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In ΔABC the following relationship holds:

$$\frac{r_b r_c}{r_a^2}+\frac{r_c r_a}{r_b^2}+\frac{r_a r_b}{r_c^2}+\frac{2nr}{R}\ge n+3\,,\ \ n\le 8$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_b r_c}{r_a^2}=\sum\frac{(s-a)^2}{(s-b)(s-c)}=\frac{\sum(s-a)^3}{r^2 s}=\frac{\sum(s^3-3s^2a+3sa^2-a^3)}{r^2 s}$$

$$=\frac{3s^3-3s^2(2s)+6s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{r^2 s}=\frac{s^2-12Rr}{r^2}$$

$$\therefore\sum\frac{r_b r_c}{r_a^2}+\frac{16r}{R}-11=\frac{s^2-12Rr}{r^2}+\frac{16r}{R}-11$$

$$=\frac{R(s^2-12Rr)+16r^3-11Rr^2}{Rr^2}\overset{\text{Gerretsen}}{\ge}\frac{R(4Rr-5r^2)+16r^3-11Rr^2}{Rr^2}$$

$$=\frac{4R^2-16Rr+16r^2}{Rr}=\frac{4(R-2r)^2}{Rr}\ge 0$$

$$\Rightarrow\sum\frac{r_b r_c}{r_a^2}\ge 11-\frac{16r}{R}\ \therefore\ \text{it suffices to prove : } 11-\frac{16r}{R}\ge n+3-\frac{2nr}{R}\Leftrightarrow 8-n$$

$$\ge\frac{2r}{R}(8-n)\Leftrightarrow(8-n)\left(1-\frac{2r}{R}\right)\ge 0$$

$$\rightarrow\text{true}\because n\le 8\text{ and }R\overset{\text{Euler}}{\ge}2r\ \therefore\ \frac{r_b r_c}{r_a^2}+\frac{r_c r_a}{r_b^2}+\frac{r_a r_b}{r_c^2}+\frac{2nr}{R}\ge n+3\ \forall\ n\le 8\ \text{(Proved)}$$