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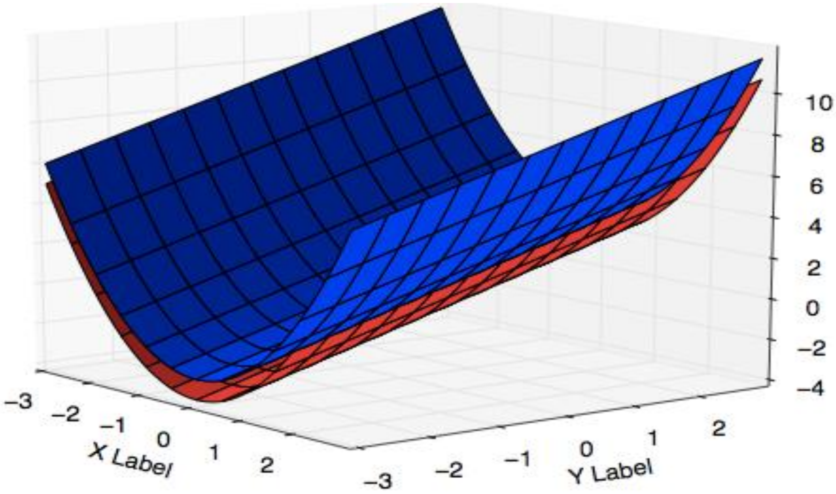

If $x,y,z>0$, then:

$$4\cdot\sqrt[4]{\left(\prod_{cyc}x\right)\cdot\left(\sum_{cyc}x\right)}+\sum_{cyc}x\le\sum_{cyc}\left(\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\right)\le 3\cdot\sum_{cyc}x$$

Proposed by Florică Anastase-Romania

Solution by proposer:

$$u=\sqrt{\frac{x^2+y^2}{2}},v=\sqrt{xy}\to\begin{cases}2u^2=x^2+y^2\\ v^2=xy\end{cases}\to\begin{cases}x^2+y^2+2xy=2u^2+2v^2\\ (x+y)^2=2(u^2+v^2)\end{cases}$$

$$0\le(u-v)^2\to(u+v)^2\le 2(u^2+v^2)\to(u+v)^2\le(x+y)^2$$

$$u+v\le x+y\to\begin{cases}\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}\le x+y\\ \sqrt{xy}\le\frac{x+y}{2}\end{cases}\to\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\le\frac{3}{2}(x+y)\to$$

$$\sum_{cyc}\left(\sqrt{\frac{x^2+y^2}{2}}+2\sqrt{xy}\right)\le 3(x+y+z)\ \ldots\ldots(1)$$

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\ge 2\cdot\sqrt[4]{xyz(x+y+z)},\forall x,y,z>0$$

Let $z = \max\{x, y, z\}$ and $a = \frac{x}{z}, b = \frac{y}{z}$, $a, b \in [0,1] \overset{(2)}{\Rightarrow} (a + b + ab)^2 \ge 4ab\sqrt{a^2 + b^2 + 1}$

But $(a + b + ab)^2 = a^2b^2 + (a + b)^2 + 2ab(a + b) \ge 2ab(a + b + 2)$, then we have:

$(a + b + 2)^2 \ge 4(a + b + 1)$, true from $(a + b + 2)^2 = (a + b)^2 + 4 + 4(a + b) \ge 4(a + b + 1)$
$\ge 4(a^2 + b^2 + 1), \forall a, b \in [0,1]$

$$\sum_{cyc}\left(\sqrt{\frac{x^2 + y^2}{2}} + 2\sqrt{xy}\right) \ge \sum_{cyc}\left(\frac{x + y}{2} + 2\sqrt{xy}\right) \ge x + y + z + 4 \cdot \sqrt[4]{xyz(x + y + z)} \ldots \ldots (2)$$