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By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$12 \leq \sum \left(\frac{b+c}{a}\right)^2 \leq 18\left(\frac{R}{2r}\right)^2 - 6$$

Proposed by George Apostolopoulos – Greece

Solution:

We prove the following lemma:

Lemma:

2) In ΔABC the following relationship holds:

$$\sum \left(\frac{b+c}{a}\right)^2 = \frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2}$$

Solution:

We have $\sum \left(\frac{b+c}{a}\right)^2 = \frac{\sum b^2c^2(b+c)^2}{a^2b^2c^2} = \frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2}$

which follows from the known identity in triangle

$$\sum b^2c^2(b+c)^2 = 4s^2[s^2(s^2+r^2-12Rr)+r^2(12R^2+4Rr+r^2)]$$

Let's get back to the main problem.

The left hand inequality.

Using the Lemma the inequality can be written:

$\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2} \geq 12$, ***which follows from Gerretsen's inequality***

$$s^2 \geq 16Rr - 5r^2$$

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2) \geq 48R^2r^2 \Leftrightarrow$$

$\Leftrightarrow 7R^2 - 16Rr + 4r^2 \geq 0 \Leftrightarrow (R-2r)(7R-2r) \geq 0$, ***obviously from Euler's inequality*** $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

The right hand inequality

Using the Lemma the inequality can be written:

$\frac{s^2(s^2+2r^2-12r)+r^2(12R^2+4Rr+r^2)}{4R^2r^2} \le 18\left(\frac{R}{2r}\right)^2 - 6$, *which follows from Gerretsen's inequality:*

$$s^2 \le 4R^2 + 4Rr + 3r^2$$

It remains to prove that:

$$\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2} \le$$

$$\le 18\left(\frac{R}{2r}\right)^2 - 6 \Leftrightarrow$$

$$\Leftrightarrow R^4 + 8R^3r - 18R^2r^2 - 8r^4 \ge 0 \Leftrightarrow (R-2r)(R^2+10R^2r-2Rr^2-4r^3) \ge 0$$

Obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

Inequality 2) can be strengthened.

3) In ΔABC the following relationship holds:

$$19 - \frac{16r}{R} + \frac{4r^2}{R^2} \le \sum\left(\frac{b+c}{a}\right)^2 \le 3 - \frac{4R}{r} + \frac{4R^2}{r^2} + \frac{4r^2}{R^2}$$

Proposed by Marin Chirciu – Romania

Solution:

The left hand inequality.

Using the Lemma the inequality can be written:

$\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2} \ge 19 - \frac{16r}{R} + \frac{4r^2}{R^2}$, *which follows from Gerretsen's inequality*

$$s^2 \ge 16Rr - 5r^2$$

It remains to prove that:

$$\frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2} \ge 19 - \frac{16r}{R} + \frac{4r^2}{R^2}$$

true with equality.

Equality holds if and only if the triangle is equilateral.

The right hand inequality.

Using the Lemma the inequality can be written:

$$\frac{s^2(s^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2}\le 3-\frac{4R}{r}+\frac{4R^2}{r^2}+\frac{4r^2}{R^2}$$

which follows from Gerrentsen's inequality $s^2\le 4R^2+4Rr+3r^2$

It remains to prove that:

$$\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^2(12R^2+4Rr+r^2)}{4R^2r^2}\le$$

$$\le 3-\frac{4R}{r}+\frac{4R^2}{r^2}+\frac{4r^2}{R^2}$$

true with equality.

Equality holds if and only if the triangle is equilateral.

Remark.

Inequality 3) is stronger than 2)

4) In ΔABC the following relationship holds:

$$12\le 19-\frac{16r}{R}+\frac{4r^2}{R^2}\le\sum\left(\frac{b+c}{a}\right)^2\le 3-\frac{4R}{r}+\frac{4R^2}{r^2}+\frac{4r^2}{R^2}\le 18\left(\frac{R}{2r}\right)^2-6$$

Solution:

See 3) and the inequalities:

i) $12\le 19-\frac{16r}{R}+\frac{4r^2}{R^2}\Leftrightarrow 7R^2-16Rr+4r^2\ge 0\Leftrightarrow (R-2r)(7R-2r)\ge 0$

obviously from Euler's inequality $R\ge 2r$.

ii) $3-\frac{4R}{r}+\frac{4R^2}{r^2}+\frac{4r^2}{R^2}\le 18\left(\frac{R}{2r}\right)^2-6\Leftrightarrow R^4+8R^3r-18R^2r^2-8r^4\ge 0\Leftrightarrow$

$\Leftrightarrow (R-2r)(R^2+10R^2r-2Rr^2-4r^3)\ge 0$, ***obviously from Euler's inequality***

$R\ge 2r$.

Refferences:
Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**