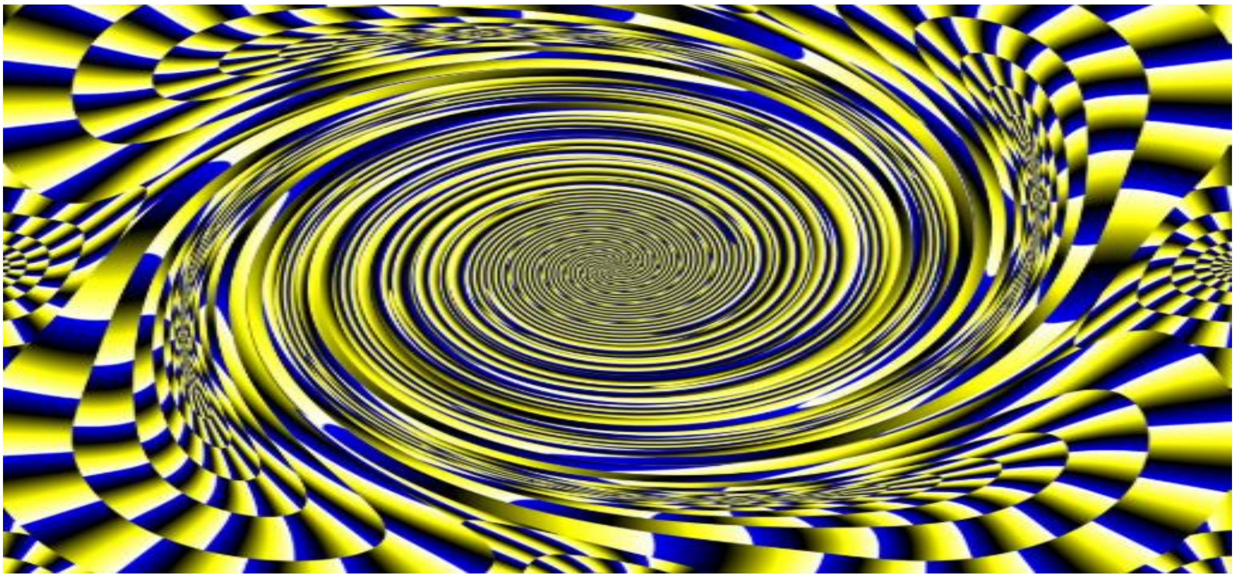
R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro



If $a, b, c > 0$ then:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+6 \geq \frac{9(a^2+b^2+c^2)}{ab+bc+ca}$$

Proposed by Marin Chirciu-Romania
Solution 1 by Rahim Shahbazov-Baku-Azerbaijan, Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+6 \geq \frac{9(a^2+b^2+c^2)}{ab+bc+ca} \ldots (1)$$

$$\Rightarrow \frac{a^3+b^3+c^3}{abc}-3 \geq 9\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}-1\right) \ldots (2)$$

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ldots (3)$$

$$\overset{(2)}{\Rightarrow} \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{abc} \geq \frac{9\left((a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right)}{ab+bc+ca}$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

$$\Rightarrow \frac{a+b+c}{abc} \ge \frac{9}{ab+bc+ca}$$

$$(a+b+c)(ab+bc+ca) \ge 9abc. \text{ Done.}$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\text{Proof: } \frac{a^3+b^3+c^3+6abc}{abc} \ge \frac{9\sum a^2}{\sum ab}$$

$$\sum ab\left(\sum a^3+6abc\right) \ge 9abc \cdot \sum a^2$$

$$\sum ab \cdot \left(\sum a \cdot \sum a^2 - \sum a \sum ab + 9abc\right) \ge 9abc \sum a^2$$

$$\sum a \sum ab\left(\sum a^2 - \sum ab\right) \ge 9abc\left(\sum a^2 - \sum ab\right)$$

$$\sum a \sum ab \ge 9abc$$

Equality for $a=b=c$

Solution 3 by Soumava Chakraborty-Kolkata-India

$Proof: Let\ b+c=A, c+a=B\ and\ a+b=C \therefore A+B>C, B+C > A\ and\ C+A>B$

$\Rightarrow A, B, C\ are\ sides\ of\ a\ triangle\ with\ semiperimeter, circumradius\ and\ inradius = s, R, r\ respectively\ (say)$

$$\therefore 2\sum a = \sum A = 2s \Rightarrow \sum a = s \therefore a = s-A, b = s-B\ and\ c = s-C$$

$$Now, \sum ab = \sum (s-B)(s-C) = \sum (s^2 - s(B+C) + BC)$$
$$= 3s^2 - s(4s) + s^2 + 4Rr + r^2 \therefore \sum ab \overset{(i)}{=} 4Rr + r^2$$

$$Again, \sum a^3 = \left(\sum a\right)^3 - 3(a+b)(b+c)(c+a) = s^3 - 3ABC$$
$$\therefore \sum a^3 \overset{(ii)}{=} s^3 - 12Rrs\ and\ abc = \prod (s-A) \overset{(ii)}{=} sr^2$$

$$Now, proposed\ inequality \Leftrightarrow \frac{\sum a^3}{abc} + 6 \ge \frac{9\left[\left(\sum a\right)^2 - 2\sum ab\right]}{\sum ab} \overset{by\ (i),(ii),(iii)}{\Leftrightarrow}$$

$$\frac{s^3-12Rrs}{sr^2}+6\ge\frac{9(s^2-8Rr-2r^2)}{4Rr+r^2}$$

$$\Leftrightarrow\frac{s^2-12Rr+6r^2}{r}\ge\frac{9(s^2-8Rr-2r^2)}{4R+r}\Leftrightarrow(4R+r)(s^2-12Rr+6r^2)\ge 9r(s^2-8Rr-2r^2)$$

$$\Leftrightarrow(4R-8r)s^2+9r(8Rr+2r^2)-(4R+r)(12Rr-6r^2)\overset{(1)}{\ge}0$$

$$Now, LHS\ of\ (1)\overset{Gerretsen}{\ge}(4R-8r)(16Rr-5r^2)+9r(8Rr+2r^2)-(4R+r)(12Rr-6r^2)\overset{?}{\ge}0\Leftrightarrow 16(R^2-4Rr+4r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow 16(R-2r)^2\overset{?}{\ge}0\to true\Rightarrow(1)\ is\ true\ \therefore\ \forall\ a,b,c>0,\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+6\ge\frac{9(a^2+b^2+c^2)}{ab+bc+ca}\ (Proved)$$

Note by Editor:
Many Thanks to Florică Anastase-Romania for typed content.