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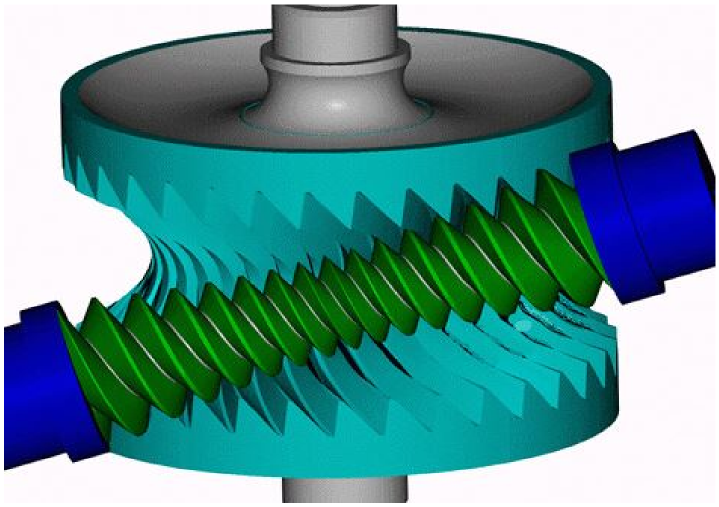

If in ΔABC, $ab+bc+ca\le 1$ then the following relationship holds:

$$\left(\sum_{cyc}\left(\frac{a}{b^3}\right)^m\right)\left(\sum_{cyc}\left(\frac{m_a}{\mu(A)}\right)^n\right)\ge 3^{m+2n+2}\cdot\left(\frac{r}{\pi}\right)^n, m,n\ge 2$$

Proposed by Radu Diaconu-Romania

Solution by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \Omega_1=\sum\left(\frac{a}{b^3}\right)^m \overset{AM-GM}{\ge} 3\left(\frac{abc}{a^3b^3c^3}\right)^{\frac{m}{3}}=3\left(\frac{1}{a^2b^2c^2}\right)^{\frac{m}{3}}$$

$$\Rightarrow \frac{ab+bc+ca}{3}\overset{AM-GM}{\ge}(a^2b^2c^2)^{\frac{1}{3}}\Rightarrow a^2b^2c^2\le\frac{(ab+bc+ca)^3}{27}\le\frac{1}{27}$$

$$\Leftrightarrow \Omega_1\ge 3\left(\frac{1}{\frac{1}{27}}\right)^{\frac{m}{3}}=3\cdot 3^m$$

$$\Leftrightarrow \Omega_2=\sum\left(\frac{m_a}{\mu(A)}\right)^n\overset{AM-GM}{\ge}3\left(\frac{m_a}{\prod\mu(A)}\right)^{\frac{n}{3}}\overset{m_a\ge\sqrt{s(s-a)}}{\ge}3\left(\frac{\sqrt{s^3(s-a)(s-b)(s-c)}}{\frac{\pi^3}{27}}\right)^{\frac{n}{3}}$$

$$=3\left(\frac{27\times 27\times r^3}{\pi^3}\right)^{\frac{n}{3}}=3\cdot\frac{3^{2n}\cdot r^n}{\pi^n}$$

$$\left[\because \sqrt{s^3(s-a)(s-b)(s-c)}=s\Delta=s^2r\overset{Mitrinovic}{\ge}\left(3\sqrt{3}r\right)^2r=27r^3\right]$$

$$\left[\because \frac{\sum \mu(A)}{3} \overset{AM-GM}{\geq} \left(\prod \mu(A)\right)^{\frac{1}{3}} \Rightarrow \prod \mu(A) \leq \frac{\pi^3}{27} \because \sum \mu(A) = \pi\right]$$

$$\Leftrightarrow \Omega = \Omega_1 \cdot \Omega_2 \geq 3 \cdot 3^m \cdot \frac{3 \cdot 3^{2n} \cdot r^n}{\pi^n} = 3^{m+2n+2} \cdot \left(\frac{r}{\pi}\right)^n$$

(proved)