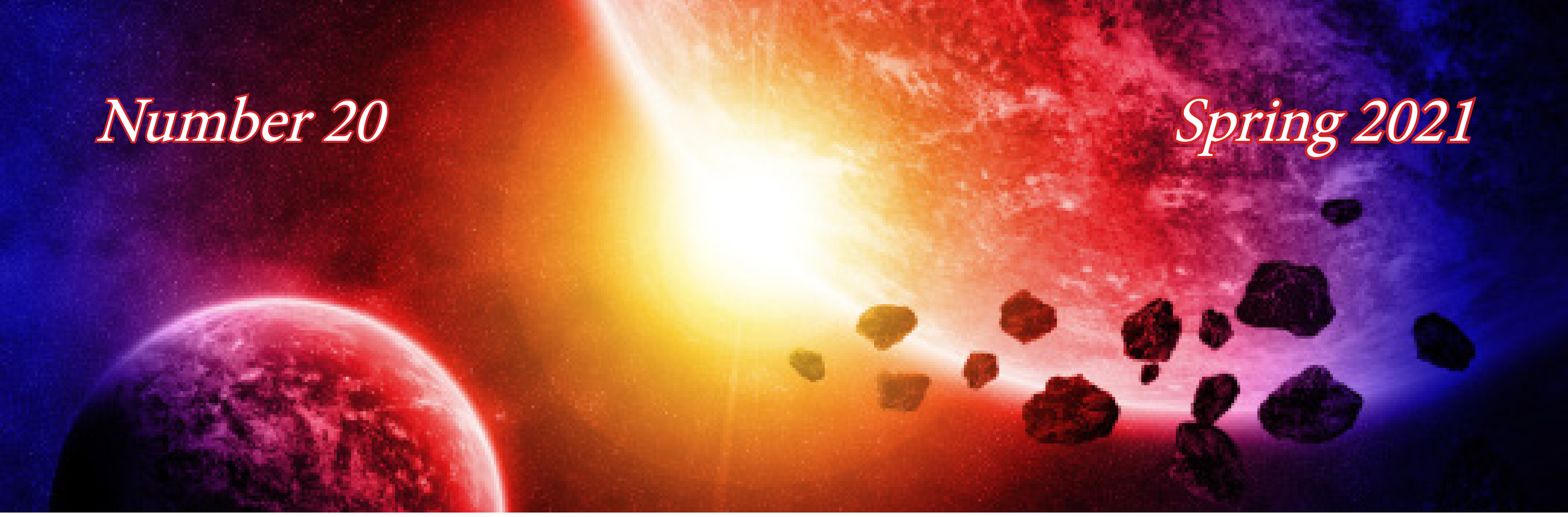

Number 20 Spring 2021

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

PROBLEMS FOR JUNIORS

JP.286. If $a, b, c > 0; ab + bc + ca = 3$ then:

$$\frac{(a^2+b^2)(ab+1)}{a+b} + \frac{(b^2+c^2)(bc+1)}{b+c} + \frac{(c^2+a^2)(ca+1)}{c+a} \geq 6$$

Proposed by Daniel Sitaru - Romania

JP.287. If $x, y, z \in (0, \frac{\pi}{2})$ then:

$$\left(\prod_{cyc} \log(1+2\sin^2 x\right)\left(\prod_{cyc} \log(1+2\cos^2 x)\right) \leq \log^6 2$$

Proposed by Daniel Sitaru - Romania

JP.288. Solve for real numbers:

$$2\sqrt{\frac{x-2}{x+3}} + \frac{x+4}{(\sqrt{x-2}+\sqrt{x+4})^2} = 1$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.289. If $a, b, c > 0; abc(a+b+c) = 3$ then:

$$(a^5-2a+4)(b^5-2b+4)(c^5-2c+4) \geq 9\sqrt{3(a^2+b^2+c^2)}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.290. In ΔABC the following relationship holds:

$$\frac{w_a w_b}{w_c} + \frac{w_b w_c}{w_a} + \frac{w_c w_a}{w_b} \geq \frac{h_a h_b}{h_c} + \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.291. Solve for real numbers:

$$\begin{cases} \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} = 3 \\ \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = 3\sqrt[3]{\frac{a^3+b^3+c^3}{3}} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.292. If $x, y, z > 0; a \geq 3; xy + yz + zx = 3$ then:

$$\frac{1}{a + x^2} + \frac{1}{a + y^2} + \frac{1}{a + z^2} \leq \frac{3}{a + 1}$$

Proposed by Marin Chirciu - Romania

JP.293. If $a, b, c \geq 0; a + b + c = 3; k \geq \frac{3}{2}$ then:

$$(k + a^2)(k + b^2)(k + c^2) \geq (k + 1)^3$$

Proposed by Marin Chirciu - Romania

JP.294. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a}{b(b + 2c)^2} + \frac{b}{c(c + 2a)^2} + \frac{c}{a(a + 2b)^2} \geq \frac{9}{(\sqrt{a} + \sqrt{b} + \sqrt{c})^3}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.295. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a^2}{b(a + 5c)^3} + \frac{b^2}{c(b + 5a)^3} + \frac{c^2}{a(c + 5b)^3} \geq \frac{1}{24(\sqrt{a} + \sqrt{b} + \sqrt{c})}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.296. If $a, b, c > 0; abc = 1; n \geq 2$ then:

$$a + b + c + \frac{1}{n + a} + \frac{1}{n + b} + \frac{1}{n + c} \geq \frac{3(n + 2)}{n + 1}$$

Proposed by Marin Chirciu - Romania

JP.297. If $a, b, c > 0; ab + ba + ca = 3$ then find:

$$\min \Omega(a, b, c); \Omega(a, b, c) = \frac{1}{(a + b)^5} + \frac{1}{(b + c)^5} + \frac{1}{(c + a)^5}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.298. If $x, y, z \in \mathbb{R}$ such that:

$$\begin{cases} \tan x(\tan y + \tan z) = 5 \\ \tan z(\tan x + \tan y) = 9 \text{ then find } \Omega = x + y + z \\ \tan y(\tan z + \tan x) = 8 \end{cases}$$

Proposed by Daniel Sitaru - Romania

JP.299. If $a, b, c > 0; abc = 4$ then:

$$\frac{2(a^5+b^5)+c^5}{(a+b)^2}+\frac{2(b^5+c^5)+a^5}{(b+c)^2}+\frac{2(c^5+a^5)+b^5}{(c+a)^2}\geq 15$$

Proposed by Daniel Sitaru - Romania

JP.300. In $\Delta ABC, I$ - incenter, ID, IE, IF - symedians in $\Delta BIC, \Delta CIA, \Delta AIB, D \in (BC), E \in (CA), F \in (AB)$. Prove that:

$$\frac{[DEF]}{[ABC]}\geq\frac{r^2}{2R^2-Rr-2r^2}$$

Proposed by Marian Ursărescu - Romania

PROBLEMS FOR SENIORS

SP.286. In ΔABC the following relationship holds:

$$\frac{\sqrt[4]{27}}{\sqrt{(n+1)R}}\leq\frac{1}{\sqrt{na+b}}+\frac{1}{\sqrt{nb+c}}+\frac{1}{\sqrt{nc+a}}\leq\frac{\sqrt[4]{27}}{\sqrt{(n+1)2r}}; n\in\mathbb{N}$$

Proposed by George Apostolopoulos - Greece

SP.287. In ΔABC the following relationship holds:

$$\frac{2}{(n+1)R}\leq\frac{1}{nm_a+mb}+\frac{1}{nm_b+m_c}+\frac{1}{nm_c+m_a}\leq\frac{1}{(n+1)r}$$

Proposed by George Apostolopoulos - Greece

SP.288. If $a, b, c > 0; a + b + c = 3$ then find:

$$\Omega=\min\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sqrt[3]{\frac{ab}{a^2-ab+b^2}}+\sqrt[3]{\frac{bc}{b^2-bc+c^2}}+\sqrt[3]{\frac{ca}{c^2-ca+c^2}}\right)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.289. If $x, y, z > 0; x + y + z = 1$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{x\sin A+y\sin B+z\sin C}\leq\sum_{cyc}\frac{1}{\sin A+\sin B-\sin C}$$

Proposed by George Apostolopoulos - Greece

SP.290. If $x, y, z > 0; x + y + z = 3$ then:

$$\sqrt[6]{x}+\sqrt[6]{y}+\sqrt[6]{z}+12\geq 5(xy+yz+zx)$$

When equality holds?

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.291. In acute ΔABC, let h_1, h_2, h_3 be the altitudes of orthic triangle. Prove that:

$$\frac{F^4}{sR^2} \le \frac{\pi^3}{486\sqrt{3}}\left(\sum_{cyc} h_1 \cdot r_a\right)\left(\sum_{cyc} \frac{h_a^3}{\mu^3(A)}\right)$$

Proposed by Proposed by Radu Diaconu - Romania

SP.292. Let $ABCD$ be a tangential quadrilateral circumscribed to a circle with radii $r = 1$. Prove that:

$$[ABCD] \cdot \sum_{cyc} \mu(A)\sqrt{\mu(A) + \mu(B) + \mu(C)} \le 2\pi\sqrt{\frac{3\pi}{2}\left(\sum_{cyc} \cot\frac{A}{2}\right)}$$

Proposed by Radu Diaconu - Romania

SP.293. If $0 < a \le b < e; 0 \le x \le y \le z; x + y + z = 3$ then:

$$(x-1)\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab}} + (y-1)(\sqrt{ab})^{\frac{1}{ab}} + (z-1)\left(\frac{a+b}{2}\right)^{\frac{2}{a+b}} \ge 0$$

Proposed by Daniel Sitaru - Romania

SP.294. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a^2}{\sqrt[3]{4(b^6+1)} + 4b} + \frac{b^2}{\sqrt[3]{4(c^6+1)} + 4c} + \frac{c^2}{\sqrt[3]{4(a^6+1)} + 4a} \ge \frac{1}{2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.295. In ΔABC the following relationship holds:

$$(a+2r)(b+2r)(c+2r) \ge 2R^3(3\sqrt{3}+5)$$

Proposed by Florentin Vişescu - Romania

SP.296. In acute $\Delta ABC, H$ - orthocenter, O - circumcenter, I - incenter the following relationship holds:

$$\sum_{cyc} \frac{(r_a + AH)^2}{\mu(A) + r_a \cdot \cos A} \ge \frac{36(4OI^2 + 12Rr + r^2)}{4\pi + 9R}$$

Proposed by Radu Diaconu - Romania

SP.297. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{12} + \sum_{cyc} \frac{a}{7bc + \sqrt[3]{4(b^6+c^6)}} \ge \frac{7}{12}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.298. Find without softs:

$$\Omega=\int_0^{\frac{\pi}{4}}\left(\log(\cos 2x)+\left(\frac{1-\tan x}{1+\tan x}\right)\log(1+\sin 2x)\right)dx$$

Proposed by Proposed by Pedro Pantoja - Natal - Brasil

SP.299. Solve for real numbers:

$$\begin{cases}\sqrt{x^2+y^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}+\sqrt{(x-4)^2+(y-3)^2}=10\\ x+2y=5z\end{cases}$$

Proposed by Daniel Sitaru - Romania

SP.300. If $A\in M_n(\mathbb{R});A^3=2A^2+7A+4I_n$ then find:

$$\Omega=\det(A^2-3A+3I_n)$$

Proposed by Marian Ursărescu - Romania

UNDERGRADUATE PROBLEMS

UP.286. If $n\in\mathbb{R};n\geq 2$ then:

$$\int_0^n (n+x)^{\frac{1}{\sqrt{n^2+x^2}}}dx\geq n\sqrt[n]{(1+\sqrt{2})^{\log(n\sqrt{2})}}$$

Proposed by Florică Anastase - Romania

UP.287. If $0<a\leq b$ then:

$$\tan^{-1}\left(\frac{2ab}{a+b}\right)+\int_{\frac{2ab}{a+b}}^{\sqrt{ab}}e^{-t^2}dt\leq\tan^{-1}(\sqrt{ab})$$

Proposed by Daniel Sitaru - Romania

UP.288. Solve for real numbers:

$$\begin{cases}\frac{x^2}{y}+\frac{y^2}{x}=\sqrt[3]{128(x^8+y^8)}\\ 4x^3-3y=\sqrt{\frac{1+\sqrt{1-xy}}{2}}\end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.289. If $0\leq k\leq m;0\leq l\leq n$ then:

$$\left(\frac{4}{\pi}\right)^{k+l}\int_0^1(\tan^{-1}(1+x^2))^{m+n}dx\geq\int_0^1\frac{(\tan^{-1}(1+x^2))^{m+n}}{(\tan^{-1}(1+x^2))^{k+l}}dx$$

Proposed by Daniel Sitaru - Romania

UP.290. If $a, b \geq 1; f \in C^2([0,1]), f(0) \leq 0, f$ convexe, then:

$$2\int_0^1 f(x)dx \geq a^2\int_0^{\frac{1}{a}} f(x)dx + b^2\int_0^{\frac{1}{b}} f(x)dx$$

Proposed by Daniel Sitaru - Romania

UP.291. Prove that:

$$\int_0^{\frac{1}{2}} \left(\log(1+x)\log\left(\frac{3}{2}+x\right)\right)dx < \frac{1}{2}\left(\int_0^1 \log(1+x)dx\right)^2$$

Proposed by Daniel Sitaru - Romania

UP.292. Prove that:

$$\int_0^1 \left(\tan^{-1} x + \frac{x}{1+x^2}\right)^2 dx + 4\int_0^1 \frac{1}{(1+x^2)^4}dx > \frac{(\pi+2)^2}{16}$$

Proposed by Daniel Sitaru - Romania

UP.293. Solve for real numbers:

$$\begin{cases} \frac{x^2}{2y} + \frac{y^2}{2x} = \sqrt[4]{\frac{x^4+y^4}{2}} \\ x^2y^2 - y^3 + 1 = \sqrt{2x^2 - 2y + 1} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.294. In $\Delta ABC, AD, BE, CF$ medians; G - centroid; $AM = MG, M \in (AG)$;

$$2\cot A = \cot B + \cot C$$

Prove that: $DEMF$ is a cyclic quadrilateral.

Proposed by Marian Ursărescu - Romania

UP.295. Find $a, b, c, d \in \mathbb{R}$ such that:

$$\begin{cases} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^3 + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{cases}$$

Proposed by Daniel Sitaru - Romania

UP.296. If $x, y, z > 0; xy + yz + zx = 3$ then:

$$\sqrt[3]{\frac{2x}{(y+z)^7}} + \sqrt[3]{\frac{2y}{(z+x)^7}} + \sqrt[3]{\frac{2z}{(x+y)^7}} \geq \frac{3}{4}$$

Proposed by Pedro Pantoja - Brazil

UP.297. If $a_n \in (0,\infty); n \in \mathbb{N}^*; \lim_{n\to\infty} \frac{a_{n+1}}{na_n} = 0$ then find:

$$\Omega = \lim_{n\to\infty} \sqrt[n^2]{a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \cdot \sqrt[n]{a_n}}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.298. Find:

$$\Omega = \lim_{n\to\infty} \left(\sqrt[2n+2]{(2n+1)!! \cdot (n+1)!} - \sqrt[2n]{(2n-1)!! \cdot n!} \right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.299. If $m, p \in \mathbb{N}^*$ - fixed values; $a_n \in (0,\infty); b_n \in (0,\infty)$, $n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{a_{n+1}}{(3n+m)a_n} = a > 0; \lim_{n\to\infty} \frac{b_{n+1}}{(4n+p)b_n} = b > 0 \text{ then find:}$$

$$\Omega = \lim_{n\to\infty} \left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1} \cdot b_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n \cdot b_n}} \right)$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

UP.300. If $f, \Gamma : (0,\infty) \to (0,\infty); \lim_{x\to\infty} \frac{f(x+1)}{x^2 f(x)} = a > 0$ and exists $\lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x^2}$ then find:

$$\Omega = \lim_{n\to\infty} \left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x^2} - \frac{(\Gamma(x))^{\frac{1}{x+1}}}{(x+1)^2} \right) \cdot (f(x))^{\frac{1}{x}}$$

Proposed by D.M. Bătineţu - Giurgiu, Neculai Stanciu - Romania

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