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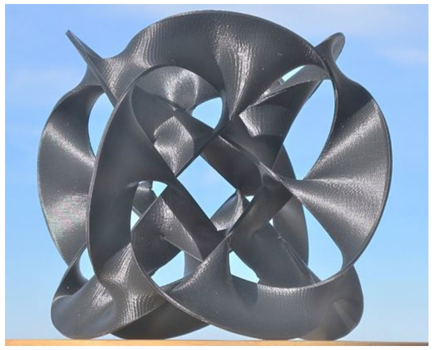

$$x^{2n+1}+1=(x+1)\prod_{k=1}^{n}(x^2+a_ix+1),\ \forall x\in\mathbb{C}$$

Find:

$$\Omega=\sum_{i=1}^{n}a_i^4,\ a_i\in\mathbb{C},\ i\in\overline{1,n}$$

Proposed by Rahim Shahbazov-Baju-Azerbaijan

Solution 1 by Gabriel Ruddy Cruz Mendez-Lima-Peru; Solution 2 by Le Van-Ho Chi Minh-Vietnam

Solution 1 by Gabriel Ruddy Cruz Mendez-Lima-Peru

$$8\cos^4 x=3+4\cos(2x)+\cos(4x)$$

$$a_1^4=\left(-2\cos\left(\frac{\pi}{2n+1}\right)\right)^4=16\cos^4\left(\frac{\pi}{2n+1}\right)=6+8\cos\left(\frac{2\pi}{2n+1}\right)+2\cos\left(\frac{4\pi}{2n+1}\right)$$

$$a_2^4=\left(-2\cos\left(\frac{3\pi}{2n+1}\right)\right)^4=16\cos^4\left(\frac{3\pi}{2n+1}\right)=6+8\cos\left(\frac{6\pi}{2n+1}\right)+2\cos\left(\frac{12\pi}{2n+1}\right)$$

$$a_3^4=\left(-2\cos\left(\frac{5\pi}{2n+1}\right)\right)^4=16\cos^4\left(\frac{5\pi}{2n+1}\right)=6+8\cos\left(\frac{10\pi}{2n+1}\right)+2\cos\left(\frac{20\pi}{2n+1}\right)$$

$$a_{n-1}^4=\left(-2\cos\left(\frac{(2n-3)\pi}{2n+1}\right)\right)^4=16\cos^4\left(\frac{(2n-3)\pi}{2n+1}\right)=$$

$$=6+8\cos\left(\frac{2(2n-3)\pi}{2n+1}\right)+2\cos\left(\frac{4(2n-3)\pi}{2n+1}\right)$$

$$a_n^4=\left(-2\cos\left(\frac{(2n-1)\pi}{2n+1}\right)\right)^4=16\cos^4\left(\frac{(2n-1)\pi}{2n+1}\right)=$$

$$=6+8\cos\left(\frac{2(2n-1)\pi}{2n+1}\right)+2\cos\left(\frac{4(2n-1)\pi}{2n+1}\right)$$

Sumando verticalmente $a_n^4=6n+8\left[\frac{2\cos\left(\frac{2n\pi}{2n+1}\right)\sin\left(\frac{2n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right]+2\left[\frac{2\cos\left(\frac{4n\pi}{2n+1}\right)\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right]$

$$a_n^4=6n+8\left[\frac{\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right]+2\left[\frac{\sin\left(\frac{8n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right]\to a_n^4=6n+4(-1)+(-1)$$

$$\therefore a_n^4=6n-5$$

Solution 2 by Le Van-Ho Chi Minh-Vietnam

Lemma I: According to Section 7 of Le Van(2019)

$$x^{2s+1}+1=\prod_{j=0}^{2s}\left[x-exp\left(\frac{(2j+1)i\pi}{2s+1}\right)\right]=\prod_{j=0}^{2s}(x-\omega_j)$$

$$=(x+1)\prod_{j=0}^{s-1}\left[(x-\omega_j)(x-\overline{\omega_j})\right]$$

$$=(x+1)\prod_{j=0}^{s-1}\left[x^2-2Re(\omega_j)x+1\right]$$

Which result in

$$a_j=-2Re(\omega_j)=-2cos\left(\frac{(2j+1)\pi}{2s+1}\right);(j=\overline{1,s-1})$$

Applying Vieta's theorem to the expansion

$$x^{2s+1}+1=(x+1)(x^{2s}-x^{2s-1}+\cdots-x+1),\ we\ get$$

$$\sum_{j=0,j\neq s}^{2s-1}\omega_j=2\sum_{j=1}^{s-1}\omega_j=\sum_{j=0}^{s-1}Re(\omega_j)=-\sum_{j=0}^{s-1}a_j=1$$

Lemma 2: Replacing $\psi_j=\frac{(2j+1)\pi}{2s+1}$*, the following function is bijective*

$$f:X=\{cos\psi_j\}_{j=\overline{0,s-1}}\to Y=\{cos2\psi_j\}_{j=\overline{0,s-1}}\equiv -X=\{-cos|\pi-\psi_j|\}_{j=\overline{1,s-1}}$$

Firstly, we shall prove that f is injective function.Indeed, for two distinct numbers K and

$$l: |\pi - \psi_l| = |\pi - \psi_k| \Leftrightarrow$$

$$\begin{cases} \psi_l = \psi_k \\ \psi_l + \psi_k = 2\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} 2s + 1 - 2(2k + 1) = 2s + 1 - 2(2l + 1) \\ 2(2k + 1) + 2(2l + 1) = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} k = l \\ 2(k + l + 1) = 1 \end{cases}$$

The above result is self-conflicted as $k \neq l$ and $k, l = \overline{0, s-1}$.Thus, $k \neq l$ result in $f(k) \neq f(l)$ which implies that f is an injection.

Secondly, we shall prove that f is a surjective function.For each given number $\lambda_0 = \overline{0, s-1}$, it is supposed to find number $j_0 = \overline{0, s-1}$ such that

$$\left|\pi - \frac{2\pi(2j_0 + 1)}{2s + 1}\right| = \frac{(2\lambda_0 + 1)\pi}{2s + 1}$$

$$\Leftrightarrow |2s + 1 - 2(2j_0 + 1)| = 2\lambda_0 + 1 \Leftrightarrow \begin{cases} 2s + 1 - 2(2j_0 + 1) = 2\lambda_0 + 1 \\ 2s + 1 - 2(2j_0 + 1) = -2\lambda_0 - 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(2j_0 + 1) = 2s - 2\lambda_0 \\ 2(2j_0 + 1) = 2s + 2\lambda_0 + 2 \end{cases} \Leftrightarrow \begin{cases} 2j_0 + 1 = s - \lambda_0 \\ 2j_0 + 1 = s + \lambda_0 + 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} j_0 = \mu = \dfrac{s - \lambda_0 - 1}{2} \\ j_0 = v = \dfrac{s + \lambda_0}{2} \end{cases}$$

The above implies that for every element λ_0, we are able to find j_0 such that $\cos\psi_{j_0} = -\cos\psi_{\lambda_0}$

Exepli gratia, for s=3, the surjective property of function j is illustrated as follow:

j_0	$\cos\psi_j$	$\cos 2\psi_j$	$-\cos\psi_j$	λ_0	illustration
0	$\cos\frac{\pi}{7}$	$\cos\frac{2\pi}{7}$	$-\cos\frac{5\pi}{7}$	2	$j_1 = \frac{s - \lambda_1 - 1}{2} = \frac{3 - 2 - 1}{2} = 0$
1	$\cos\frac{3\pi}{7}$	$\cos\frac{6\pi}{7}$	$-\cos\frac{\pi}{7}$	0	$j_2 = \frac{s - \lambda_2 - 1}{2} = \frac{3 - 0 - 1}{2} = 1$
2	$\cos\frac{5\pi}{7}$	$\cos\frac{10\pi}{7}$	$-\cos\frac{3\pi}{7}$	1	$j_3 = \frac{s - \lambda_3}{2} = \frac{3 + 1}{2} = 2$

As either μ or v is a natural number.Thus, f is a surjection, and consequently a bijection.

Lemma III:

Lemma II results in

$$\sum_{j=0}^{s-1} cos2\psi_j = -\sum_{j=0}^{s-1} cos\psi_j = -\frac{1}{2}\sum_{j=0}^{s-1} Re(\omega_j) = -\frac{1}{2}$$

And therefore

$$\sum_{j=0}^{s-1}(cos\psi_j)^2 = \frac{1}{2}\sum_{j=0}^{s-1}(1 + cos2\psi_j) = \frac{1}{2}\sum_{j=0}^{s-1} 1 + \frac{1}{2}\sum_{j=0}^{s-1} cos2\psi_j = \frac{2s-1}{4}$$

For $a_j = -2cos\psi_j$ *, we get*

$$\sum_{j=1}^{s-1} a_j = 1; \sum_{j=1}^{s-1} a_j^2 = 2s - 1$$

The function f also result in the generalized cased(of which w is a natural number) as follow:

$$\sum_{j=1}^{s-1} cos2^w\psi_j = \frac{(-1)^w}{2}; \sum_{j=1}^{s-1}(cos2^w\psi_j)^2 = \frac{2s-1}{4}$$

Solution: Applying lemma III:

$$\Omega = \sum_{j=0}^{s-1} a_j^4 = \sum_{j=0}^{s-1}\left[\left(-2cos\psi_j\right)^2\right]^2 = 4\sum_{j=0}^{s-1}\left(1 + cos2\psi_j\right)^2$$

$$= 4\sum_{j=0}^{s-1}\left[1 + 2cos2\psi_j + (cos2\psi_j)^2\right]$$

Following the expansion

$$\Omega = 4\sum_{j=1}^{s-1} 1 + 8\sum_{j=0}^{s-1} cos2\psi_j + 4\sum_{j=0}^{s-1}\left(cos2\psi_j\right)^2 = 6s - 5$$

Note by Editor:

Many thanks to Florică Anastase-Romania for typed solutions.