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If $a, b, c > 0$ then:

$$\left(\sum_{cyc} \sqrt[4]{a^3} \right)^4 \left(\sum_{cyc} \sqrt[5]{a^4} \right)^5 \left(\sum_{cyc} \sqrt[6]{a^5} \right)^6 \leq 27 \left(\sum_{cyc} a \right)^{12}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria; Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand; Solution 3 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\varphi = \left(\sum_{cyc} \sqrt[4]{a^3} \right)^4 \left(\sum_{cyc} \sqrt[5]{a^4} \right)^5 \left(\sum_{cyc} \sqrt[6]{a^5} \right)^6$$

Suppose $f, g, h: [0, \infty) \rightarrow [0, \infty)$; $f(x) = \sqrt[4]{x^3}$; $g(x) = \sqrt[5]{x^4}$; $h(x) = \sqrt[6]{x^5}$ it's very easy to prove f is convex functions

So,

$$\begin{aligned} \varphi &= \left(\sum_{cyc} \sqrt[4]{a^3} \right)^4 \left(\sum_{cyc} \sqrt[5]{a^4} \right)^5 \left(\sum_{cyc} \sqrt[6]{a^5} \right)^6 \\ &\leq \left(3 \cdot \sqrt[4]{\left(\frac{a+b+c}{3}\right)^3} \right)^4 \left(\sqrt[5]{\left(\frac{a+b+c}{3}\right)^4} \right)^5 \left(\sqrt[6]{\left(\frac{a+b+c}{3}\right)^5} \right)^6 \end{aligned}$$

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 $\leq 27(a + b + c)^{12}$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For, $b, c > 0$, we give $a = x^{60}, b = y^{60}, c = z^{60}$

We have:

$$\begin{aligned} 3(x^{60} + y^{60} + z^{60})^3 &\geq (x^{45} + y^{45} + z^{45})^4 \\ 3(x^{60} + y^{60} + z^{60})^4 &\geq (x^{48} + y^{48} + z^{48})^5 \\ 3(x^{60} + y^{60} + z^{60})^5 &\geq (x^{50} + y^{50} + z^{50})^6 \\ (x^{45} + y^{45} + z^{45})^4 (x^{48} + y^{48} + z^{48})^5 (x^{50} + y^{50} + z^{50})^6 &\leq 3^3 (x^{60} + y^{60} + z^{60})^{12} \\ \Leftrightarrow \left(\sum_{cyc} \sqrt[4]{a^3} \right)^4 \left(\sum_{cyc} \sqrt[5]{a^4} \right)^5 \left(\sum_{cyc} \sqrt[6]{a^5} \right)^6 &\leq 27 \left(\sum_{cyc} a \right)^{12} \end{aligned}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

Let: $f(a) = \sqrt[4]{a^3}; a > 0 \Rightarrow$

$$f'(a) = \frac{3}{4} a^{-\frac{1}{4}}$$

$$f''(x) = -\frac{3}{16} a^{-\frac{5}{4}} < 0, \forall a > 0$$

$$\sum_{cyc} \sqrt[4]{a^3} \stackrel{\text{Jensen}}{\geq} 3 \cdot \sqrt[3]{\left(\frac{a+b+c}{3}\right)^3}$$

Similary

$$\sum_{cyc} \sqrt[5]{a^4} \stackrel{\text{Jensen}}{\geq} 3 \cdot \sqrt[5]{\left(\frac{a+b+c}{3}\right)^4}$$

$$\sum_{cyc} \sqrt[6]{a^5} \stackrel{\text{Jensen}}{\geq} 3 \cdot \sqrt[6]{\left(\frac{a+b+c}{3}\right)^5}$$

$$\begin{aligned} &\left(\sum_{cyc} \sqrt[4]{a^3} \right)^4 \left(\sum_{cyc} \sqrt[5]{a^4} \right)^5 \left(\sum_{cyc} \sqrt[6]{a^5} \right)^6 \\ &\leq 3^4 \left(\frac{a+b+c}{3}\right)^3 \cdot 3^5 \left(\frac{a+b+c}{3}\right)^4 \cdot 3^6 \left(\frac{a+b+c}{3}\right)^5 = 27(a+b+c)^{12} \end{aligned}$$

Note by Editor:

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