrow \frac{\pi}{4} < \cot^{-1}(\sin x) < \frac{\pi}{2} \\ &\Rightarrow \int_0^2 \frac{\pi}{4} dx < \int_0^2 \cot^{-1}(\sin x) dx < \int_0^2 \frac{\pi}{2} dx \\ &\Rightarrow \frac{\pi}{2} < \int_0^2 \cot^{-1}(\sin x) dx < \pi\end{aligned}$$

873. If $a > 1$ then:

$$\frac{4\log 2}{\pi} + \int_1^a \frac{2x \tan^{-1} x - \log(1+x^2)}{(1+x^2)(\tan^{-1} x)^2} dx < \frac{a^2}{\tan^{-1} a}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sujit Bhowmick-North Bengal-India

$$\text{Put } \tan^{-1} x = u; \frac{1}{1+x^2} dx = du \Rightarrow x = \tan(u)$$

$$\frac{4\log 2}{\pi} + \int_1^a \frac{2x \tan^{-1} x - \log(1+x^2)}{(1+x^2)(\tan^{-1} x)^2} dx = \frac{4\log 2}{\pi} + \int_{\pi/4}^{\tan^{-1} a} \frac{2 \tan(u) - \log(\sec^2 u)}{u^2} du$$

$$\text{Put: } \frac{\log(\sec^2 u)}{u^2} = t$$

$$\Rightarrow \left(\frac{u \cdot \frac{1}{\sec^2 u} \cdot 2 \sec(u) \sec(u) \tan(u) - \log(\sec^2(u))}{u^2} \right) du = dt$$

$$\Rightarrow \frac{2u \tan(u) - \log(\sec^2(u))}{u^2} du = dt$$

$$\begin{aligned}\frac{4\log 2}{\pi} + \int_{\frac{4\log 2}{\pi}}^{\frac{\log(1+a^2)}{\tan^{-1} a}} dt &= \frac{4\log 2}{\pi} + t \left| \frac{\log(1+a^2)}{\frac{\tan^{-1} a}{\frac{4\log 2}{\pi}}} \right| = \frac{4\log 2}{\pi} + \frac{\log(1+a^2)}{\tan^{-1} a} - \frac{4\log 2}{\pi} \\ &= \frac{\log(1+a^2)}{\tan^{-1} a} < \frac{a^2}{\tan^{-1} a}\end{aligned}$$



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Solution 2 by Ravi Prakash-New Delhi-India

For $x \geq 1$

$$\begin{aligned}
 & \frac{2x \tan^{-1} x - \log(1+x^2)}{(1+x^2)(\tan^{-1} x)^2} = \frac{\frac{2x}{1+x^2} \tan^{-1} x - (\log(1+x^2)) \frac{1}{1+x^2}}{(\tan^{-1} x)^2} \\
 &= \frac{\left(\frac{d}{dx}(\log(1+x^2))(\tan^{-1} x) - (\log(1+x^2)) \frac{d}{dx}(\tan^{-1} x) \right)}{(\tan^{-1} x)^2} = \frac{d}{dx} \left(\frac{\log(1+x^2)}{\tan^{-1} x} \right) \\
 & \int_1^a \frac{2x \tan^{-1} x - \log(1+x^2)}{(1+x^2)(\tan^{-1} x)^2} dx = \int_1^a \frac{d}{dx} \left(\frac{\log(1+x^2)}{\tan^{-1} x} \right) dx \\
 &= \frac{\log(1+a^2)}{\tan^{-1} a} \Big|_1^a = \frac{\log(1+a^2)}{\tan^{-1} a} - \frac{\log 2}{\pi/4} \\
 &\Rightarrow \frac{4 \log 2}{\pi} + \int_1^a \frac{2x \tan^{-1} x - \log(1+x^2)}{(1+x^2)(\tan^{-1} x)^2} dx = \frac{\log(1+a^2)}{\tan^{-1} a} < \frac{a^2}{\tan^{-1} a} \\
 & \text{as } \log(1+x) < x, \forall x > 0
 \end{aligned}$$

874. If $f: [0, 1] \rightarrow \mathbb{R}$, f –continuous then:

$$\int_0^1 \int_0^1 \int_0^1 \sqrt[3]{(3f(x)f(y)f(z) - f^3(x) - f^3(y) - f^3(z))^2} dx dy dz \leq 3 \int_0^1 f^2(x) dx$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Vișescu-Romania

$$3abc - a^3 - b^3 - c^3 = (a+b+c)(ab+bc+ca - a^2 - b^2 - c^2)$$

$$(3abc - a^3 - b^3 - c^3)^2 = (a+b+c)^2(ab+bc+ca - a^2 - b^2 - c^2)^2$$

We must show: $(a+b+c)^2(ab+bc+ca - a^2 - b^2 - c^2)^2 \leq (a^2 + b^2 + c^2)^3$

$$(a+b+c)^2(3(ab+bc+ca) - (a+b+c)^2)^2 \leq ((a+b+c)^2 - 2(ab+bc+ca))^3$$

Let $a+b+c = p$; $ab+bc+ca = q$

$$p^2(3q-p^2)^2 \leq (p^2-2q)^3; 8q^3 \leq 3p^2q^2 \Leftrightarrow 8q \leq 3p^2$$

$$8(ab+bc+ca) \leq 3(a^2 + b^2 + c^2)$$

$$8(ab+bc+ca) \leq 3(a^2 + b^2 + c^2) + 6(ab+bc+ca)$$



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$2(ab + bc + ca) \leq 3(a^2 + b^2 + c^2)$ true from:

$$\xrightarrow{Am-Gm} \begin{cases} a^2 + b^2 \geq 2ab \\ b^2 + c^2 \geq 2bc \\ c^2 + a^2 \geq 2ca \end{cases}$$

$3(a^2 + b^2 + c^2) \geq 2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$. So,

$$(3abc - a^3 - b^3 - c^3)^2 \leq (a^2 + b^2 + c^2)^3$$

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 \sqrt[3]{(3f(x)f(y)f(z) - f^3(x) - f^3(y) - f^3(z))^2} dx dy dz \leq \\ & \leq \int_0^1 \int_0^1 \int_0^1 (f^2(x) + f^2(y) + f^2(z)) dx dy dz \leq 3 \int_0^1 f^2(x) dx \end{aligned}$$

875. Evaluate the following sum:

$$\sum_{k=0}^{\infty} \left(\frac{1}{(4k+1)^2} + \frac{1}{(4k+2)^2} - \frac{1}{(4k+3)^2} - \frac{1}{(4k+4)^2} \right)$$

Proposed by Prem Kumar-India

Solution 1 by Avishek Mitra-West Bengal-India

$$\begin{aligned} & \Leftrightarrow \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} = \frac{1}{16} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{1}{4}\right)^2} = \frac{1}{16} \psi_1\left(\frac{1}{4}\right) = \frac{1}{16} (\pi^2 + 8G) = \frac{\pi^2}{16} + \frac{G}{2} \\ & \Leftrightarrow \sum_{k=0}^{\infty} \frac{1}{(4k+2)^2} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{4} \lambda(2) = \frac{1}{4} (1 - 2^{-2}) \zeta(2) = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{32} \\ & \Leftrightarrow \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} = \frac{1}{16} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{3}{4}\right)^2} = \frac{1}{16} \psi_1\left(\frac{3}{4}\right) = \frac{1}{16} (\pi^2 - 8G) = \frac{\pi^2}{16} - \frac{G}{2} \\ & \Leftrightarrow \sum_{k=0}^{\infty} \frac{1}{(4k+4)^2} = \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{16} \zeta(2) = \frac{1}{16} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{96} \\ & \Leftrightarrow \Omega = \frac{\pi^2}{16} + \frac{G}{2} + \frac{\pi^2}{32} - \frac{\pi^2}{16} + \frac{G}{2} - \frac{\pi^2}{96} = G + \frac{\pi^2}{48} \end{aligned}$$



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Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\begin{aligned}
 S &= \sum_{k=0}^{\infty} \left[\frac{1}{(4k+1)^2} + \frac{1}{(4k+2)^2} - \frac{1}{(4k+3)^2} - \frac{1}{(4k+4)^2} \right] = \\
 &= \sum_{k=0}^{\infty} \left[\frac{1}{(4k+4-3)^2} + \frac{1}{4(2k+1)^2} - \frac{1}{(4k+4-1)^2} - \frac{1}{16(k+1)^2} \right] \\
 &= \sum_{k=0}^{\infty} \left\{ \frac{1}{[4(k+1)-3]^2} + \frac{1}{4[2(k+1)-1]^2} - \frac{1}{[4(k+1)-1]^2} - \frac{1}{16(k+1)^2} \right\} = \\
 &= \sum_{k=1}^{\infty} \left[\frac{1}{(4k-3)^2} + \frac{1}{4(2k-1)^2} - \frac{1}{(4k-1)^2} - \frac{1}{16k^2} \right] \\
 &= \sum_{k=1}^{\infty} \left[\frac{1}{(4k-3)^2} - \frac{1}{(4k-1)^2} \right] + \frac{1}{4} \left[\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} \right] = \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} + \frac{1}{4} \left[\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} \right] \\
 &= G + \frac{1}{4} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \right) = G + \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k^2} = G + \frac{\zeta(2)}{8} = G + \frac{\pi^2}{48} \\
 \zeta(x) &= \sum_{k=1}^{\infty} \frac{1}{k^x}
 \end{aligned}$$

876. Find:

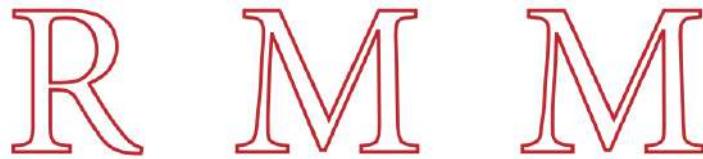
$$\Omega = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k \frac{1}{ijkl(i+1)(j+1)(k+1)(l+1)} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\text{Let } M = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{ijkl(i+1)(j+1)(k+1)(l+1)}$$

$$N = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{ij(i+1)(j+1)(k+1)^2 k^2}$$



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$$P = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{i^2(i+1)^2(k+1)^2 k^2}; S = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{i(i+1)(k+1)^3 k^3}; T = \sum_{i=1}^{\infty} \frac{1}{i^4(i+1)^4}$$

$$\text{We will prove that: } \Omega = \frac{M+6N+3P+8S+12T}{24}$$

In first note that there are $4!$ of permutations of $\frac{1}{i(1+i)} \cdot \frac{1}{j(1+j)} \cdot \frac{1}{k(1+k)} \cdot \frac{1}{l(1+l)}$ let's

discuss the 04 cases:

Case 01: if $l < k < j < i$ all $4!$ combinations are then contained with in M

Case 02: if $l = k$ and i, j are distinct we need 6 of N because some possibilities are contained in M

Case 03: if $i = j \neq l = k$ we need 3 of P because some possibilities are contained in M and N

Case 03: if $i = j = k = l$ we need 8 of S because some possibilities are contained in M, N and P

Case 03: if $i = j = k = l$ this appears exactly once in every sum, which thus us 12 copies of T

$$\text{Since: } A = \sum_{i=1}^{\infty} \frac{1}{i(1+i)} = 1 \Rightarrow M = 1$$

$$\therefore \sum_{i=1}^{\infty} \frac{1}{i^2(1+i)^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} + \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} - 2A = 2\zeta(2) - 3 \Rightarrow N = 2\zeta(2) - 3 \Rightarrow P = (2\zeta(2) - 3)^2$$

$$\begin{aligned} \therefore \sum_{i=1}^{\infty} \frac{1}{i^3(1+i)^3} &= \sum_{i=1}^{\infty} \left(\frac{1}{i^2} - \frac{1}{(1+i)^3} \right) - 3 \sum_{i=1}^{\infty} \left(\frac{1}{i^2} + \frac{1}{(1+i)^2} \right) + 6A = 10 - \pi^3 \Rightarrow \\ &\Rightarrow S = 10 - \pi^2 \therefore T = \sum_{i=1}^{\infty} \frac{1}{i^4(1+i)^4} = \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{\infty} \left(\frac{1}{i^4} + \frac{1}{(1+i)^4} \right) - 4 \sum_{i=1}^{\infty} \left(\frac{1}{i^2} - \frac{1}{(1+i)^3} \right) + 10 \sum_{i=1}^{\infty} \left(\frac{1}{i^2} + \frac{1}{(1+i)^2} \right) - 20A = \\ &= 2\zeta(4) + 20\zeta(2) - 35 \therefore \Omega = \frac{9\zeta(4) + 28\zeta(2) - 55}{4} \end{aligned}$$

Solution 2 by Kartick Chandra Betal-India

$$\sum_{i=1}^{\infty} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k \frac{1}{ijkl(1+i)(1+j)(1+k)(1+l)}$$



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$$\begin{aligned}
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \sum_{k=1}^j \sum_{l=1}^k \frac{1}{ijk(1+i)(1+j)(1+k)} \left\{ \frac{1}{l} - \frac{1}{1+l} \right\} \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \sum_{k=1}^j \frac{1}{ijk(1+i)(1+j)(1+k)} \left(1 - \frac{1}{1+k} \right) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \frac{1}{ij(1+i)(1+j)} \cdot \left(H_{j+1}^{(2)} - 1 \right) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^i \frac{1}{i(1+i)} \left\{ \frac{H_{j+1}^{(2)}}{j} - \frac{H_{j+1}^{(2)}}{j+1} \right\} - \sum_{i=1}^{\infty} \sum_{j=1}^i \frac{1}{i(1+i)} \left\{ \frac{1}{j} - \frac{1}{j+1} \right\} \\
 &= \sum_{i=1}^{\infty} \frac{1}{i(1+i)} \sum_{j=1}^i \left\{ \left(\frac{H_j^{(2)}}{j} - \frac{H_{j+1}^{(2)}}{j+1} \right) - \frac{1}{j(1+j)^2} \right\} - \sum_{i=1}^{\infty} \frac{1}{i(1+i)} \left(1 - \frac{1}{1+i} \right) \\
 &= \sum_{i=1}^{\infty} \frac{1}{i(1+i)} \left[1 - \frac{H_{j+1}^{(2)}}{i+1} + \sum_{j=1}^i \left\{ \frac{1}{j} - \frac{1}{(j+1)} \right\} - \left\{ H_{j+1}^{(2)} - 1 \right\} \right] - \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \\
 &= \sum_{i=1}^{\infty} \frac{1}{i(1+i)} \left[1 - \frac{H_{j+1}^{(2)}}{i+1} + 1 - \frac{1}{i+1} - H_{j+1}^{(2)} + 1 \right] - \zeta(2) + 1 \\
 &= 1 - \frac{\pi^2}{6} + \sum_{i=1}^{\infty} \frac{1}{i(1+i)} \left[3 - \frac{1}{i+1} - \frac{H_{j+1}^{(2)}}{i+1} - H_{j+1}^{(2)} \right] \\
 &= 1 - \frac{\pi^2}{6} + 3 - \sum_{i=1}^{\infty} \frac{1}{i(1+i)^2} - \sum_{i=1}^{\infty} \frac{H_{j+1}^{(2)}}{i(1+i)^2} - \sum_{i=1}^{\infty} \frac{H_{j+1}^{(2)}}{i(1+i)} \\
 &= 4 - \frac{\pi^2}{6} - [1 - \{\zeta(2) - 1\}] - \sum_{i=1}^{\infty} \left\{ \frac{1}{i} - \frac{1}{1+i} - \frac{1}{(1+i)^2} \right\} H_{j+1}^{(2)} - \sum_{i=1}^{\infty} \left\{ \frac{H_{j+1}^{(2)}}{i} - \frac{H_{j+1}^{(2)}}{i+1} \right\} \\
 &= 4 - \frac{\pi^2}{6} - 2 + \frac{\pi^2}{6} - 2 \sum_{i=1}^{\infty} \left\{ \frac{H_{j+1}^{(2)}}{i} - \frac{H_{j+1}^{(2)}}{i+1} \right\} + \sum_{i=1}^{\infty} \frac{H_{j+1}^{(2)}}{(1+i)^2} \\
 &= 2 - 2 \sum_{i=1}^{\infty} \left[\left\{ \frac{H_i^{(2)}}{i} - \frac{H_{j+1}^{(2)}}{i+1} \right\} + \frac{1}{i(1+i)^2} \right] + \sum_{i=1}^{\infty} \frac{H_i^{(2)}}{i^2} - 1
 \end{aligned}$$



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$$\begin{aligned}
 &= 1 - 2 \left\{ 1 + \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{1+i} \right) - \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \right\} + \sum_{i=1}^{\infty} \frac{H_n^{(2)}}{n^2} \\
 &= 1 + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - 2 - 2 + 2\{\zeta(2) - 1\} = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \frac{\pi^2}{3} - 5 = \frac{7\pi^4}{360} + \frac{\pi^3}{3} - 5
 \end{aligned}$$

877. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\int_0^1 \left(e^{\frac{x^2}{n}} \right) dx \right)^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

Here

$$I(n) = \int_0^1 e^{\frac{x^2}{n}} dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{n} \right)^k \right) dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)k! n^k}$$

$$\begin{aligned}
 \text{and hence we have the limit: } &\lim_{n \rightarrow \infty} (I(n))^n = \exp \left(\lim_{n \rightarrow \infty} n \log \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)k! n^k} \right) \right) \\
 &= \exp \left(\lim_{n \rightarrow \infty} n \log \left(1 + \sum_{k=1}^{\infty} \frac{1}{(2k+1)k! n^k} \right) \right) \\
 &= \exp \left(\lim_{n \rightarrow \infty} n \left((-1)^{k+1} \sum_{k=1}^{\infty} \frac{1}{(2k+1)n^k} \right)^k \right) \\
 &= \exp \left(\lim_{n \rightarrow \infty} n \left(\frac{1}{3n} + \frac{1}{2! 5n^2} + \dots \right) \right) - O(n^2) = \sqrt[3]{e}
 \end{aligned}$$

Solution 2 by Abdul Hafeez Ayinde-Nigeria

Let $\Omega = \int_0^1 e^{\frac{x^2}{n}} dx$ and $\Lambda = \lim_{n \rightarrow \infty} (\Omega)^n$

$$\Omega = \int_0^1 e^{\frac{x^2}{n}} dx; \quad \Lambda = \sum_{k=0}^{\infty} \frac{1}{k! n^k} \int_0^1 x^{2k} dx$$



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$$\Lambda = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^{\infty} \frac{1}{k! n^k (2k+1)} \right)^n$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp n \log \left(\sum_{k=0}^{\infty} \frac{1}{k! n^k (2k+1)} \right)$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp n \log \left(1 + \sum_{k=1}^{\infty} \frac{1}{k! n^k (2k+1)} \right)$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp n \sum_{z=0}^{\infty} \frac{(-1)^z}{z+1} \left(\sum_{k=1}^{\infty} \frac{1}{k! n^k (2k+1)} \right)^{z+1}$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp n \left\{ \sum_{z=0}^{\infty} \frac{(-1)^z}{z+1} \left(\frac{1}{3n} + \frac{1}{5 \cdot n^2 \cdot 2!} + \dots \right)^{z+1} \right\}$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp n \left\{ \frac{1}{3n} + \frac{1}{5 \cdot n^2 \cdot 2!} + \dots - \frac{1}{2} \left(\frac{1}{3n} + \frac{1}{5 \cdot n^2 \cdot 2!} + \dots \right)^2 + \dots \right\}$$

$$\Lambda = \lim_{n \rightarrow \infty} \exp \left\{ \frac{1}{3} + \frac{1}{5 \cdot n \cdot 2!} + O(n^2) \right\}; \quad \Lambda = \lim_{n \rightarrow \infty} \exp \left\{ \frac{1}{3} + \frac{1}{5 \cdot n \cdot 2!} + O(n^2) \right\} = e^{\frac{1}{3}}$$

878. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \left(\sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{i \cdot j}} \right) \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Remus Florin Stanca – Romania

$$\begin{aligned}
 \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} &= \sum_{k=1}^n \left(\frac{1}{\sqrt{k}} \cdot \sqrt{k} \right) + \frac{1}{\sqrt{1}} (\sqrt{2} + \dots + \sqrt{n}) + \frac{1}{\sqrt{2}} (\sqrt{3} + \dots + \sqrt{n}) + \dots + \\
 &+ \frac{1}{\sqrt{n-1}} \cdot \sqrt{n} + \sqrt{1} \left(\frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \right) + \sqrt{2} \left(\frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right) + \dots + \sqrt{n-1} \cdot \frac{1}{\sqrt{n}} = \\
 &= n + \sum_{1 \leq i < j \leq n} \sqrt{\frac{j}{i}} + \sum_{1 \leq i < j \leq n} \sqrt{\frac{i}{j}} =
 \end{aligned}$$



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$$\begin{aligned}
 &= n + \sum_{1 \leq i < j \leq n} \left(\sqrt{\frac{j}{i}} + \sqrt{\frac{i}{j}} - 2 + 2 \right) = n + \sum_{1 \leq i < j \leq n} \left(\frac{i+j-2\sqrt{ij}}{\sqrt{ij}} + 2 \right) = \\
 &= n + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} + 2 \cdot \frac{n(n-1)}{2} = n + n^2 - n + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} = \\
 &= n^2 + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} \Rightarrow \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} = n^2 + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} \Rightarrow \\
 &\Rightarrow \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} = \\
 &= n^2 + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} = n^2 \Rightarrow \\
 &\Rightarrow \frac{1}{n^2} \left(\sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i}-\sqrt{j})^2}{\sqrt{ij}} \right) = \frac{n^2}{n^2} = 1 \Rightarrow \Omega = 1
 \end{aligned}$$

879. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\int_0^{\frac{1}{\sqrt{n}} e^{\sum_{k=1}^n (\frac{1}{2k-1})}} \left(\frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} \right) dx + \int_{\frac{1}{\sqrt{n+1}} e^{\sum_{k=1}^{n+1} (\frac{1}{2k-1})}}^{\frac{\pi}{2}} \left(\frac{\cos^2 x + \cos x}{\cos x + \sin x + 1} \right) dx \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\begin{aligned}
 &\text{Let's compute } I = \int \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx \text{ and } J = \int \frac{\cos^2 x + \cos x}{\cos x + \sin x + 1} dx \Rightarrow \\
 &\Rightarrow I + J = \int \frac{\sin x + \cos x + 1}{\cos x + \sin x + 1} dx = \int dx = x \\
 &J - I = \int \frac{(\cos x - \sin x)(\cos x + \sin x) + \cos x - \sin x}{\cos x + \sin x + 1} dx = \\
 &= \int \frac{(\cos x - \sin x)(\cos x + \sin x + 1)}{\cos x + \sin x + 1} dx = \int (\cos x - \sin x) dx = \sin x + \cos x
 \end{aligned}$$

So, $I + J = x$



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$$J - I = \sin x + \cos x$$

----- "+" -----

$$2J = \sin x + \cos x + x \Rightarrow J = \frac{\sin x + \cos x + x}{2} \Rightarrow$$

$$\Rightarrow I = x - \frac{\sin x + \cos x + x}{2} = \frac{x - \sin x - \cos x}{2} \Rightarrow$$

$$\begin{aligned} \Rightarrow 2\Omega &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \cdot e^{\sum_{k=1}^n \frac{1}{2k-1}} - \sin\left(\frac{1}{\sqrt{n}} e^{\sum_{k=1}^n \frac{1}{2k-1}}\right) - \cos\left(\frac{1}{\sqrt{n}} e^{\sum_{k=1}^n \frac{1}{2k-1}}\right) + 1 + \frac{\pi}{2} + \\ &\quad + 1 - \frac{1}{\sqrt{n+1}} e^{\sum_{k=1}^{n+1} \frac{1}{2k-1}} - \sin\left(\frac{1}{\sqrt{n+1}} e^{\sum_{k=1}^{n+1} \frac{1}{2k-1}}\right) - \cos\left(\frac{1}{\sqrt{n+1}} e^{\sum_{k=1}^{n+1} \frac{1}{2k-1}}\right) \quad (1) \end{aligned}$$

$$\begin{aligned} \text{Let's compute } \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} e^{\sum_{k=1}^n \frac{1}{2k-1}} &= \lim_{n \rightarrow \infty} \frac{e^{\sum_{k=1}^n \frac{1}{2k-1}}}{e^{\ln(\sqrt{n})}} = \lim_{n \rightarrow \infty} e^{\sum_{k=1}^n \frac{1}{2k-1} - \ln(\sqrt{n})} = \\ &= \lim_{n \rightarrow \infty} e^{H_{2n-1} - \frac{H_{n-1}}{2} - \ln(\sqrt{n})} = \\ &= \lim_{n \rightarrow \infty} e^{H_{2n-1} - \ln(2n-1) + \ln(2n-1) - \frac{H_{n-1} - \ln(n-1)}{2} - \frac{\ln(n-1)}{2} - \ln(\sqrt{n})} = e^{\frac{\gamma}{2}} \cdot \lim_{n \rightarrow \infty} e^{\ln\left(\frac{2n-1}{\sqrt{n^2-n}}\right)} = \\ &= e^{\frac{\gamma}{2}} \cdot \lim_{n \rightarrow \infty} e^{\ln\left(\frac{n(2-\frac{1}{n})}{n\sqrt{1-\frac{1}{n}}}\right)} = e^{\frac{\gamma}{2}} \cdot 2 \stackrel{(1)}{\Rightarrow} \Omega = 1 + \frac{\pi}{4} - \sin\left(2e^{\frac{\gamma}{2}}\right) - \cos\left(2e^{\frac{\gamma}{2}}\right) \end{aligned}$$

Solution 2 by Ali Jaffal-Lebanon

We have: $\sum_{k=1}^n \frac{1}{2k-1} + \sum_{k=1}^n \frac{1}{2k} = \sum_{k=1}^{2n} \frac{1}{k}$. So,

$$\sum_{k=1}^{2n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2} H_n = \log(2n) - \frac{1}{2} \log(n) + \frac{\gamma}{2} + \varphi(n)$$

Where $\lim_{n \rightarrow \infty} \varphi(n) = 0$

$$So, \frac{1}{\sqrt{n}} e^{\sum_{k=1}^n \frac{1}{2k-1}} = \frac{1}{\sqrt{n}} \times \frac{2n}{\sqrt{n}} \times e^{\frac{\gamma}{2}} \times e^{\varphi(n)} = 2e^{\frac{\gamma}{2}} e^{\varphi(n)}$$

Then, $\frac{1}{\sqrt{n}} e^{\sum_{k=1}^n \frac{1}{2k-1}}$ tends to $2e^{\frac{\gamma}{2}}$ when $n \rightarrow +\infty$

$$and \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} e^{\sum_{k=1}^n \frac{1}{2k-1}} = \lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2+n}} e^{\frac{\gamma}{2}} e^{\varphi(n)} = 2e^{\frac{\gamma}{2}}$$

Let $a = 2e^{\frac{\gamma}{2}}$ then



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$$I = \int_0^a \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx + \int_a^{\frac{\pi}{2}} \frac{\cos^2 x + \cos x}{\cos x + \sin x + 1} dx$$

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx + \int_{\frac{\pi}{2}}^a \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx + \int_a^{\frac{\pi}{2}} \frac{\cos^2 x + \cos x + 1}{\sin x + \cos x + 1} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx + \int_a^{\frac{\pi}{2}} \frac{(\cos x - \sin x)(\cos x + \sin x + 1)}{\sin x + \cos x + 1} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx + \int_a^{\frac{\pi}{2}} (\cos x - \sin x) dx \end{aligned}$$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \sin x}{\sin x + \cos x + 1} dx$ and $k = \int_a^{\frac{\pi}{2}} (\cos x - \sin x) dx$

$$J = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x + \cos x}{\sin x + \cos x + 1} dx$$

We have $k = \sin x + \cos x \Big|_a^{\frac{\pi}{2}} = 1 - \sin a - \cos a$

$$I + J = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$I - J = \int_0^{\frac{\pi}{2}} \frac{(\sin x - \cos x)(\sin x + \cos x + 1)}{\sin x + \cos x + 1} dx = \int_0^{\frac{\pi}{2}} (\sin x - \cos x) dx = 0$$

So, $2I = \frac{\pi}{2}$ then $I = \frac{\pi}{4}$ Therefore $\Omega = \frac{\pi}{4} + 1 - \sin a - \cos a$

880. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\binom{n}{3} \cdot \lim_{m \rightarrow \infty} \left(\frac{1}{m} \sum_{k=0}^m \left(\frac{\binom{m}{k}}{\binom{m+n}{k+3}} \right) \right) \right)$$

Proposed by Daniel Sitaru – Romania



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Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned}
 \text{Let } \omega &= \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{m+n}{k+3}} = \sum_{k=0}^m \binom{m}{n} \frac{(k+3)!(m+n-k-3)!}{(m+n)!} = \\
 &= (n+m+1) \sum_{k=0}^m \binom{m}{k} \frac{\Gamma(k+4)\Gamma(m+n-k-2)}{\Gamma(m+n+2)} \\
 &= (n+m+1) \sum_{k=0}^m \binom{m}{k} \int_0^1 x^{k+3} (1-x)^{m+n-k-3} dx = \\
 &= (m+n+1) \int_0^1 x^3 (1-x)^{n-3} \sum_{k=0}^m \binom{m}{k} x^k (1-x)^{m-k} dx = \\
 &= (n+m+1) \int_0^1 x^3 (1-x)^{n-3} dx \\
 &= \frac{(n+m+1)\Gamma(4)\Gamma(n-2)}{\Gamma(n+2)} = \frac{6(m+n+1)}{(n-1)n(n+1)(n-2)} \Rightarrow \\
 \Rightarrow \lim_{m \rightarrow \infty} \frac{\omega}{m} &= \lim_{m \rightarrow \infty} \frac{6(m+n+1)}{m(n-1)n(n+1)(n-2)} = \frac{6}{(n-1)n(n+1)(n-2)} \\
 \text{Now: } \Omega &= \lim_{n \rightarrow \infty} \left(\binom{n}{3} \lim_{m \rightarrow \infty} \frac{\omega}{m} \right) = \lim_{n \rightarrow \infty} \frac{n!}{6(n-3)!} \cdot \frac{6}{(n-1)n(n+1)(n-2)} = 0
 \end{aligned}$$

Solution 2 by Ali Jaffal-Lebanon

$$\begin{aligned}
 \text{Let } \Omega(n, m) &= \frac{1}{m} \sum_{k=0}^m \frac{C_m^k}{C_{m+n}^{k+3}} = \frac{1}{m} \sum_{k=0}^m C_m^k \times \frac{(k+3)!(m+n-k-3)!}{(m+n)!} \\
 &= \frac{1}{m} \sum_{k=0}^{k=m} C_m^k \times \frac{\Gamma(k+1) \cdot \Gamma(m+n-k-2)}{\Gamma(m+n+2)} \times (m+n+1) \\
 &= \frac{m+n+1}{m} \sum_{k=0}^{k=m} C_m^k \int_0^1 x^{k+3} \cdot (1-x)^{m+n-k-3} dx \\
 &= \frac{m+n+1}{m} \int_0^1 \sum_{k=0}^{k=m} C_m^k x^k (1-x)^{m-n-k} C_m^k x^3 \cdot (1-x)^{n-3} dx \\
 &= \frac{m+n+1}{m} \int_0^1 (x+1-x)^m \cdot x^3 \cdot (1-x)^{n-3} dx = \frac{m+n+1}{m} \times \int_0^1 x^3 (1-x)^{n-3} dx
 \end{aligned}$$



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$$\lim_{n \rightarrow +\infty} \Omega(n, m) = \int_0^1 x^3(1-x)^{n-3} dx$$

$$\text{Let } I_n = C_n^3 \times \int_0^1 x^3(1-x)^{n-3} dx$$

$$= C_n^3 \times \frac{\Gamma(4) \cdot \Gamma(n-2)}{\Gamma(n+2)} = \frac{n!}{(n-3)! \times 3!} \times \frac{3! \times (n-3)!}{(n+1)!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

So, $\lim_{n \rightarrow \infty} I_n = 0$. Then $\Omega = 0$

881. $x_n > 0, n \in \mathbb{N}, n \geq 1$

$$\lim_{n \rightarrow \infty} \left(\left(\frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \right)^{n-1} \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \right) = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \sum_{i=1}^n x_i \right)^{n-1} \left(\frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \right) \right) = \omega$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n x_i \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ali Jaffal-Lebanon

$$\text{Let } S_n = \frac{1}{n} \sum_{i=1}^n x_i ; h_n = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \text{ and } P_n = \sqrt[n]{\prod_{i=1}^n x_i}$$

By GM-AM inequality: $h_n < P_n < S_n$ then: $(h_n)^n < (P_n)^n < (S_n)^n$

We have $\lim_{n \rightarrow \infty} (S_n^{n-1} h_n) = \lim_{n \rightarrow \infty} S_n \cdot (h_n)^{n-1} = \omega$. So,

$$(n-1) \log S_n + \log h_n = \log \omega + \varphi(n) \quad (*)$$

$$(n-1) \log h_n + \log S_n = \log \omega + \varphi_2(n) \quad (**)$$

where $\lim_{n \rightarrow \infty} \varphi(n) = \lim_{n \rightarrow \infty} \varphi_2(n) = 0$

by (*) and (**) we have: $n \log S_n + n \log h_n = \log \omega^2 + \varphi_1(n) + \varphi_2(n)$

So: $\lim_{n \rightarrow \infty} n \log(S_n \cdot h_n) = \log \omega^2$ then $\lim_{n \rightarrow \infty} (S_n h_n)^n = \omega^2$

And also, by () and (**) we have:*



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$$\log S_n - (n-1)^2 \log S_n = (\log \omega)(2-n) + \varphi_2(n) - (n-1)\varphi_1(n)$$

$$\text{then } n(2-n) \log S_n = (2-n) \log \omega + \varphi_2(n) - (n-1)\varphi_1(n)$$

$$n \log S_n = \log \omega + \frac{\varphi_2(n)}{2-n} + \frac{1-n}{2-n} \varphi_1(n)$$

So, $\lim_{n \rightarrow +\infty} n \log S_n = \log \omega$ then $\lim_{n \rightarrow \infty} (S_n)^n = \omega$ but $\lim_{n \rightarrow \infty} (S_n h_n)^n = \omega^2$ then

$$\lim_{n \rightarrow \infty} (h_n)^n = \omega. \text{ we have } (h_n)^n \leq (P_n)^n \leq (S_n)^n$$

then by Sandwich theorem: $\lim_{n \rightarrow \infty} (h_n)^n \leq \lim_{n \rightarrow \infty} (P_n)^n \leq \lim_{n \rightarrow \infty} (S_n)^n$ then

$$\lim_{n \rightarrow \infty} (P_n^n) = \omega \text{ and } \lim_{n \rightarrow \infty} \prod_{i=1}^{i=n} x_i = \omega$$

882. If $x, y, z > 0$, different in pairs, then:

$$3 + \Omega(x, y) + \Omega(y, z) + \Omega(z, x) > \log \left(\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right)$$

$$\Omega(x, y) = \sum_{k=1}^{\infty} \left(\frac{1}{2k} \left(\frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} \right)^k \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \Omega(x, y) &= \sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} \right)^k = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} \right)^k \\ &= -\frac{1}{2} \ln \left(1 - \frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} \right) = -\frac{1}{2} \ln \left(\frac{4xy}{x^2 + 2xy + y^2} \right) = \\ &= \ln \left(\frac{x+y}{2\sqrt{xy}} \right) = \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) - \ln 2 \\ \therefore 3 \ln e + \sum_{cyc} \Omega(x, y) &= \sum_{cyc} \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) + 3 \ln \left(\frac{e}{2} \right) = \\ &= \sum_{cyc} \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) + 3 \ln(1.3591) \end{aligned}$$



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$$3 + \sum_{cyc} \Omega(x, y) > \sum_{cyc} \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right)$$

Solution 2 by Adrian Popa-Romania

$$\begin{aligned} \therefore 1 + x + x^2 + \cdots + x^{n-1} &= \frac{x^n - 1}{x - 1} \\ \text{If } x \in (0, 1) \text{ and } n \rightarrow \infty \Rightarrow x^n \rightarrow 0 \quad \left. \right\} \Rightarrow \\ \Rightarrow 1 + x + x^2 + \cdots + x^{n-1} + \cdots &= \frac{1}{1-x} | \int \Rightarrow \\ \Rightarrow x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} &= -\ln(1-x) \Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x) \\ \Omega(x, y) &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{(x-y)^2}{(x+y)^2} \right)^k = -\frac{1}{2} \cdot \ln \left(1 - \frac{(x-y)^2}{(x+y)^2} \right) = -\frac{1}{2} \ln \frac{4xy}{(x+y)^2} = \\ &= \frac{1}{2} \ln \frac{(x+y)^2}{4xy} = \frac{1}{2} \ln \left(\frac{x+y}{2\sqrt{xy}} \right)^2 = \ln \frac{x+y}{2\sqrt{x \cdot y}} = \ln \left(\frac{\sqrt{x}}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{x}} \right) \end{aligned}$$

We must prove that: $3 + \sum \ln \left(\frac{\sqrt{x}}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{x}} \right) \geq \sum \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right)$

$$\begin{aligned} \ln e + \ln \left(\frac{1}{2} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right) &= \ln \left(\frac{e}{2} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right) \stackrel{?}{>} \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \Leftrightarrow \\ \Leftrightarrow \ln \frac{e}{2} + \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) &> \ln \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \Rightarrow \ln \frac{e}{2} \geq 1 \text{ (True) because } e > 2 \quad (1) \end{aligned}$$

Similarly: $1 + \ln \left(\frac{1}{2} \left(\sqrt{\frac{x}{z}} + \sqrt{\frac{z}{x}} \right) \right) > \ln \left(\sqrt{\frac{x}{z}} + \sqrt{\frac{z}{x}} \right) \quad (2) \text{ and}$

$1 + \ln \left(\frac{1}{2} \left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}} \right) \right) > \ln \left(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}} \right) \quad (3)$

$$(1)+(2)+(3) \Rightarrow 3 + \Omega(x, y) + \Omega(y, z) + \Omega(x, z) \geq \ln \left(\prod \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right)$$

Solution 3 by Remus Florin Stanca-Romania

$$\Omega(x, y) = \sum_{k=1}^{\infty} \left(\frac{1}{2k} \left(\frac{x-y}{x+y} \right)^{2k} \right), \text{ let } \frac{x-y}{x+y} = \alpha, \text{ we also know that } x, y, z > 0 \Rightarrow$$



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$$\Rightarrow -y < y \Rightarrow x - y < x + y \Rightarrow \frac{x-y}{x+y} < 1 \text{ because } x + y > 0, \text{ so } \alpha < 1 \quad (1)$$

$$\begin{aligned}
\Omega(x, y) &= \sum_{k=1}^{\infty} \left(\frac{\alpha^{2k}}{2k} \right) = \sum_{k=1}^{\infty} \left(\int \left(\frac{\alpha^{2k}}{2k} \right)' d\alpha \right) = \\
&= \sum_{k=1}^{\infty} \left(\int \alpha^{2k-1} d\alpha \right) = \int \left(\sum_{k=1}^{\infty} \alpha^{2k-1} \right) d\alpha = \int \lim_{n \rightarrow \infty} \left(\alpha \cdot \frac{(\alpha^2)^n - 1}{\alpha^2 - 1} \right) d\alpha = - \int \frac{\alpha}{\alpha^2 - 1} d\alpha = \\
&= -\frac{1}{2} \int \frac{2\alpha}{\alpha^2 - 1} d\alpha = -\frac{1}{2} \ln(|\alpha^2 - 1|) \stackrel{\alpha < 1}{=} -\frac{1}{2} \ln(1 - \alpha^2) = -\frac{1}{2} \ln \left(1 - \frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} \right) = \\
&= -\frac{1}{2} \ln \left(\frac{4xy}{(x+y)^2} \right) = -\ln \left(\frac{2\sqrt{xy}}{x+y} \right) \text{ because } x, y > 0 \\
\Rightarrow \Omega(x, y) &= -\ln \left(\frac{2\sqrt{xy}}{x+y} \right) \Rightarrow 3 + \Omega(x, y) + \Omega(y, z) + \Omega(z, x) = \\
&= 3 - \sum_{cyc} \ln \left(\frac{2\sqrt{xy}}{x+y} \right) = 3 + \ln \left(\frac{\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right)}{8} \right) = \\
&= 3 - \ln(8) + \ln \left(\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right) \quad (2)
\end{aligned}$$

we know that $2 < e \Rightarrow 8 < e^3 \Rightarrow \ln(8) < 3 \Rightarrow 3 - \ln(8) > 0 \Rightarrow$

$$\begin{aligned}
\Rightarrow 3 + \ln \left(\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right) - \ln(8) &> \ln \left(\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right) \stackrel{(2)}{\Rightarrow} \\
\Rightarrow 3 + \Omega(x, y) + \Omega(y, z) + \Omega(z, x) &> \ln \left(\prod_{cyc} \left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) \right)
\end{aligned}$$

883. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{\log_n^n \left(\frac{(1+H_1)^2 + (1+H_2)^n + \dots + (1+H_n)^2}{n} \right)}{\log_n(1+H_1) \cdot \log_n(1+H_2) \cdot \dots \cdot \log_n(1+H_n)} \right)$$

Proposed by Daniel Sitaru – Romania



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Solution 1 by Ali Jaffal-Lebanon

We have by GM-AM inequality:

$$\begin{aligned} \log^n \left(\frac{(1+H_1)^2 + (1+H_2)^2 + \dots + (1+H_n)^2}{n} \right) &\geq \\ \log^n \left(\sqrt[n]{(1+H_1)^2(1+H_2)^2 \cdot \dots \cdot (1+H_n)^2} \right) &\geq \\ \left(\frac{2}{n}\right)^n [\log(1+H_1)(1+H_2) \dots (1+H_n)]^n &\geq \\ \left(\frac{2}{n}\right)^n [\log(1+H_1) + \log(1+H_2) + \dots + \log(1+H_n)]^n &(*) \end{aligned}$$

$$\text{And } \log(1+H_1) + \dots + \log(1+H_n) \leq \left(\frac{1}{n}\right)^n [\log(1+H_1) + \dots + \log(1+H_n)]^n \quad (**)$$

$$\begin{aligned} \text{Let } U_n &= \frac{\log_n((1+H_1)^2 + (1+H_2)^2 + \dots + (1+H_n)^2)}{\log_n(1+H_1) \cdot \log_n(1+H_2) \cdot \dots \cdot \log_n(1+H_n)} \\ &= \frac{\frac{1}{(\ln n)^n} \log^n \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)}{\frac{1}{(\ln n)^n} [\log(1+H_1) \log(1+H_2) \dots \log(1+H_n)]} \quad \text{by (*) and (**)} \geq \frac{\left(\frac{2}{n}\right)^n}{\left(\frac{1}{n}\right)^n} \geq 2^n, \text{ So, } U_n \geq 2^n \end{aligned}$$

but $\lim_{n \rightarrow \infty} 2^n = +\infty$ then $\lim_{n \rightarrow \infty} U_n \geq +\infty$ so, $\lim_{n \rightarrow \infty} U_n = +\infty$

Solution 2 by Remus Florin Stanca-Romania

$$\begin{aligned} \Rightarrow \Omega &= \lim_{n \rightarrow \infty} \left(\frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{(\ln(n))^n} \cdot \frac{(\ln(n))^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)}, \text{ let } x_n = \frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)} \Rightarrow \\ &\Rightarrow \frac{x_{n+1}}{x_n} = \left(\frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)} \right)^n \cdot \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln(1+H_{n+1})} \quad (1) \\ &\lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln(1+H_{n+1})} = \\ &= \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)}{\ln(1+H_n)} \underset{\text{Stolz-Cesaro}}{=} \end{aligned}$$



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$$\begin{aligned}
 & \underset{\text{Stolz-Cesaro}}{=} \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} \right)}{\ln \left(\frac{1+H_{n+1}}{1+H_n} \right)} = \\
 & = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1 + 1 \right)}{\ln \left(\frac{1+H_{n+1}}{1+H_n} - 1 + 1 \right)} = \\
 & = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1 + 1 \right)}{\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1} \cdot \\
 & \cdot \frac{n((1+H_1)^2 + \dots + (1+H_{n+1})^2) - (n+1)((1+H_1)^2 + \dots + (1+H_n)^2)}{(n+1)((1+H_1)^2 + \dots + (1+H_n)^2)} \cdot \\
 & \cdot \frac{\frac{1}{(n+1)(1+H_n)}}{\ln \left(\frac{1}{(n+1)(1+H_n)} + 1 \right)} \cdot (n+1)(1+H_n) = \lim_{n \rightarrow \infty} (n+1) \frac{1+H_n}{\ln(n)} \cdot \ln(n) \cdot \\
 & \cdot \frac{n((1+H_1)^2 + \dots + (1+H_{n+1})^2) - (n+1)((1+H_1)^2 + \dots + (1+H_n)^2)}{(n+1)((1+H_1)^2 + \dots + (1+H_n)^2)} = \\
 & = \lim_{n \rightarrow \infty} \ln(n) \cdot \frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{(1+H_1)^2 + \dots + (1+H_n)^2} = \\
 & = \lim_{n \rightarrow \infty} \frac{n \ln^2(n)}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{1}{n \ln(n)} \cdot (n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2) = \\
 & \underset{\text{Stolz-Cesaro}}{=} \lim_{n \rightarrow \infty} \left(\frac{n \ln^2(n) \left(\frac{n+1}{n} \left(\frac{\ln(n+1)^2}{\ln(n)} \right) - 1 \right)}{(1+H_{n+1})^2} \right) \cdot \\
 & \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) = \\
 & = \lim_{n \rightarrow \infty} \left(\left(\frac{\ln(n)}{1+H_{n+1}} \right)^2 \right) \cdot \lim_{n \rightarrow \infty} \left(n \ln \left(\frac{n+1}{n} \cdot \left(\frac{\ln(n+1)}{\ln(n)} \right)^2 \right) \right) \cdot \\
 & \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) =
 \end{aligned}$$



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$$\begin{aligned}
 &= 1 \cdot \ln(e) \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) \underset{\text{Stolz Cesaro}}{=} \\
 &\underset{\text{Stolz Cesaro}}{=} \lim_{n \rightarrow \infty} \frac{(n+1)(1+H_{n+2})^2 - n(1+H_{n+1})^2 - (1+H_{n+1})^2}{(n+1) \ln(n+1) - n \ln(n)} = \\
 &= \lim_{n \rightarrow \infty} (n+1) \cdot \frac{\left(2+2H_{n+1}+\frac{1}{n+2}\right) \cdot \frac{1}{n+1}}{n \ln(n) \ln\left(\frac{n+1}{n} \frac{\ln(n+1)}{\ln(n)}\right)} = \lim_{n \rightarrow \infty} \frac{2+2H_{n+1}+\frac{1}{n+2}}{\ln(n)} \underset{\text{Stolz Cesaro}}{=} 2 \quad (2) \\
 &\quad \lim_{n \rightarrow \infty} e^{\frac{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_{n+1})^2}{(1+H_1)^2+\dots+(1+H_n)^2} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_n)^2}{n}\right)}} = \\
 &= \lim_{n \rightarrow \infty} e^{\frac{n \ln\left(\frac{(1+H_1)^2+\dots+(1+H_{n+1})^2}{(1+H_1)^2+\dots+(1+H_n)^2} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_n)^2}{n}\right)} \frac{\ln(1+H_n)}{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_n)^2}{n}\right)} \frac{1}{\ln(1+H_n)}} = \\
 &= \lim_{n \rightarrow \infty} e^{\frac{\frac{1}{2} \cdot n \cdot \frac{1}{\ln(1+H_n)} \cdot \frac{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_{n+1})^2}{(1+H_1)^2+\dots+(1+H_n)^2} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)}}{\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)}} = \\
 &= \lim_{n \rightarrow \infty} e^{\frac{n}{\ln(1+H_n)} \ln\left(\frac{1+H_{n+1}}{1+H_n}\right)} = \lim_{n \rightarrow \infty} e^{\frac{n}{\ln(1+H_n)} \frac{1}{(n+1)(1+H_n)} (n+1)(1+H_n)} = \\
 &= e^{\frac{1}{\infty}} = e^0 = 1 \quad (3) \stackrel{(1);(2);(3)}{\Rightarrow} \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 2 > 1 \Rightarrow \lim_{n \rightarrow \infty} x_n = \infty \Rightarrow \Omega = \infty \\
 &\text{Because } \lim_{n \rightarrow \infty} \left(\frac{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_{n+1})^2}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_n)^2}{n}\right)} \right)^n = \lim_{n \rightarrow \infty} e^{\frac{n \cdot \ln\left(\frac{(1+H_1)^2+\dots+(1+H_{n+1})^2}{(1+H_1)^2+\dots+(1+H_n)^2} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\dots+(1+H_n)^2}{n}\right)}}
 \end{aligned}$$

884. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\log \left(\sum_{k=0}^n \binom{2^{-k}}{k} (n+k) \right) \right) - \sum_{k=1}^n \frac{n}{n+k}$$

Proposed by Florică Anastase – Romania

Solution 1 by Marian Ursărescu – Romania

Let $x_n = \sum_{k=0}^n \frac{c_{n+k}^k}{2^k}$. We have: $x_1 = \frac{c_1^0}{1} + \frac{c_2^1}{2} = 2$



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$$x_{n+1} = \sum_{k=0}^{n+1} \frac{C_{n+k+1}^k}{2^k} = \sum_{k=0}^n \frac{C_{n+k}^k + C_{n+k}^{k-1}}{2^k} = \\ = \sum_{k=0}^{n+1} \frac{C_{n+k}^k}{2^k} + \sum_{k=0}^{n+1} \frac{C_{n+k}^{k-1}}{2^k} = x_n + \sum_{k=1}^{n+2} \frac{C_{n+k}^{k-1}}{2^k} \quad (1)$$

$$\text{But } \sum_{k=1}^n \frac{C_{n+k}^{k-1}}{2^k} = \frac{C_{n+1}^0}{2} + \frac{C_{n+2}^1}{2^2} + \dots + \frac{C_{2n+2}^{n+1}}{2^{n+2}} = \\ = \frac{1}{2} \left(\frac{C_{n+1}^0}{2^0} + \frac{C_{n+2}^1}{2^1} + \dots + \frac{C_{2n+2}^{n+1}}{2^{n+1}} \right) = \frac{1}{2} x_{n+1} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow x_{n+1} = x_n + \frac{1}{2} x_{n+1} \Rightarrow \frac{1}{2} x_{n+1} = x_n \Rightarrow$$

$x_{n+1} = 2x_n$ and $x_1 = 2 \Rightarrow x_n = 2^n$ (geometric progress). We must find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\ln 2^n - \sum_{k=1}^n \frac{n}{n+k} \right) = \lim_{n \rightarrow \infty} n \left(\ln 2 - \sum_{k=1}^n \frac{1}{n+k} \right) = \\ = \lim_{n \rightarrow \infty} \frac{\ln 2 - \sum_{k=1}^n \frac{1}{n+k}}{\frac{1}{n}} \quad (3)$$

Now, using Cesaro-Stolz for $\frac{0}{0}$:

$$a_n = \ln 2 - \sum_{k=1}^n \frac{1}{n+k}, b_n = \frac{1}{n}$$

a) b_n strictly decreasing

$$b) \lim_{n \rightarrow \infty} b_n = 0, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln 2 - \frac{1}{n} \sum_{k=1}^n \frac{1}{1+\frac{k}{n}} = \ln 2 - \int_0^1 \frac{1}{1+x} dx = \ln 2 -$$

$$\ln 2 = 0$$

$$c) \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2n+2} - \frac{1}{2n+1} + \frac{1}{n+1}}{\frac{1}{n+1} - \frac{1}{n}} = \\ = \lim_{n \rightarrow \infty} \frac{-\frac{1}{2n+1} + \frac{1}{2n+2}}{\frac{n-n-1}{n(n+1)}} = \frac{\frac{-2n-2+2n+1}{(2n+1)(2n+2)}}{-\frac{1}{n(n+1)}} = \\ = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{(2n+1)(2n+2)}}{-\frac{1}{n(n+1)}} = \frac{1}{4} \quad (4)$$

$$\text{From (3)+(4)} \Rightarrow \Omega = \frac{1}{4}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Let:



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$$a_n = \sum_{k=0}^n \binom{n+k}{k} 2^{-k} = \sum_{k=0}^n \binom{n+k-1}{k} 2^{-k} + \sum_{k=0}^{n-1} \binom{n+k-1}{k} 2^{-k-1} = 2^{-n} \binom{2n-1}{n} +$$

$$+ \overbrace{\sum_{k=0}^{n-1} \binom{n+k-1}{k} 2^{-k}}^{\alpha_{n-1}} - \frac{1}{2} \left(2^{-n} \binom{2n}{n} - \overbrace{\sum_{k=0}^n \binom{n+k}{k} 2^{-k}}^{\alpha_n} \right)$$

$$\begin{aligned} \Rightarrow \alpha_n &= 2\alpha_{n-1} + \binom{2n-1}{n} 2^{1-n} - \binom{2n}{n} 2^n = \\ &= 2\alpha_{n-1} + \binom{2n-1}{n} 2^{1-n} - \left(\binom{2n-1}{n} + \binom{2n-1}{n-1} \right) 2^{-n} = \\ &= 2\alpha_{n-1} + \binom{2n-1}{n} 2^{-n} - \binom{2n-1}{n-1} 2^{-n} = 2\alpha_{n-1} \end{aligned}$$

$\therefore \alpha_n = 2\alpha_{n-1} = 2^2\alpha_{n-1} = \dots 2^n\alpha_0 = 2^n$. Also:

$$\beta_n = \sum_{k=1}^n \frac{n}{n+k} = n \sum_{k=n+1}^{2n} \frac{1}{k} = n \left(\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k} \right) = n(H_{2n} - H_n)$$

$$\begin{aligned} \text{Now: } \Omega &= \lim_{n \rightarrow \infty} (\log(\alpha_n) - \beta_n) = \lim_{n \rightarrow \infty} n(H_n - H_{2n} + \log 2) = \\ &= \lim_{n \rightarrow \infty} \frac{H_n - H_{2n} + \log 2}{\frac{1}{n}} \stackrel{s-C(\frac{0}{0})}{=} \lim_{n \rightarrow \infty} \frac{H_{n+1} - H_{2n+2} + \log 2 - H_n + H_{2n} - \log 2}{\frac{1}{n+1} - \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} n(n+1) \left(\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} \frac{n}{2(2n+1)} = \frac{1}{4} \\ \Omega &= \frac{1}{4} \text{ (answer)} \end{aligned}$$

885. Dedicated to teacher Mehmet Şahin

$$f: \mathbb{R} \rightarrow \mathbb{R}, 2f^4(x) + 2f^2(x) + 2 \leq 3f^3(x) + 3f(x), \forall x \in \mathbb{R}$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \sum_{1 \leq i < j \leq n} \left(f\left(e + \frac{\pi i}{n}\right) f\left(e + \frac{\pi j}{n}\right) \right) \right)$$

Proposed by Daniel Sitaru – Romania



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Solution by Florentin Vișescu – Romania

$$f: \mathbb{R} \rightarrow \mathbb{R}, 2f^4(x) + 2f^2(x) + 2 \leq 3f^3(x) + 3f(x), \forall x \in \mathbb{R}$$

$$2(f^4(x) + f^2(x) + 1) \leq 3[f^3(x) + f(x)]$$

$$2f^4(x) - 3f^3(x) + 2f^2(x) - 3f(x) + 2 \leq 0; \forall x \in \mathbb{R}$$

$$f(x) = 0 \text{ not verify; } f(x) \neq 0, \forall x \in \mathbb{R}$$

$$2f^2(x) - 3f(x) + 2 - 3\frac{1}{f(x)} + \frac{2}{f^2(x)} \leq 0; \forall x \in \mathbb{R}$$

$$2\left(f^2(x) + \frac{1}{f^2(x)}\right) - 3\left(f(x) + \frac{1}{f(x)}\right) + 2 \leq 0$$

$$f(x) + \frac{1}{f(x)} = g(x); 2(g^2(x) - 2) - 3g(x) + 2 \leq 0; 2g^2(x) - 3g(x) - 2 \leq 0$$

$$\Delta = 9 + 16 = 25; g(x)_{1,2} = \frac{3 \pm 5}{4} = \begin{cases} 2 \\ -1; g(x) \in \left[-\frac{1}{2}; 2\right] \end{cases}$$

$$f(x) + \frac{1}{f(x)} \in \left[-\frac{1}{2}; 2\right]; -\frac{1}{2} \leq f(x) + \frac{1}{f(x)} \leq 2 \Rightarrow f(x) = 1; \forall x \in \mathbb{R}$$

$$\Omega = \frac{1}{2} \left(\int_0^1 \mathbf{1} dx \right)^2 = \frac{1}{2} \left(x \Big|_0^1 \right)^2 = \frac{1}{2}$$

$$\sum_{1 \leq i < j \leq n} \left(f\left(e + \frac{\pi i}{n}\right) f\left(e + \frac{\pi j}{n}\right) \right) = \frac{\left(\sum_{i=1}^n f\left(e + \frac{\pi i}{n}\right) \right)^2 - \sum_{i=1}^n f^2\left(e + \frac{\pi i}{n}\right)}{2}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \left(\sum_{1 \leq i < j \leq n} \left(f\left(e + \frac{\pi i}{n}\right) f\left(e + \frac{\pi j}{n}\right) \right) \right) =$$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{i=1}^n f\left(e + \frac{\pi i}{n}\right) \right]^2 - \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{1}{n} \sum_{i=1}^n f^2\left(e + \frac{\pi i}{n}\right)$$

$$= \frac{1}{2} \left(\int_0^1 f(e + \pi x) dx \right)^2 - \frac{1}{2} \int_0^1 f^2(e + \pi x) dx \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{2} \left(\int_0^1 f(e + \pi x) dx \right)^2$$



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$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx = ?$$

Proposed by Jalil Hajimir-Canada

Solution 1 by Ali Jaffal-Lebanon

$$\begin{aligned} \text{Let } I_n &= \int_0^{+\infty} \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx \\ &= \int_0^1 \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx + \int_1^{+\infty} \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx \end{aligned}$$

$$\text{We have } \left| \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} \right| \leq \frac{nx}{n^3 x + x(1+x^3)} \leq \frac{n}{n^3 + x^3 + 1}$$

Since $|\sin t| \leq |t|, \forall t \in \mathbb{R}$

$$\left| \int_0^1 \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx \right| \leq \int_0^1 \frac{n}{n^3 + x^3 + 1} dx \leq \int_0^1 \frac{1}{n^2} dx \leq \frac{1}{n^2}$$

then $\lim_{n \rightarrow \infty} \int_0^1 \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx = 0$ we have:

$$\begin{aligned} \left| \int_1^{+\infty} \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx \right| &\leq \int_1^{+\infty} \frac{n}{n^3 + x^3 + 1} dx \leq \int_1^{+\infty} \frac{n}{n^3 + x^2} dx \leq \frac{1}{n^2} \int_1^{+\infty} \frac{dx}{1 + \left(\frac{x}{n\sqrt{n}}\right)^2} \\ &\leq \frac{1}{\sqrt{n}} \left[\frac{\pi}{2} - \arctan\left(\frac{1}{n\sqrt{n}}\right) \right] = \frac{\pi}{2\sqrt{n}} \end{aligned}$$

then $\lim_{n \rightarrow +\infty} \int_1^{+\infty} \frac{n^2 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx = 0$ therefore $\lim_{n \rightarrow +\infty} I_n = 0$

Solution 2 by Naren Bhandari-Bajura-Nepal

Here $\lim_{n \rightarrow \infty} \int_0^\infty \frac{n^3 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)} dx$

Consider $f_n(x) = \frac{n^3 \sin\left(\frac{x}{n}\right)}{n^3 x + x(1+x^3)}$. We note that at $x = 0$

$\lim_{n \rightarrow \infty} f_n(x) = 0$ and $\forall x > 0$ we have $\lim_{n \rightarrow \infty} f_n(1) = \lim_{n \rightarrow \infty} \frac{n^3 \sin\frac{x}{n}}{n^3 x + x(x^3+1)} =$



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$$= \lim_{n \rightarrow \infty} \frac{n^3 \sin \frac{x}{n}}{n^3 x + x(1+x^3)} = \lim_{n \rightarrow \infty} \frac{n^3 \left(\frac{x}{n} - 0 \left(\frac{x}{n} \right)^3 \right)}{n^3 x + n(1+x^3)} = \lim_{n \rightarrow \infty} \frac{n^2 x}{n^3 x + x(1+x^3)} = 0$$

$|\lim_{n \rightarrow \infty} f_n(n)| \leq g(n) = 0$. Then, by dominating convergence, we have:

$$\lim_{n \rightarrow \infty} \int_0^n f_n(x) dx = \int_0^\infty g(n) = \int_0^\infty 0 dx = 0$$

887. Find:

$$\Omega = \sum_{n=2}^{\infty} \left(\frac{H_n}{(\sum_{i=1}^{n-1} H_i)(\sum_{j=1}^n H_j)} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa – Romania

$$\begin{aligned} \Omega &= \sum_{n=2}^{\infty} \left(\frac{H_n}{(\sum_{i=1}^{n-1} H_i)(\sum_{j=1}^n H_j)} \right) = \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{H_1 + H_2 + \dots + H_{n-1}} - \frac{1}{H_1 + H_2 + \dots + H_{n-1} + H_n} \right) = \\ &= \frac{1}{H_1} - \frac{1}{H_1 + H_2} + \frac{1}{H_1 + H_2} - \frac{1}{H_1 + H_2 + H_3} + \dots + \frac{1}{H_1 + H_2 + \dots + H_{n-1}} - \\ &\quad - \frac{1}{H_1 + H_2 + \dots + H_n} + \dots = \lim_{n \rightarrow \infty} \frac{1}{H_1} - \frac{1}{H_1 + H_2 + \dots + H_n} \end{aligned}$$

$$\begin{aligned} \text{We know that } H_n &= \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \rightarrow \infty \Rightarrow H_1 + H_2 + \dots + H_n \rightarrow \infty \Rightarrow \\ &\Rightarrow \frac{1}{H_1 + H_2 + \dots + H_n} \rightarrow 0 \Rightarrow \Omega = \frac{1}{H_1} = \frac{1}{1} = 1 \end{aligned}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

We write the sum without notation

$$\begin{aligned} \Omega &= \frac{H_2}{H_1(H_1 + H_2)} + \frac{H_3}{(H_1 + H_2)(H_1 + H_2 + H_3)} + \dots + \\ &\quad + \frac{H_n}{(H_1 + H_2 + \dots + H_{n-1})(H_1 + H_2 + \dots + H_n)} \end{aligned}$$

Now, by partial fraction decomposition we have:



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$$\Omega = \left(\frac{1}{H_1} - \frac{1}{H_1 + H_2} \right) + \left(\frac{1}{H_1 + H_2} - \frac{1}{H_1 + H_2 + H_3} \right) + \cdots + \frac{1}{H_1 + H_2 + \cdots + H_{n-1}} - \frac{1}{H_1 + H_2 + \cdots + H_n}$$

Since Ω is a telescoping series with n^{th} partial sum as

$$\Omega = \frac{1}{H_1} - \frac{1}{H_1 + H_2 + \cdots + H_n}$$

$$\lim_{n \rightarrow \infty} \Omega = H_1 - \lim_{n \rightarrow \infty} \left[\frac{1}{\sum_{k=1}^n H_k} \right] = 1 - 0 = 1$$

Solution 3 by Naren Bhandari-Bajura-Nepal

$$\begin{aligned} \text{Here } \sum_{n=2}^{\infty} \left(\frac{H_n}{\sum_{e=1}^{n-1} H_i \sum_e^n H_i} \right) &= \sum_{n=2}^{\infty} \left(\frac{\psi(n+1)+\gamma}{\sum_{i=1}^{n-1} (\psi(i+1)+\gamma) \sum_{j=1}^{n-1} (\psi(j+1)+\gamma)} \right) \\ &= \frac{\psi(3)+\gamma}{(\psi(2)+\gamma)(\psi(2)+\gamma+\psi(3)+\gamma)} + \\ &+ \frac{\psi(4)+\gamma}{(\psi(2)+\gamma+\psi(3)+\gamma)(\psi(2)+\gamma+\psi(3)+\gamma+\psi(4)+\gamma)} + \cdots + \\ &+ \frac{\psi(n)+\gamma}{(\psi(2)+\gamma+\psi(3)+\gamma+\cdots+\psi(n)+\gamma)(\psi(3)+\gamma+\psi(4)+\gamma+\cdots+\psi(n+1)+\gamma)} \\ &= \frac{1}{\psi(2)+\gamma} - \frac{1}{\psi(2)+\psi(3)+2\gamma} + \frac{1}{\psi(2)+\psi(3)+2\gamma} - \frac{1}{\psi(2)+\psi(3)+\psi(4)+3\gamma} + \\ &\quad + \cdots \end{aligned}$$

We have telescoping sum give us

$$= \frac{1}{\psi(2)+\gamma} = \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{1+k} \right) \right)^{-1} = 1$$

888. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n(n+1)} \sum_{k=1}^n \left(k \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) \right) \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

$$\text{Let } f(x) = 1 - \frac{1}{x+1}, x \in [0, +\infty[$$



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$$f'(x) = \frac{1}{(x+1)^2} > 0 \text{ then } f \text{ is increasing on } [0, +\infty[$$

Let $n \in \mathbb{N}^$ and $1 \leq k \leq n$, so, $f(k) \leq f(n)$, then $1 - \frac{1}{k+1} \leq 1 - \frac{1}{n+1}$*

then $\frac{k}{k+1} \leq \frac{n}{n+1}; \frac{k}{n} \leq \frac{k+1}{n+1}$ then $\frac{k(k+1)}{n(n+1)} \leq \frac{(k+1)^2}{(n+1)^2}$ () but $\frac{k}{k+1} \leq \frac{n}{n+1} \Rightarrow \frac{k^2}{k^2+k} \leq \frac{n^2}{(n+1)n}$*

*So, $\frac{k^2}{n^2} \leq \frac{k^2+k}{n^2+n}$ (**). Then by (*) ; (**) we have: $\frac{k^2}{n^2} \leq \frac{k(k+1)}{n(n+1)} \leq \frac{(k+1)^2}{(n+1)^2}$*

but $x \rightarrow \arctan x$ is increasing, so: $\arctan\left(\frac{k^2}{n^2}\right) \leq \arctan\left(\frac{k(k+1)}{n(n+1)}\right) \leq \arctan\left(\left(\frac{k+1}{n+1}\right)^2\right)$

So, $\frac{1}{n(n+1)} \sum_{k=1}^n k \arctan\left(\frac{k}{n}\right)^2 \leq S_n \leq \frac{1}{n(n+1)} \sum_{k=1}^n (1+k) \arctan\left(\left(\frac{k+1}{n+1}\right)^2\right)$

Where $S_n = \frac{1}{n(n+1)} \sum_{k=1}^n k \arctan\left(\frac{k(k+1)}{n(n+1)}\right)$

We have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \arctan\left(\left(\frac{k}{n}\right)^2\right) = \int_0^1 x \arctan(x^2) dx$

Let $u = \arctan(x^2); u' = \frac{2x}{1+x^4}; v' = x; v = \frac{x^2}{2}$

then $\int_0^1 x \arctan(x^2) dx = \frac{\pi}{8} - \int_0^1 \frac{x^3}{1+x^4} dx = \frac{\pi}{8} - \frac{1}{4} [\ln(1+x^4)]_0^1 = \frac{\pi}{8} - \frac{1}{4} \ln 2$

We have $\lim_{n \rightarrow \infty} \frac{1}{n(n+1)} \sum_{k=1}^n k \arctan\left(\left(\frac{k}{n}\right)^2\right) = \frac{\pi}{8} - \frac{1}{4} \ln 2$

and $\lim_{n \rightarrow \infty} \frac{1}{n(n+1)} \sum_{k=1}^n (1+k) \arctan\left(\left(\frac{k+1}{n+1}\right)^2\right) =$

$\lim_{n \rightarrow \infty} \frac{1}{n(n+1)} \sum_{t=2}^{t=n+1} t \arctan\left(\left(\frac{t}{n+1}\right)^2\right) = \frac{\pi}{8} - \frac{1}{4} \ln 2$

Then $\lim_{n \rightarrow \infty} S_n = \frac{\pi}{8} - \frac{1}{4} \ln 2$

Solution 2 by Remus Florin Stanca-Romania

We know that $\lim_{n \rightarrow \infty} \sum_{k=1}^n (x_{k+1} - x_k) f(\zeta_k) = \int_a^b f(x) dx, f = \text{continuous and}$

$\zeta_k \in [x_k; x_{k+1}]$ and $\lim_{n \rightarrow \infty} |\Delta_n| = 0$ where $|\Delta_n| \stackrel{\max}{\equiv} (x_{k+1} - x_k)$, and $x_1 = a$ and

$x_n = b$, let $x_k = \frac{k(k+1)}{n(n+1)} \Rightarrow x_{k+1} - x_k = \frac{(k+1)(k+2)}{n(n+1)} - \frac{k(k+1)}{n(n+1)} = \frac{k^2+2k+k+2-k^2-k}{(n+1)} =$

$= 2 \cdot \frac{k+1}{n(n+1)} \stackrel{k \leq n}{\Rightarrow} \max(x_{k+1} - x_k) = 2 \cdot \frac{n+1}{n(n+1)} = \frac{2}{n}$ and $\lim_{n \rightarrow \infty} \max(x_{k+1} - x_k) = 0 \Rightarrow$

$\Rightarrow \lim_{n \rightarrow \infty} |\Delta_n| = 0$ and $\zeta_k = \frac{k(k+1)}{n(n+1)}$ and $f(x) = \tan^{-1}(x) = \text{continuous}$



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$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2(k+1)}{n(n+1)} \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) = \int_0^1 \tan^{-1}(x) dx \Rightarrow$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n(n+1)} \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) = \frac{1}{2} \int_0^1 \tan^{-1}(x) dx - \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n(n+1)} \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right)$$

(1)

$$\tan^{-1}(x) = \text{increasing} \Rightarrow \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) \leq \tan^{-1} \left(\frac{n^2+n}{n^2+n} \right) = \frac{\pi}{4} \Rightarrow$$

$$\Rightarrow \frac{1}{n(n+1)} \sum_{k=1}^n \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) \leq \frac{1}{n(n+1)} \sum_{k=1}^n \frac{\pi}{4} = \frac{\frac{\pi}{4}}{n+1} \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{\pi}{4}}{n+1} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} \sum_{k=1}^n \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) = 0 \stackrel{(1)}{\Rightarrow}$$

$$\stackrel{(1)}{\Rightarrow} \lim_{n \rightarrow \infty} \left(\frac{1}{n(n+1)} \sum_{k=1}^n \left(k \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) \right) \right) = \frac{1}{2} \int_0^1 \tan^{-1}(x) dx \quad (2)$$

$$\begin{aligned} \int \tan^{-1}(x) dx &= \int \tan^{-1}(x) \cdot x' dx = x \tan^{-1}(x) - \int \frac{x}{x^2+1} dx = \\ &= x \tan^{-1}(x) - \frac{1}{2} \int \frac{2x}{x^2+1} dx = x \tan^{-1}(x) - \frac{1}{2} \ln(x^2+1) \stackrel{(2)}{\Rightarrow} \end{aligned}$$

$$\begin{aligned} \stackrel{(2)}{\Rightarrow} \lim_{n \rightarrow \infty} \left(\frac{1}{n(n+1)} \sum_{k=1}^n \left(k \tan^{-1} \left(\frac{k^2+k}{n^2+n} \right) \right) \right) &= \frac{1}{2} \left(\frac{\pi}{4} - \frac{\ln(2)}{2} \right) = \\ &= \frac{\pi}{8} - \frac{\ln(2)}{4} = \frac{\pi - \ln(4)}{8} \Rightarrow \Omega = \frac{\pi - \ln(4)}{8} \end{aligned}$$

Solution 3 by Marian Ursărescu-Romania

(another approach) Let $f: [0, 1] \rightarrow \mathbb{R}, f(x) = \arctan x ; f - \text{Riemann integrability}$

$$\text{Let } \Delta_n = \left(0, \frac{1-2}{n(n+1)}, \dots, \frac{k(k+1)}{n(n+1)}, \dots, \frac{n(n+1)}{n(n+1)} = 1 \right)$$

$$||\Delta_n|| = \frac{1}{n(n+1)} \max_{1 \leq k \leq n} (k(k+1) - (k-1)(k)) = \frac{1}{n(n+1)} \cdot 2k = \frac{2n}{n(n+1)} \Rightarrow$$

$||\Delta_n|| \rightarrow 0$. Let $\xi_k^n \in [x_{k-1}^n, x_k^n]$, so that:

$$\begin{aligned} \xi_k^n = \frac{k(n+1)}{n(n+1)} \Rightarrow \sigma_{\Delta_n}(f, \xi_k^n) &= \sum_{k=1}^n f \left(\frac{k(k+1)}{n(n+1)} \right) \cdot \frac{2k}{n(n+1)} = \\ &= \frac{2}{n(n+1)} \sum_{k=1}^n k \cdot \arctan \frac{k^2+k}{n^2+n} \Rightarrow \end{aligned}$$



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$$\begin{aligned}
 \Omega &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} \sum_{k=1}^n k \arctan \frac{k^2 + k}{n^2 + n} = \frac{1}{2} \int_0^1 \arctan x \, dx \\
 &= \frac{1}{2} \int_0^1 x' \arctan x \, dx = \frac{1}{2} x \arctan x \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x}{1+x^2} \, dx \\
 &= \frac{\pi}{8} - \frac{1}{4} \ln(1+x^2) \Big|_0^1 = \frac{\pi}{8} - \frac{1}{4} \ln 2
 \end{aligned}$$

889. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \int_1^2 \left(\log \left(1 + e^{\frac{n}{x}} \right) \right) dx \right)$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Ali Jaffal-Lebanon

$$\begin{aligned}
 \text{Let } I_n &= \frac{1}{n} \int_1^2 \log \left(1 + e^{\frac{n}{x}} \right) dx; \quad I_n - \int_1^2 \frac{1}{x} dx = \frac{1}{n} \int_1^2 \log \left(1 + e^{\frac{n}{x}} \right) dx - \frac{1}{n} \int_1^2 \log \left(e^{\frac{n}{x}} \right) dx \\
 I_n - \log 2 &= \frac{1}{n} \int_1^2 \log \left(\frac{1+e^{\frac{n}{x}}}{e^{\frac{n}{x}}} \right) dx \text{ then } |I_n - \log 2| = \frac{1}{n} \int_1^2 \log \left(1 + e^{-\frac{n}{x}} \right) dx
 \end{aligned}$$

We have $\log(1+u) \leq u, \forall u > 0$ and $e^u > u, \forall u > 0$ then $e^{-u} < \frac{1}{u}, \forall u > 0$

$$\text{so, } \log \left(1 + e^{-\frac{n}{x}} \right) \leq \frac{x}{n} \text{ then } |I_n - \log 2| \leq \frac{1}{n^2} \int_1^2 x \, dx \leq \frac{3}{2n^2}$$

$$\text{so } \lim_{n \rightarrow \infty} |I_n - \log 2| = 0 \text{ then } \lim_{n \rightarrow \infty} I_n = \log 2$$

Solution 2 by Naren Bhandari-Bajura-Nepal

We consider $f_n(x) = \frac{\log(1+e^{\frac{n}{x}})}{n}$. We observe that at $x = 1$, the limit of $f_n(1) = 1, \frac{1}{2}$ respectively when $n \rightarrow \infty$. Limit of $f_n(x) = \frac{1}{x}$ when $n \rightarrow \infty$. Hence by Dominated

Convergence Theorem, we have:

$$\lim_{n \rightarrow \infty} \int_1^2 f_n(x) \, dx = \int_1^2 \frac{1}{x} \, dx = \ln 2$$

890. Find the closed form:



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$$\Omega = \sum_{n=1}^{\infty} \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdot \dots \cdot (\pi^e + n - 1)}{(e^\pi + 1)(e^\pi + 2)(e^\pi + 3) \cdot \dots \cdot (e^\pi + n)}$$

Proposed by Florică Anastase-Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$S = \sum_{n=1}^{+\infty} \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdot \dots \cdot (\pi^e + n - 1)}{(e^\pi + 1)(e^\pi + 2) \cdot \dots \cdot (e^\pi + n)}$$

Consider the series $s(u_n, \sum_{n \geq 1} u_n)/u_n = \frac{(a+1)(a+2)\dots(a+n)}{(b+1)(b+2)\dots(b+n)}$, so:

$$\begin{aligned} S(a, b) &= \sum_{n=1}^{+\infty} \frac{(a+1)(a+2)\dots(a+n)}{(b+1)(b+2)\dots(b+n)} / 0 < a+1 < b \\ &= \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a)} \sum_{n=1}^{+\infty} \frac{\Gamma(n+a+1)\Gamma(b-a)}{\Gamma(n+b+1)} \\ &= \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a)} \sum_{n=1}^{+\infty} \int_0^1 t^{n+a} (1-t)^{b-a-1} dt \\ &= \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a)} \int_0^1 \sum_{n=1}^{+\infty} t^{n+a} (1-t)^{b-a-1} dt \\ &= \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a)} \int_0^1 \frac{t^{a+1}(1-t)^{b-a-1}}{1-t} dt \\ &= \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a)} \cdot \frac{\Gamma(a+2)\Gamma(b-a-1)}{\Gamma(b+1)} \\ \therefore S(a, b) &= \frac{a+1}{(b-a-1)} \quad (1) \end{aligned}$$

Consider the function $f(t) = \frac{\ln(t)}{t}$; $f \in C^1([0; +\infty[)$ and $f'(t) = \frac{1-\ln(t)}{t^2}$

So, for $t > e$; $f(t) < f(e) \Leftrightarrow \frac{\ln(\pi)}{\pi} < \frac{\ln(e)}{e} \Rightarrow e^\pi > \pi^e \Rightarrow e^\pi > \pi^e$

Let $a = \pi^e - 1$, $b = e^\pi$. $e > 0 \Rightarrow \pi^e > 1 \Leftrightarrow \pi^e - 1 > 0$

So, $0 < \pi^e < e^\pi$, using the result (1):



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$$\sum_{n=1}^{+\infty} \frac{\pi^e(\pi^e+1)(\pi^e+2) \cdot \dots \cdot (\pi^e+n-1)}{(e^\pi+1)(e^\pi+2) \cdot \dots \cdot (e^\pi+n)} = \frac{\pi^e}{e^\pi - \pi^e}$$

Solution 2 by Samir HajAli-Damascus-Syria

$$\text{Let put: } \Omega(p, q) = \sum_{k=0}^{\infty} \prod_{m=0}^k \frac{p+m}{q+m}$$

$$\begin{aligned} \text{We have: } \prod_{m=0}^k \frac{p+m}{q+m} &= \frac{\Gamma(p+k+1)}{\Gamma(q+k+1)} \times \frac{\Gamma(q)}{\Gamma(p)} = \frac{\Gamma(p+k+1) \times \Gamma(q-p)}{\Gamma(q+k+1)} \times \frac{\Gamma(q)}{\Gamma(p) \times \Gamma(q-p)} \\ &= \frac{1}{B(p; q-p)} \times B(p+k+1; q-p); q > p > 0 \end{aligned}$$

Where $B(*; *)$ is Beta function. Now: $\Omega(p, q) = \frac{1}{B(p; q-p)} \cdot \sum_{k=0}^{\infty} B(p+k+1; q-p)$

$$\text{But: } B(p+k+1; q-p) = \int_0^1 x^{p+k} \cdot (1-x)^{q-p-1} dx.$$

Since: $\sum_{k=0}^{\infty} x^k$ uniformly converge

$$\begin{aligned} B(p; q-p)\Omega(p, q) &= \int_0^1 \sum_{k=0}^{\infty} x^{p+k} \cdot (1-x)^{q-p-1} dx = \int_0^1 \frac{x^p(1-x)^{q-p-1}}{1-x} dx \\ &= \int_0^1 x^{(p+1)-1} \cdot (1-x)^{(q-p-1)-1} dx = B(p+1; q-p-1) \end{aligned}$$

$$\text{So, } \Omega(p, q) = \frac{B(p+1; q-p-1)}{B(p; q-p)} = \frac{\Gamma(p+1) \times \Gamma(q-p-1)}{\Gamma(p) \cdot \Gamma(q-p)} \times \frac{\Gamma(q)}{\Gamma(q)} = \frac{p}{q-p-1}; q > p > 0$$

Now, for: $p = \pi^e, q = e^\pi + 1, k = n - 1$

Where: $e^\pi > \pi^e$

$$\text{We find: } \Omega = \sum_{n=1}^{\infty} \frac{\pi^e(\pi^e+1) \dots (\pi^e+n-1)}{(e^\pi+1)(e^\pi+2) \dots (e^\pi+n)} = \frac{\pi^e}{e^\pi+1-\pi^e-1} = \frac{\pi^e}{e^\pi-\pi^e}$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned} \Omega &= \sum_{n=1}^{\infty} \frac{\pi^e(\pi^e+1)(\pi^e+2) \cdot \dots \cdot (\pi^e+n-1)}{(e^\pi+1)(e^\pi+2) \cdot \dots \cdot (e^\pi+n)} = \\ &= \frac{\Gamma(e^\pi+1)}{\Gamma(e^\pi-\pi^e+1)\Gamma(\pi^e)} \sum_{n=1}^{\infty} B(\pi^e+n, e^\pi-\pi^e+1) = \\ &= \frac{\Gamma(e^\pi+1)}{\Gamma(e^\pi-\pi^e+1)\Gamma(\pi^e)} \int_0^1 x^{\pi^e} (1-x)^{e^\pi-\pi^e} \sum_{n=1}^{\infty} x^{n-1} dx \end{aligned}$$



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$$\begin{aligned}
 &= \frac{\Gamma(e^\pi + 1)}{\Gamma(e^\pi - \pi^e + 1)\Gamma(\pi^e)} \int_0^1 x^{\pi^e} (1-x)^{e^\pi - \pi^e - 1} dx = \\
 &= \frac{\Gamma(e^\pi + 1)}{\Gamma(e^\pi - \pi^e + 1)\Gamma(\pi^e)} \cdot \frac{\Gamma(\pi^e + 1)\Gamma(e^\pi - \pi^e)}{\Gamma(e^\pi + 1)} = \frac{\pi^e}{e^\pi - \pi^e}
 \end{aligned}$$

Solution 4 by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned}
 \Omega &= \sum_{n=1}^{\infty} \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n - 1)}{(e^\pi + 1)(e^\pi + 2) \cdots (e^\pi + n)} = \\
 &= \frac{e^\pi}{e^\pi - \pi^e} \sum_{n=1}^{\infty} \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n - 1)(e^\pi + n) - \pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n - 1)(\pi^e + n)}{e^\pi(e^\pi + 1)(e^\pi + 2) \cdots (e^\pi + n)} \\
 &= \frac{e^\pi}{e^\pi - \pi^e} \sum_{n=1}^{\infty} \left(\frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n - 1)}{e^\pi(e^\pi + 1)(e^\pi + 2) \cdots (e^\pi + n - 1)} - \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n)}{e^\pi(e^\pi + 1)(e^\pi + 2) \cdots (e^\pi + n)} \right) \\
 &= \frac{e^\pi}{e^\pi - \pi^e} \left(\frac{\pi^e}{e^\pi} - \lim_{n \rightarrow \infty} \frac{\pi^e(\pi^e + 1)(\pi^e + 2) \cdots (\pi^e + n)}{e^\pi(e^\pi + 1)(e^\pi + 2) \cdots (e^\pi + n)} \right) = \frac{\pi^e}{e^\pi - \pi^e}
 \end{aligned}$$

891. $\omega_n = (2n+1) \left(\sum_{k=0}^{2n} \tan \left(x + \frac{k\pi}{2n+1} \right) \right)^{-1}$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}} \right)^{\omega_n} \right)$$

Proposed by Florică Anastase – Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\text{Since: } \prod_{k=0}^{n-1} \cos \left(x + \frac{k\pi}{n} \right) = \frac{\sin(n(x + \frac{\pi}{2}))}{2^{n-1}} \Rightarrow \sum_{k=0}^{n-1} \log \left(\cos \left(x + \frac{k\pi}{n} \right) \right) = \log \left(\frac{\sin(n(x + \frac{\pi}{2}))}{2^{n-1}} \right)$$

Now, by differentiable both side with respect to x we get:

$$\sum_{k=0}^{n-1} \frac{d}{dx} \log \left(\cos \left(x + \frac{k\pi}{n} \right) \right) = \frac{d}{dx} \log \left(\frac{\sin \left(n \left(x + \frac{\pi}{2} \right) \right)}{2^{n-1}} \right)$$



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$$\Rightarrow \sum_{k=0}^{n-1} \tan\left(x + \frac{k\pi}{n}\right) = -n \cot\left(n\left(x + \frac{\pi}{2}\right)\right) \Rightarrow \omega_n = \frac{2n+1}{\sum_{k=0}^{2n} \left(x + \frac{k\pi}{2n+1}\right)} = \\ = \frac{2n+1}{-(2n+1) \cot\left((2n+1)\left(x + \frac{\pi}{2}\right)\right)} = \cot((2n+1)x)$$

$$\text{Let: } \varphi_n = \lim_{x \rightarrow \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot(\frac{\pi}{2n+1})} \right)^{\omega_n} = \exp \left(\lim_{x \rightarrow \frac{\pi}{2n+1}} \cot((2n+1)x) \log \left(\frac{\cot x}{\cot(\frac{\pi}{2n+1})} \right) \right) = \\ = \exp \left(\lim_{x \rightarrow \frac{\pi}{2n+1}} \frac{\log \left(\frac{\cot x}{\cot(\frac{\pi}{2n+1})} \right)}{\tan((2n+1)x)} \right) \stackrel{\text{Hospital}}{=} \\ = \exp \left(\lim_{x \rightarrow \frac{\pi}{2n+1}} \frac{-\sec(x) \csc(x)}{(2n+1) \sec^2((2n+1)x)} \right) = \exp \left(-\frac{\csc(\frac{2\pi}{2n+1})}{2n+1} \right). \text{ Now:}$$

$$\Omega = \lim_{n \rightarrow \infty} \varphi_n = \lim_{n \rightarrow \infty} \exp \left(-\frac{\csc(\frac{2\pi}{2n+1})}{2n+1} \right) = \exp \left(-\frac{1}{2\pi} \lim_{n \rightarrow \infty} \frac{2\pi}{2n+1} \csc \left(\frac{2\pi}{2n+1} \right) \right) = \frac{1}{\sqrt[2\pi]{e}}$$

892. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\sqrt[4]{(2n+3)^3 \cdot \sqrt[n+1]{(n+1)!}} - \sqrt[4]{(2n+1)^3 \cdot \sqrt[n]{n!}} \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Naren Bhandari-Bajura-Nepal

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\sqrt[4]{(2n+3)^3 \cdot \sqrt[n+1]{(n+1)!}} - \sqrt[4]{(2n+1)^3 \cdot \sqrt[n]{n!}} \right) \\ & \sim \lim_{n \rightarrow \infty} \left((2n+3)^{\frac{3}{4}} \left(\sqrt{2\pi(n+1)} \left(\frac{n+1}{e} \right)^{n+1} \right)^{\frac{1}{4(n+1)}} - (2n+1)^{\frac{3}{4}} \left(\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \right)^{\frac{1}{4}} \right) \\ & = \sqrt[4]{\frac{8}{e}} \lim_{n \rightarrow \infty} \left(\left(n + \frac{3}{2} \right)^{\frac{3}{4}} \sqrt[4]{n+1} (n+1)^{\frac{1}{8(n+1)}} - \left(n + \frac{1}{2} \right)^{\frac{3}{4}} \sqrt[4]{n} n^{\frac{1}{8n}} \right) \end{aligned}$$



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$$\begin{aligned}
 &= \sqrt[4]{\frac{8}{e}} \lim_{n \rightarrow \infty} \left((n+1)^{\frac{3}{4} + \frac{1}{4}} \left(1 + \frac{1}{2(n+1)} \right)^{\frac{3}{4}} e^{\frac{\log(n+1)}{8(n+1)}} - n^{\frac{3}{4} + \frac{1}{4}} \left(1 + \frac{1}{2n} \right)^{\frac{3}{4}} e^{\frac{\log n}{8n}} \right) \\
 &= \sqrt[4]{\frac{8}{e}} \lim_{n \rightarrow \infty} \left(n+1 \left(1 + \frac{3}{8(n+1)} + O\left(\frac{1}{(n+1)^2}\right) \right) - n \left(1 + \frac{3}{8n} + O\left(\frac{1}{n^2}\right) \right) \right) \\
 &= \sqrt[4]{\frac{8}{e}} \lim_{n \rightarrow \infty} \left(n+1 + \frac{3}{8} - n - \frac{3}{8} \right) = \sqrt[4]{\frac{8}{e}}
 \end{aligned}$$

893. If: $\omega_n = 1 - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} - \dots + \frac{(-1)^n \binom{n}{n}}{2n+1}$

Find: $\Omega = \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{e^n}}$

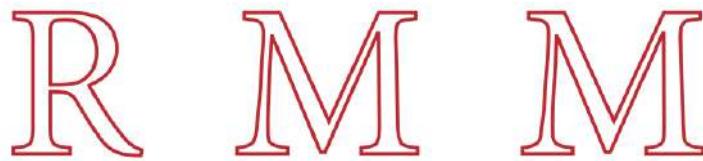
Proposed by Florică Anastase – Romania

Solution 1 by proposer

$$\begin{aligned}
 (1-x^2)^n &= \binom{n}{0} - \binom{n}{1}x^2 + \binom{n}{2}x^4 - \dots + (-1)^n \binom{n}{n}x^{2n} \\
 I_n &= \int_0^1 (1-x^2)^n \cdot x' dx = (1-x^2)^n \cdot x \Big|_0^1 + 2n \int_0^1 (1-x^2)^{n-1} \cdot x^2 dx = \\
 &= -2n \int_0^1 (1-x^2-1)(1-x^2)^{n-1} dx = -2n \int_0^1 (1-x^2)^n dx + 2n \int_0^1 (1-x^2)^{n-1} dx = \\
 &= -2nI_n + 2nI_{n-1} \Rightarrow I_n = \frac{2^{2n} \cdot (n!)^2}{(2n+1)!}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sqrt[n]{\omega_n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^{2n} \cdot (n!)^2}{(2n+1)!}} \stackrel{C-D'Alembert}{=} \lim_{n \rightarrow \infty} \frac{2^{2(n+1)}((n+1)!)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{2n}(n!)^2} = 1 \Rightarrow \\
 \Omega &= \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{e^n}} = e^{\lim_{n \rightarrow \infty} \frac{\sqrt[n]{\omega_n} \cdot n!}{n! \cdot e^n}} = e^0 = 1
 \end{aligned}$$

Solution 2 by Naren Bhandari-Bajura-Nepal



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$$\begin{aligned}
 \text{We write: } \omega_n &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{2k+1} = \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^1 x^{2k} dx \\
 &= \int_0^1 \left(\sum_{k=0}^n (-1)^k \binom{n}{k} x^{2k} \right) dx = \int_0^1 (1-x^2)^n dx \\
 &\stackrel{x^2=x}{=} \int_0^1 (1-x)^n x^{-\frac{1}{2}} dx = \frac{1}{2} \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \Gamma\left(\frac{1}{2}\right) \right] \\
 &= \frac{1}{2} \left[\frac{n! \sqrt{\pi}}{(2n+1)!! \sqrt{\pi}} \right] = \frac{2^n n!}{(2n+1)!!} = \frac{4^n (n!)^2}{(2n+1)!}
 \end{aligned}$$

Now, we solve for

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{e^n}} = \exp \left(\lim_{n \rightarrow \infty} \frac{\sqrt[n]{\omega_n}}{e^n} \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{4}{\sqrt[n]{(2n+1)e^n}} \sqrt[n]{\frac{(n!)^2}{(2n)!}} \right) \\
 &\sim \exp \left(\lim_{n \rightarrow \infty} \frac{4}{\sqrt[n]{2n+1} e^n} \frac{\sqrt[2n]{\pi}}{4} \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{\sqrt[2n]{\pi}}{e^n} \right) = e^0 = 1
 \end{aligned}$$

Note for all $k \geq 1$ inductively we can show that: $1 \leq n^{\frac{1}{kn}} \leq n^{\frac{1}{n}}$

$$\therefore \lim_{n \rightarrow \infty} 1 \leq \lim_{n \rightarrow \infty} n^{\frac{1}{kn}} \leq \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Set $k = 2$ and hence by Squeeze theorem $\lim_{n \rightarrow \infty} n^{\frac{1}{2n}} = 1$

894. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[3]{n}} - \sum_{k=1}^n \sin \left(\frac{2k}{\sqrt[3]{n}} \right) \right)$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Florentin Vișescu-Romania

We know that: $x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\frac{2k}{\sqrt[3]{n}} - \frac{1}{3!} \left(\frac{2k}{\sqrt[3]{n}} \right)^3 \leq \sin \frac{2k}{\sqrt[3]{n}} \leq \frac{2k}{\sqrt[3]{n}} - \frac{1}{3!} \left(\frac{2k}{\sqrt[3]{n}} \right)^3 + \frac{1}{5!} \left(\frac{2k}{\sqrt[3]{n}} \right)^5$$



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$$\begin{aligned}
 & \frac{2 \sum_{k=1}^n k}{n\sqrt[3]{n}} - \frac{1}{3!} \cdot \frac{2^3}{n^4} \sum_{k=1}^n k^3 \leq \sum_{k=1}^n \sin \frac{2k}{n\sqrt[3]{n}} \leq \\
 & \frac{2}{n\sqrt[3]{n}} \sum_{k=1}^n k - \frac{1}{3!} \cdot \frac{2^3}{n^4} \sum_{k=1}^n k^3 + \frac{1}{5!} \cdot \frac{2^5}{n^6\sqrt[3]{n^2}} \sum_{k=1}^n k^5 \\
 & \frac{n(n+1)}{n\sqrt[3]{n}} - \frac{1}{3!} \cdot \frac{2^3}{n^4} \cdot \frac{n^2(n+1)^2}{4} \leq \sum_{k=1}^n \sin \frac{2k}{n\sqrt[3]{n}} \leq \\
 & \frac{n(n+1)}{n\sqrt[3]{n}} - \frac{1}{3} \cdot \frac{2^3}{n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{1}{5!} \cdot \frac{2^5}{n^6\sqrt[3]{n^2}} P_6(n) \\
 & \underbrace{\frac{1}{3} \cdot \frac{2^3}{n^5} \cdot \frac{n^2(n+1)^2}{4}}_{\frac{1}{3}} - \underbrace{\frac{1}{5!} \cdot \frac{2^5}{n^6\sqrt[3]{n^2}} P_6(n)}_0 \leq \frac{n+1}{\sqrt[3]{n}} - \sum_{k=1}^n \sin \frac{2k}{n\sqrt[3]{n}} \leq \underbrace{\frac{1}{3!} \cdot \frac{2^3}{n^4} \cdot \frac{n^2(n+1)^2}{4}}_{\frac{1}{3}}
 \end{aligned}$$

$P_6(n)$ grade 6 polynomial in n .

Solution 2 by Naren Bhandari-Bajura-Nepal

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[3]{n}} - \sum_{k=1}^n \sin \left(\frac{2k}{n\sqrt[3]{n}} \right) \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n\sqrt[3]{n}} - \sin \left(\frac{2k}{n\sqrt[3]{n}} \right) \right) \\
 & = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n\sqrt[3]{n}} - \left(\frac{2k}{n\sqrt[3]{n}} - \frac{8k^3}{6n^4} + \frac{32k^5}{120n^5\sqrt[3]{n^5}} \right) - \dots \right) \\
 & = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{8k^3}{6n^4} - \lim_{n \rightarrow \infty} \sum_{k=1}^n A(k) = \frac{4}{3} \cdot \frac{n^2(n+1)^2}{4n^4} - S(A) = \frac{1}{3} - S(A)
 \end{aligned}$$

We note that:

$$\begin{aligned}
 S(A) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sum_{r=2}^{\infty} \frac{2^{2r+1}(-1)^{r+1}k^{2r+1}}{(2r+1)! n^{2r+1}\sqrt[3]{n^{2r+1}}} \right) \stackrel{C.S.t}{=} \\
 &= \sum_{r=2}^n \frac{2^{2r+1}}{(2r+1)!} \left(\lim_{n \rightarrow \infty} \frac{(n+1)^{2r+1}}{(n+1)^{2r+1}\sqrt[3]{(n+1)^{2r+1}} - n^{2r+1}\sqrt[3]{n^{2r+1}}} \right) \\
 &= \sum_{r=2}^n \frac{2^{2r+1}(-1)^{r+1}}{(2r+1)!} (0) = 0
 \end{aligned}$$

895. Find:



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$$\Omega(a) = \prod_{n=1}^{\infty} \left(\frac{n + a^{\tan a}}{n + (\tan a)^a} \right), 0 < a < 1$$

Proposed by Florică Anastase – Romania

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega(a) = \prod_{n=1}^{+\infty} \left(\frac{n + a^{\tan a}}{n + (\tan a)^a} \right). \text{ Let } \alpha = a^{\tan a}, \beta = (\tan a)^a$$

$$\therefore \Gamma_m(a) = \prod_{m=1}^m \frac{n + \alpha}{n + \beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + 1)} \cdot \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \beta + 1)}$$

If $\alpha < \beta$ then: $\lim_{m \rightarrow +\infty} P_m(a) = 0$. If $\alpha > \beta$ then $\lim_{m \rightarrow +\infty} P_m(a) = \infty$

If $\alpha = \beta$ then $\lim_{m \rightarrow +\infty} P_m(a) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}$. Let $f(t) = \frac{\ln(t)}{t}$ then $f'(t) = \frac{1-\ln(t)}{t^2}$

So, for: $t \geq e, f' \leq 0$ and $f \searrow$ for: $t \leq e, f' \geq 0$ and $f \nearrow$

so: $\ln(\alpha) = \tan(a) \ln(a)$ and $\ln(\beta) = a \ln(\tan(a))$

we know: $\tan(a) \geq a$. For $0 < a < 1$

$$\left(g(t) = \tan(t) - t \Rightarrow g'(t) = \frac{1}{\cos^2(t)} - 1 \geq 0 \text{ and } g(t) \geq g(0) = 0 \right)$$

for: $a < 1; \tan(a) < \tan(1) < e$ so: $a < \tan(a) < e$, then $\frac{\ln(a)}{a} < \frac{\ln(\tan(a))}{\tan(a)} < \frac{1}{e}$

$\therefore \tan(a) \ln(a) < a \ln(\tan(a)) \therefore a^{\tan(a)} < (\tan(a))^a$. So: $\Omega(a) = 0$

$$\prod_{n=1}^{+\infty} \left(\frac{n + a^{\tan a}}{n + (\tan a)^a} \right) = 0$$

896. Evaluate the limit:

$$\lim_{n \rightarrow \infty} \left(\int_0^\infty (\sin(x^n) + \cos(x^n)) dx \right)^n$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\Omega = \lim_{n \rightarrow +\infty} \left(\int_0^{+\infty} (\sin(x^n) + \cos(x^n)) dx \right)^n. \text{ Let } I(n) = \int_0^{+\infty} (\sin(x^n) + \cos(x^n)) dx$$



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$$t = x^n \Rightarrow x = t^{\frac{1}{n}} \Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt \therefore I = \frac{1}{n} \int_0^{+\infty} \frac{1}{t^{1-\frac{1}{n}}} (\sin(t) + \cot(t)) dt$$

$$= \frac{1}{n\Gamma\left(1 - \frac{1}{n}\right)} \int_0^{+\infty} \frac{t^{-\frac{1}{n}} + t^{1-\frac{1}{n}}}{1+t^2} dt = \frac{1}{2n\Gamma\left(1 - \frac{1}{n}\right)} \int_0^{+\infty} \frac{-\frac{1}{2}\left(\frac{1}{n} + 1\right) + t^{-\frac{1}{2n}}}{1+t} dt$$

$$\frac{1}{2}\left(1 + \frac{1}{n}\right), \frac{1}{2n} \in]0, 1[, \forall n > 1$$

$$\therefore I = \frac{\pi}{2n\Gamma\left(1 - \frac{1}{n}\right)} \left(\frac{1}{\sin\left(\frac{\pi}{2}\left(1 + \frac{1}{n}\right)\right)} + \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \right) = \frac{\pi}{2n\Gamma\left(1 - \frac{1}{n}\right)} \left(\frac{1}{\cos\left(\frac{\pi}{2n}\right)} + \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \right)$$

$$= \frac{\pi(\cos(\frac{\pi}{2n}) + \sin(\frac{\pi}{2n}))}{n\Gamma\left(1 - \frac{1}{n}\right) \sin\left(\frac{\pi}{n}\right)}. So: I = \left(\cos\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right) \right) \Gamma\left(1 + \frac{1}{n}\right)$$

$$\therefore \Omega = \lim_{n \rightarrow \infty} \left[\Gamma\left(1 + \frac{1}{n}\right) \left(\cos\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right) \right) \right]^n$$

$$= \lim_{n \rightarrow +\infty} e^{n \left(\ln\left(\Gamma\left(1 + \frac{1}{n}\right)\right) + \ln\left(\cos\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right)\right) \right)} = \lim_{x \rightarrow 0^+} e^{\frac{1}{x} \left(\ln(\Gamma(1+x)) + \ln(\cos(\frac{\pi}{2}x) + \sin(\frac{\pi}{2}x)) \right)}$$

$$= \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} \left(\ln(\Gamma(1+x)) + \frac{1}{2} \ln(1 + \sin(\pi x)) \right)$$

Then, we have: $\lim_{x \rightarrow 0^+} \frac{\ln(\Gamma(1+x))}{x} = \Psi(1) = -\gamma$ and, $\lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin(\pi x))}{x} = \pi$

$$So: \Omega = e^{\frac{\pi}{2} - \gamma} \therefore \lim_{n \rightarrow +\infty} \left(\int_0^{+\infty} (\sin(x^n) + \cos(x^n)) dx \right)^n = e^{\frac{\pi}{2} - \gamma}$$

(γ denote Euler – Mascheroni constant)

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Since: $\int_0^\infty \sin(ax^n) dx = \frac{\Gamma(\frac{1}{n}) \sin(\frac{\pi}{2n})}{n^{n\sqrt{a}}}$ and $\int_0^\infty \cos(ax^n) dx = \frac{\Gamma(\frac{1}{n}) \cos(\frac{\pi}{2n})}{n^{n\sqrt{a}}}$ where: $a > 0$ and

$n > 1$ we obtain:

$$\Omega = \lim_{n \rightarrow \infty} \left(\int_0^\infty (\sin(x^n) + \cos(x^n)) dx \right)^n = \lim_{n \rightarrow \infty} \left(\frac{\Gamma(\frac{1}{n})}{n} \left(\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) \right) \right)^n =$$

$$= \exp \left(\lim_{n \rightarrow 0} \frac{1}{n} \left(\log \left(\sin\left(\frac{\pi}{2}n\right) + \cos\left(\frac{\pi}{2}n\right) \right) + \log(\Gamma(1+n)) \right) \right)$$



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$$\underset{\text{Hospital}}{=} \exp \left(\frac{\pi}{2} \lim_{n \rightarrow 0} \frac{\cos\left(\frac{\pi}{2}n\right) - \sin\left(\frac{\pi}{2}n\right)}{\sin\left(\frac{\pi}{2}n\right) + \cos\left(\frac{\pi}{2}n\right)} + \Psi(1+n) \right) = \exp\left(\frac{\pi}{2} - \gamma\right)$$

897. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(n \left(\log 2 - \sum_{i=1}^n \frac{(n+i)^4}{3 + (n+i)^5 + \cot^{-1}(n+i)} \right) \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Remus Florin Stanca – Romania

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)} - \sum_{i=1}^n \frac{1}{1+n} \right) = \\ & = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{(i+n)^5 - 3 - (i+n)^5 - \cot^{-1}(i+n)}{(3 + (i+n)^5 + \cot^{-1}(i+n))(i+n)} \right) = \\ & = - \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{3 + \cot^{-1}(i+n)}{(3 + (i+n)^5 + \cot^{-1}(i+n))(i+n)} \right) \end{aligned}$$

We know that $\frac{3+\cot^{-1}(i+n)}{(3+(i+n)^5+\cot^{-1}(i+n))(i+n)} \leq \frac{3+\pi}{(3+(i+n)^5)(i+n)} \leq \frac{3+\pi}{(3+(n+1)^5)(n+1)} \Rightarrow$

$$\Rightarrow \sum_{i=1}^n \frac{3 + \cot^{-1}(i+n)}{(3 + (i+n)^5 + \cot^{-1}(i+n))(i+n)} \leq \frac{n(3 + \pi)}{(n+1)(3 + (n+1)^5)}$$

$$\lim_{n \rightarrow \infty} \frac{n(3 + \pi)}{(n+1)(3 + (n+1)^5)} = 0 \Rightarrow \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)} - \sum_{i=1}^n \frac{1}{1+n} \right) = 0 \quad (*)$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i+n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{\frac{i}{n}+1} \quad (1)$$

$$f(x) = \frac{1}{x+1} = \text{continuous, let } x_k = \frac{k}{n} \Rightarrow x_{k+1} - x_k = \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} |\Delta_n| = 0 \Rightarrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_{k+1} - x_k) f(\zeta_k) = \int_a^b f(x) dx \text{ where } \zeta_k \in [x_k; x_{k+1}] \text{ and}$$

$$a = \lim_{n \rightarrow \infty} x_1 \text{ and } b = \lim_{n \rightarrow \infty} x_n \text{ and let } \zeta_k = \frac{k}{n} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{\frac{i}{n}+1} = \int_0^1 \frac{1}{x+1} dx =$$

$$\ln(2) \stackrel{(1)}{\Rightarrow}$$



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$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{i+n} = \ln(2) \stackrel{(*)}{\Rightarrow} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)} = \ln(2) \Rightarrow$$

$$a = \lim_{n \rightarrow \infty} x_1 \text{ and } b = \lim_{n \rightarrow \infty} x_n \text{ and let } \zeta_k = \frac{k}{n} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{\frac{i}{n}+1} = \int_0^1 \frac{1}{x+1} dx =$$

$$\ln(2) \stackrel{(1)}{\Rightarrow}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i+n} = \ln(2) \stackrel{(*)}{\Rightarrow} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)} = \ln(2) \Rightarrow$$

$$\Rightarrow \Omega \stackrel{\infty-0}{=} \lim_{n \rightarrow \infty} n \left(\ln(2) - \sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)} \right) \stackrel{\frac{0}{0}}{=}$$

$$= \lim_{n \rightarrow \infty} \frac{\ln(2) - \sum_{i=1}^n \frac{(i+n)^4}{3 + (i+n)^5 + \cot^{-1}(i+n)}}{\frac{1}{n}} \stackrel{\text{Stolz-Cesaro}}{=} \frac{0}{0}$$

$$\stackrel{\text{Stolz-Cesaro}}{=} \lim_{n \rightarrow \infty} \left(\frac{\frac{(2n+2)^4}{3 + (2n+2)^5 + \cot^{-1}(2n+2)} + \frac{(2n+1)^4}{3 + (2n+1)^5 + \cot^{-1}(2n+1)} -}{-\frac{(n+1)^4}{3 + (n+1)^5 + \cot^{-1}(n+1)}} \right) n(n+1) =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\frac{n(2n+2)^4}{3 + (2n+2)^5 + \cot^{-1}(2n+2)} - \frac{1}{2} + \frac{n(2n+1)^4}{3 + (2n+1)^5 + \cot^{-1}(2n+1)} - \frac{1}{2} +}{+\frac{n(n+1)^4}{3 + (n+1)^5 + \cot^{-1}(n+1)}} \right) n =$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n(2n+2)^4}{3 + (2n+2)^5 + \cot^{-1}(2n+2)} - \frac{1}{2} \right) n +$$

$$+ \lim_{n \rightarrow \infty} \left(\frac{n(2n+1)^4}{3 + (2n+1)^5 + \cot^{-1}(2n+1)} \right) n +$$

$$+ \lim_{n \rightarrow \infty} \left(1 - \frac{n(n+1)^4}{3 + (n+1)^5 + \cot^{-1}(n+1)} \right) n =$$

$$= -\frac{1}{2} - \frac{1}{4} + 1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow \Omega = \frac{1}{4}$$

898. Find (ϕ – golden ratio):



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$$\Omega = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n\phi} + \sqrt{1 + \frac{1}{n\phi^2}} + \sqrt[3]{1 + \frac{1}{n\phi^3}} + \cdots + \sqrt[n]{1 + \frac{1}{n\phi^n}} - n \right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ali Jaffal-Lebanon

We have by Bernoulli's inequality:

$$1 \leq (1+n)^\alpha \leq 1 + \alpha n \text{ for all } n > 0 \text{ and } 0 \leq \alpha \leq 1 \text{ then } 1 \leq \left(1 + \frac{1}{n\phi^k}\right)^{\frac{1}{k}} \leq 1 + \frac{1}{nk\phi^k}$$

$$\text{Let } \Omega_n = \sum_{k=1}^{n-1} \left(1 + \frac{1}{n\phi^k}\right)^{\frac{1}{k}} - n. \text{ So, } 0 \leq \Omega_n \leq \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k\phi^k}$$

but $\frac{1}{k} \leq 1$ then $0 \leq \Omega_n \leq \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{\phi^k}$ we know that $\sum_{k=1}^{n-1} \left(\frac{1}{\phi}\right)^k = \frac{1}{\phi} \cdot \frac{1 - \left(\frac{1}{\phi}\right)^n}{1 - \frac{1}{\phi}}$

and $\lim_{n \rightarrow \infty} \left(\frac{1}{\phi}\right)^n = 0$ since $0 < \frac{1}{\phi} < 1$ then $0 \leq \lim_{n \rightarrow \infty} \Omega_n \leq 0$ therefore

$$\lim_{n \rightarrow \infty} \Omega_n = 0$$

899. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k}} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

$$\begin{aligned} \sum_{k=0}^{n-1} (n^2 - nk)^2 \cdot \binom{2n}{k} &= n^2 \binom{2n}{0} + (n^2 - n)^2 \cdot \binom{2n}{1} + (n^2 - 2n)^2 \cdot \binom{2n}{2} + \\ &\quad + (n^2 - 3n)^2 \cdot \binom{3n}{3} + \cdots + (n^2 - n(n-1))^2 \cdot \binom{2n}{n-1} \\ &= n^2 \binom{2n}{0} + n^2(n-1)^2 \binom{2n}{1} + n^2(n-2)^2 \binom{2n}{2} + n^2(n-3)^2 \binom{2n}{3} + \\ &\quad + \cdots + n^2 \cdot (n - (n-1))^2 \cdot \binom{2n}{n-1} \end{aligned}$$



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$$\begin{aligned}
 &= n^2 \cdot \left[\binom{2n}{0} + (n-1)^2 \binom{2n}{1} + (n-2)^2 \binom{2n}{2} + (n-3)^2 \binom{2n}{3} + \cdots + (1)^2 \binom{2n}{n-1} \right] \\
 &\leq n^2 \left[\binom{2n}{n-1} [(n-1)^2 + (n-2)^2 + (n-3)^2 + \cdots + 1] \right]
 \end{aligned}$$

$$\frac{1}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k}} \leq n^{\frac{2}{n}-2} \sqrt[n]{\binom{2n}{n-1} \cdot \frac{n(n+1)(2n+1)}{6}}$$

Degree of $\binom{2n}{n-1} \cdot \frac{n(n+1)(2n+1)}{6}$ is equal to $\frac{3}{n}$

So: degree of $n^{\frac{2}{n}-2} \cdot \sqrt[n]{\binom{2n}{n-1} \cdot \frac{n(n+1)(2n+1)}{6}}$ is $n^{\frac{2}{n}-2} \cdot n^{\frac{3}{n}} = n^{\frac{5}{n}-2}$

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n^2} \cdot \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k}} \right) \leq \lim_{n \rightarrow +\infty} \left(\frac{\frac{5}{n}}{n^2} \right) \quad (*)$$

$$\text{But } \lim_{n \rightarrow +\infty} \left(\frac{\frac{5}{n}}{n^2} \right) = \lim_{n \rightarrow +\infty} \left(\frac{e^{\frac{5}{n} \ln(n)}}{n^2} \right) = \lim_{n \rightarrow +\infty} \left(\frac{5 - \frac{\ln(n)}{n}}{n^2} \right) = 0$$

$$\text{So: } \lim_{n \rightarrow +\infty} \sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k} = 0$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

We know that $\sum_{k=0}^{2n} C_{2n}^k \cdot 2^{2n} = 4^n$ then $0 < C_{2n}^k \leq 4^n$ for $0 \leq k \leq 2n$

Let $\Omega_n = \frac{1}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k}}$ we have $0 < \Omega_n \leq \frac{1}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 \cdot 4^n}$

$0 < \Omega_n \leq \frac{4}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2}$. We know that:

$$\sum_{k=0}^{n-1} (n^2 - nk)^2 = n^2 \sum_{k=0}^{n-1} (n - k)^2 = n^2 \sum_{k=0}^{n-1} k^2 = \frac{n^3(n+1)(2n+1)}{6}$$

$$\text{Then } 0 < \Omega_n \leq \frac{4}{n^2} \left(\frac{n^3(n+1)(2n+1)}{6} \right)^{\frac{1}{n}}$$

$$\text{But } \lim_{n \rightarrow +\infty} \left(\frac{n^3(n+1)(2n+1)}{6} \right)^{\frac{1}{n}} = \lim_{n \rightarrow +\infty} e^{\frac{1}{n} \ln \left(\frac{n^3(n+1)(2n+1)}{6} \right)} = 1$$

$$\text{then } \lim_{n \rightarrow +\infty} \Omega_n = 0$$

Solution 3 by Artan Ajredini-Presheva-Serbie



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Let $a_n = \sum_{k=0}^{n-1} (n^2 - nk) \binom{2n}{k}$ and $b_k(n) = (n^2 - nk) \binom{2n}{k}$

Since $\lim_{n \rightarrow \infty} b_k(n) = \infty$ we have that: $\lim_{n \rightarrow \infty} a_n = \infty$. Now,

$$\frac{a_{n+1}}{a_n} = \frac{a_n + O(n)}{a_n} \xrightarrow{n \rightarrow \infty} 1. \text{ Hence, } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1. \text{ Definitely, } \Omega = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} = 0$$

Solution 4 by Ali Jaffal-Lebanon

$$\text{Let } \Omega_n = \frac{1}{n^2} \sqrt[n]{\sum_{k=0}^{n-1} (n^2 - nk)^2 C_{2n}^k}$$

we know that $\sum_{k=0}^{2n} C_{2n}^k = 2^{2n} = 4^n$ then $C_{2n}^k \leq \sum_{k=0}^{2n} C_{2n}^k 4^n$

So, $C_{2n}^k \leq 4^n$ for all $0 \leq k \leq 2n$ then

$$\begin{aligned} \Omega_n &\leq \frac{1}{n^2} \left(\sum_{k=0}^{n-1} (n^2 - nk)^2 \cdot 4^n \right)^{\frac{1}{n}} \leq \frac{4}{n^2} \left(\sum_{k=0}^{n-1} (n^2 - nk)^2 \right)^{\frac{1}{n}} \\ &\leq \frac{4}{n^2} n^{\frac{2}{n}} \left(\sum_{k=0}^{n-1} (n - k)^2 \right)^{\frac{1}{n}} \leq \frac{4 \cdot n^{\frac{2}{n}}}{n^2} \left(\sum_{k=1}^n k^2 \right)^{\frac{1}{n}} \leq \frac{4 \cdot n^{\frac{2}{n}}}{n^2} \times \left(\sum_{k=1}^n n^2 \right)^{\frac{1}{n}} \end{aligned}$$

$$0 \leq \Omega_n \leq \frac{4 \cdot n^{\frac{2}{n}}}{n^2} \times n^{\frac{3}{n}} \leq \frac{4 \cdot n^{\frac{5}{n}}}{n^2} \text{ but } \lim_{n \rightarrow \infty} n^{\frac{5}{n}} = 1; \lim_{n \rightarrow +\infty} \Omega_n = 0$$

Solution 5 by Naren Bhandari-Bajura-Nepal

Let L be the limit and we write $S = \sum_{k=0}^{n-1} (n^2 - nk)^2 \binom{2n}{k}$. Now $\forall 0 \leq r \leq 1$ from

Bernoulli inequality we have: $\sqrt[n]{S} \leq 1 + \frac{s}{n} \Rightarrow \frac{s}{n} \leq \sqrt[n]{S} \leq 1 + \frac{s}{n}$ and hence we have

$$\lim_{n \rightarrow \infty} \frac{s}{n^3} \leq L \leq \lim_{n \rightarrow \infty} \frac{1}{n^2} \left(1 + \frac{s}{n} \right) \text{ and by Stolz Cesaro theorem we have}$$

$$\lim_{n \rightarrow \infty} \frac{1}{(n+1)^3 - n^3} \left(\sum_{k=0}^n (n^2 - nk)^2 \binom{2n}{k} - S \right) \leq L \leq 0 +$$

$$\lim_{n \rightarrow \infty} \frac{1}{(n+1)^3 - n^3} \left(\sum_{k=0}^n (n^2 - nk)^2 \binom{2n}{k} - S \right)$$

$$\lim_{n \rightarrow \infty} \frac{(n^2 - n^2)^2 \binom{2n}{n}}{(n+1)^3 - n^3} \leq L \leq 0 + \lim_{n \rightarrow \infty} \frac{(n^2 - n^2)^2 \binom{2n}{n}}{(n+1)^3 - n^3}$$

$0 \leq L \leq 0$ and hence by Squeeze theorem we have $L = 0$.



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$$900. \Omega(a) = \lim_{b \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{n(n+1)(n+2) \cdots (n+a-1)}{(-b)^{n-1}} \right), a \in \mathbb{N} - \{0, 1\}$$

Find:

$$\Omega = \sum_{a=2}^{\infty} \frac{1}{\Omega(a)}$$

Proposed by Daniel Sitaru – Romania

Solution by Samir HajAli-Damascus-Syria

$$\text{We have: } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2} \text{ because the series converge uniformly when } |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) \cdot x^n = \frac{1}{(1-x)^2} \text{ (Similarly)}$$

$$\text{We find: } \sum_{n=1}^{\infty} n(n+1) x^{n-1} = \frac{2(1-x)}{(1-x)^4} = \frac{1 \cdot 2}{(1-x)^3} = \frac{2!}{(1-x)^3}$$

$$\text{Then: } \sum_{n=0}^{\infty} (n+1)(n+2) x^n = \frac{1 \cdot 2}{(1-x)^3}; |x| < 1$$

$$\text{Therefore: } \sum_{n=1}^{\infty} n(n+1)(n+2)x^{n-1} = \frac{1 \cdot 2 \cdot 3}{(1-x)^4} = \frac{3!}{(1-x)^4}$$

Similarly step by step we find:

$$\sum_{n=1}^{\infty} n(n+1)(n+2) \dots (n+a-1) x^{n-1} = \frac{a!}{(1-x)^{a+1}}; |x| < 1$$

$$\text{Hence: } f(x) = \sum_{n=1}^{\infty} \prod_{k=0}^{a-1} (n-k) x^{n-1} = \frac{a!}{(1-x)^{a+1}}$$

$$\lim_{b \rightarrow \infty} \Omega(a) = \lim_{b \rightarrow \infty} f\left(\frac{1}{b}\right) = \lim_{b \rightarrow \infty} \frac{a!}{\left(1 + \frac{1}{b}\right)^{a+1}} = a!$$

$$\text{and: } \Omega = \sum_{a=2}^{\infty} \frac{1}{\Omega(a)} = \sum_{a=2}^{\infty} \frac{1}{a!} = \sum_{a=0}^{\infty} \frac{1}{a!} - 2$$

$$= e - 2 \Rightarrow \Omega = e - 2$$



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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru