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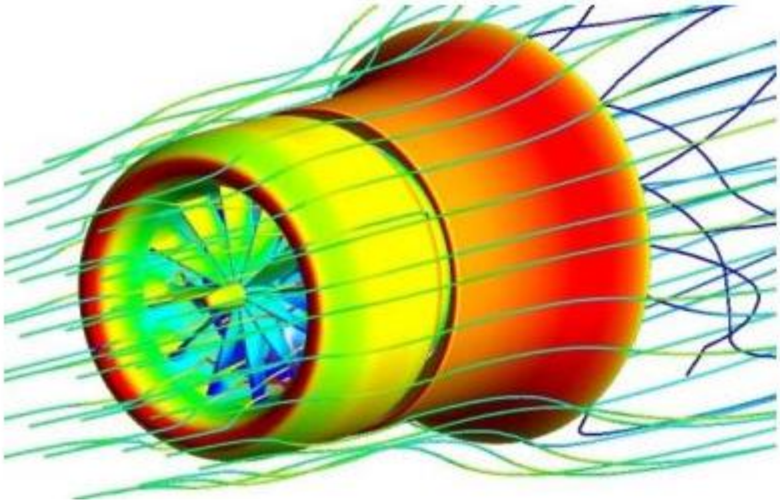

If $f:[0,1]\to\mathbb{R}, f$ −continuous then:

$$\int_0^1\int_0^1\int_0^1 \sqrt[3]{(3f(x)f(y)f(z)-f^3(x)-f^3(y)-f^3(z))^2}\,dxdydz \le 3\int_0^1 f^2(x)dx$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Vişescu-Romania

$$3abc-a^3-b^3-c^3=(a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

$$(3abc-a^3-b^3-c^3)^2=(a+b+c)^2(ab+bc+ca-a^2-b^2-c^2)^2$$

We must show:

$$(a+b+c)^2(ab+bc+ca-a^2-b^2-c^2)^2\le(a^2+b^2+c^2)^3$$

$$(a+b+c)^2(3(ab+bc+ca)-(a+b+c)^2)^2\le((a+b+c)^2-2(ab+bc+ca))^3$$

$$Let\ a+b+c=p;\ ab+bc+ca=q$$

$$p^2(3q-p^2)^2\le(p^2-2q)^3$$

$$8q^3\le 3p^2q^2\Leftrightarrow 8q\le 3p^2$$

$$8(ab+bc+ca)\le 3(a+b+c)^2$$

$$8(ab+bc+ca)\le 3(a^2+b^2+c^2)+6(ab+bc+ca)$$

$$2(ab+bc+ca)\le 3(a^2+b^2+c^2)\text{ true from:}$$

$$\overset{Am-Gm}{\Longrightarrow}\begin{cases}a^2+b^2\geq 2ab\\ b^2+c^2\geq 2bc\\ c^2+a^2\geq 2ca\end{cases}\Rightarrow$$

$$3(a^2+b^2+c^2)\geq 2(a^2+b^2+c^2)\geq 2(ab+bc+ca)$$

So,

$$(3abc-a^3-b^3-c^3)^2\leq (a^2+b^2+c^2)^3$$

$$\int_0^1\int_0^1\int_0^1\sqrt[3]{(3f(x)f(y)f(z)-f^3(x)-f^3(y)-f^3(z))^2}dxdydz$$

$$\leq \int_0^1\int_0^1\int_0^1(f^2(x)+f^2(y)+f^2(z))dxdydz\leq 3\int_0^1 f^2(x)dx$$

Note by Editor:

Many thanks to Florică Anastase for typed solution.