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Define for all $n \geq 1$, $f(\theta) = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos(2^n\theta)}}}}}_{n\ times\ radicals}$ then

prove that

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} (\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta)\frac{f(\theta)}{4\sin\theta}\,d\,\theta$$

$$= \frac{1}{2} + \frac{11\pi}{96} - \frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}+\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}$$

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*Solution by Sergio Esteban-Argentina*

***i)*** ***Probamos que si*** $A \in [0,\pi]$***, se cumpie que***

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos A}}}} = 2\cos\left(\frac{A}{2^n}\right)$$

***Demonstration:***

$$\sqrt{2+2\cos A} = \sqrt{2(1+\cos A)} = \sqrt{2\cdot 2\cos^2\frac{A}{2}} = 2\left|\cos\left(\frac{A}{2}\right)\right| = 2\cos\frac{A}{2}$$

$$\textit{Lado que } 0 \leq \tfrac{A}{2} \leq \tfrac{\pi}{2} \Rightarrow \cos\left(\tfrac{A}{2}\right) = 0$$

$$\sqrt{2+\sqrt{2+2\cos A}} = \sqrt{2+2\cos\left(\frac{A}{2}\right)} = 2\cos\frac{A}{2^2}$$

***Reemplarando lo intenso:***

$$\sqrt{2+\sqrt{2+\sqrt{2+2\cos A}}} = \sqrt{2+2\cos\left(\frac{A}{2^2}\right)} = 2\cos\left(\frac{A}{2^3}\right)$$

***Analogamente***

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos A}}}}}_{\text{"}n\text{" radecales}} = 2\cos\left(\frac{A}{2^n}\right), \forall n\in\mathbb{N}, n\geq 1$$

$\Rightarrow$ ***en el problema*** $A = 2^n\theta \to f(\theta) = 2\cos\theta$ ***(1)***

***ii) Probamos que*** $\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cos 8\theta = \cot\theta$ ***(2)***

***Es facel probar que*** $\tan\theta - \cot\theta = -2\cot 2\theta$

***Entonces:*** $\underbrace{\underbrace{\underbrace{\tan\theta - \cot\theta}_{=-2\cot 2\theta} + 2\tan 2\theta}_{=-4\cot 4\theta} + 4\tan 4\theta} + 8\cot 8\theta + \cot\theta$

$$= -8\cot 8\theta + \quad +8\cot 8\theta + \cot\theta$$

$$= \cot\theta$$

***Nota:***

$$\cot\theta - \tan\theta = \frac{1}{\tan\theta} - \tan\theta = \frac{1-\tan^2\theta}{\tan\theta}\cdot\frac{2}{2} = \frac{2}{\tan 2\theta} = 2\cot 2\theta$$

***Usando (1) y (2) en la integral***

$$I = \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \cot\theta\cdot\frac{1}{2}\cdot\cot\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \cot^2\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} (\csc^2\theta - 1)\, d\theta$$

$$I = \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} \csc^2\theta\, d\theta - \int_{\frac{\pi}{4}}^{\frac{5\pi}{240}} d\theta\right) = \frac{1}{2}\left(-\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{240}} - \theta\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{240}}\right)$$

$$I = \frac{1}{2}\left(\cot\frac{\pi}{4} - \cot\left(\frac{5\pi}{240}\right) - \frac{5\pi}{240} + \frac{\pi}{4}\right)$$

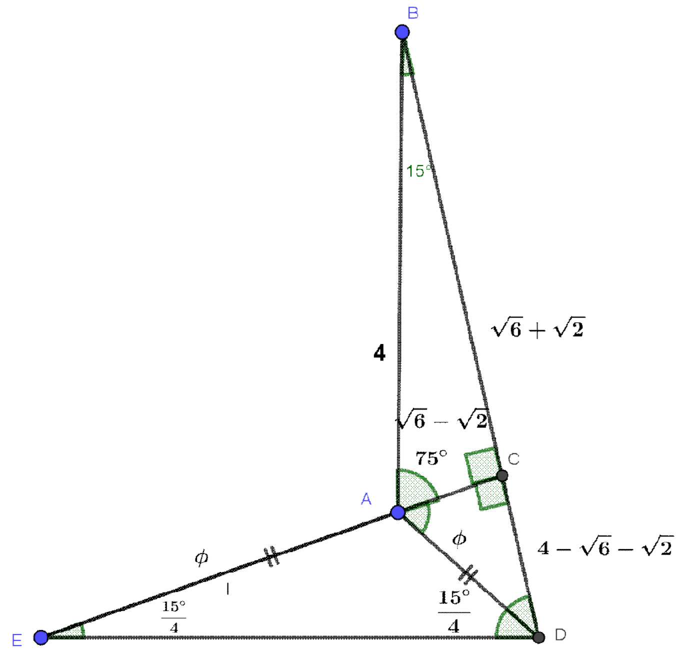
$$I = \frac{1}{2} + \frac{11\pi}{96} - \frac{1}{2}\cot\left(\frac{5\pi}{240}\right)$$

***Resolvemos*** $\cot\left(\frac{5\pi}{240}\right)$

***Consideramos el triangulo notable de*** $75'$ ***y*** $15'$

***i)*** $AB = BD$

***ii)*** $EA = AD = \emptyset$




$$\emptyset = \sqrt{\left[4-\left(\sqrt{6}+\sqrt{2}\right)\right]^2+\left(\sqrt{6}-\sqrt{2}\right)^2}$$

$$\theta = \sqrt{16+\left(\sqrt{6}+\sqrt{2}\right)^2-8\left(\sqrt{6}+\sqrt{2}\right)+\left(\sqrt{6}-\sqrt{2}\right)^2} = \sqrt{32-8\sqrt{6}-8\sqrt{2}}$$

***y SABEMOS que*** $\sqrt{6}-\sqrt{2} = 2\sqrt{2-\sqrt{3}}$ ***esto se deduce de*** $\left(\sqrt{6}-\sqrt{2}\right)^2$

***Entonces***

$$\cot\left(\frac{5\pi}{240}\right) = \cot\left(\frac{15^\circ}{4}\right) = \frac{\emptyset+\left(\sqrt{6}-\sqrt{2}\right)}{4-\sqrt{6}-\sqrt{2}} = \frac{\sqrt{32-8\sqrt{6}-8\sqrt{2}}+2\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}$$

***Por lo tanto*** $\frac{1}{2}\cot\left(\frac{15^\circ}{4}\right) = \frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}+\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}$

$$\therefore I = \frac{1}{2} = \frac{11\pi}{96} - \left(\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}+\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}\right)$$