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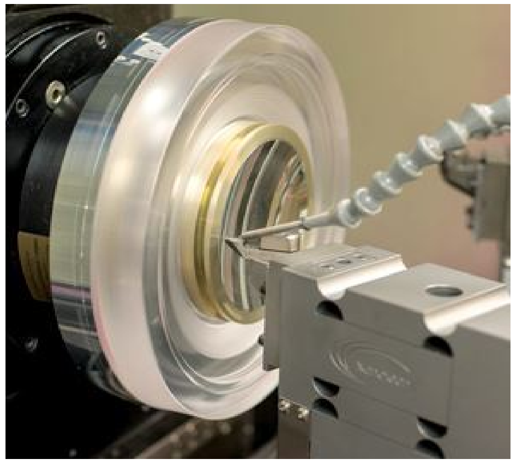

Find without softs:

$$\Omega=\int_0^{\infty}\frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega=\int_0^{+\infty}\frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx=\int_0^{+\infty}\frac{\log\left(1+x^4+x^6(1+x^4)\right)dx}{(1+x)(1-x+x^2)+3x(1+x)}$$

$$=\int_0^{+\infty}\frac{\log\left((1+x^4)(1+x^6)\right)dx}{(1+x)^3}=\int_0^{+\infty}\frac{\log(1+x^4)dx}{(1+x)^3}+\int_0^{\infty}\frac{\log(1+x^6)dx}{(1+x)^3}$$

$$\overset{I.B.P.}{=}2\int_0^{+\infty}\frac{x^3dx}{(1+x)^2(1+x^4)}+3\int_0^{+\infty}\frac{x^5dx}{(1+x)^2(1+x^6)}$$

$$\frac{x^3}{(1+x)^2(1+x^4)}=\frac{a_1x^3+a_2x^2+a_3x+a_4}{1+x^4}+\frac{a_5}{1+x}+\frac{a_6}{(1+x)^2}$$

By comparison, we get:

$$\frac{x^3}{(1+x)^2(1+x^4)}=\frac{1}{2}\left(\frac{-x^3+2x^2-x}{1+x^4}+\frac{1}{1+x}-\frac{1}{(1+x)^2}\right)$$

And:

$$\frac{x^5}{(1+x)^2(1+x^6)}=\frac{b_1x^5+b_2x^4+b_3x^3+b_4x^2+b_5x+b_6}{1+x^6}+\frac{b_7}{1+x}+\frac{b_8}{2(1+x)^2}$$

By comparison, we get:

$$\frac{x^5}{(1+x)^2(1+x^6)} = \frac{1}{6}\cdot\frac{-6x^5+9x^4-6x^3+3x^2-3}{1+x^6} + \frac{1}{1+x} - \frac{1}{2(1+x)^2}$$

So:

$$\Omega = \int_0^{+\infty}\left(\frac{1}{1+x} - \frac{x^3}{1+x^4}\right)dx + 3\int_0^{+\infty}\left(\frac{1}{1+x} - \frac{x^5}{1+x^6}\right)dx$$

$$+\int_0^{+\infty}\frac{2x^2-x}{1+x^4}dx - \frac{5}{2}\int_0^{+\infty}\frac{dx}{(1+x)^2} + \frac{1}{2}\int_0^{+\infty}\frac{9x^4-6x^3+3x^2-3}{1+x^6}dx$$

$$= log\left(\frac{1+x}{\sqrt[4]{1+x^4}}\right)\Bigg|_0^{+\infty} + 3log\left(\frac{1+x}{\sqrt[6]{1+x^6}}\right)\Bigg|_0^{+\infty} + \int_0^{+\infty}\frac{2x^2-x}{1+x^4}dx + \frac{5}{2}\cdot\frac{1}{1+x}\Bigg|_0^{+\infty}$$

$$+\frac{1}{2}\int_0^{+\infty}\frac{9x^4-6x^3+3x^2-3}{1+x^6}dx$$

$$= \frac{1}{4}\int_0^{+\infty}\frac{2t^{-\frac{1}{4}} - t^{-\frac{1}{2}}}{1+t}dt + \frac{1}{12}\int_0^{+\infty}\frac{9t^{-\frac{1}{6}} - 6t^{-\frac{2}{3}} + 3t^{-\frac{1}{2}} - 3t^{-\frac{5}{6}}}{1+t}ddt - \frac{5}{2}$$

$$= \frac{\pi}{4}\left(\frac{2}{sin\frac{\pi}{4}} - \frac{1}{sin\frac{\pi}{2}}\right) + \frac{\pi}{4}\left(\frac{3}{sin\frac{\pi}{6}} - \frac{2}{sin\frac{2\pi}{3}} + \frac{1}{sin\frac{\pi}{2}} - \frac{1}{sin\frac{5\pi}{6}}\right) - \frac{5}{2}$$

$$= \frac{\pi}{4}\left(2\sqrt{2} - 1 + 6 - \frac{4\sqrt{3}}{3} + 1 - 2\right) - \frac{5}{2} = \frac{\pi}{12}\left(12 + 6\sqrt{2} - 4\sqrt{3}\right) - \frac{5}{2}$$

So,

$$\Omega = \int_0^{\infty}\frac{log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3}dx = \frac{\pi}{12}\left(12+6\sqrt{2}-4\sqrt{3}\right) - \frac{5}{2}$$